AST513 Physical Cosmology in 2024

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In this course (formerly known as theoretical cosmology), we study the history of our universe on large scales. We first discuss key cosmological observations that led to our standard model of cosmology, and we study in detail the origin and the evolution of the Universe such as inflation, big bang nucleosynthesis, and cosmic microwave background anisotropies. In the second part we learn (relativistic) perturbation theory and apply it to initial conditions, large-scale structure and weak gravitational lensing. It is recommended to have good understanding of general relativity and quantum field theory, albeit not mandatory.

Contents

To students: This lecture note includes (almost) all the equations needed to understand the lecture. However, the physical meaning of those equations is described in this note at the minimum level, purposefully leaving the full descriptions to the lectures in the class room. Also note that one has to derive all the equations to be able to fully grasp the meaning. However, the oral exam will cover only the material discussed in the lectures.

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1 Early Universe Physics

1.1 Chronology of the Early Universe

t	$T \propto \rho^{1/4}$	Redshift	Event
10^{-43} s	10^{19} GeV	∞	Planck energy, Quantum gravity? Big Bang Singularity?
10^{-38} s	10^{16} GeV	∞	Inflation ends? Grand Unification Scale? Baryogenesis?
10^{-11} s	100 GeV	10^{15}	Electroweak phase transition (spontaneous symmetry breaking)
10^{-5} s	150 MeV	10^{12}	Quark-hadron (QCD) phase transition ($T_c \simeq \Lambda_{\rm QCD}$)
1 sec	1 MeV	6×10^9	ν_e decoupling ($\nu_e \approx 1$ MeV, $\nu_\mu, \nu_\tau \approx 3$ MeV)
6 sec	500 keV	2×10^9	$e\bar{e}$ annihilation
3 min	100 keV	4×10^8	Nucleosynthesis (BBN)
60 kyr	0.75 eV	3200	Matter-radiation equality
300 kyr	0.3 eV	1100	Atom formation, photon decoupling (CMB)
400 Myr	5 meV	~ 10	Reionization
9 Gyr	0.33 meV	0.4	Dark energy-matter equality
Now	10^{-4} eV (2.73 K)	0	now

At the early Universe, the Universe was denser and hotter, dominated by the relativistic particles and radiation. Because of its high energy, particles and anti-particle pairs are created and annihilated. This process depends on the particle contents of our nature. The standard model of particle physics is well tested and understood up to ~ 1 TeV (horizontal line in the table), beyond which the predictions from the standard model are somewhat uncertain and other beyond-the-standard-model physics have vastly different predictions. Our discussion will be limited to the standard model physics.

In this radiation dominated era, almost all particles behave like massless particles, and their energy density evolves as radiation $\rho \propto 1/a^4$. In RDE, the Hubble expansion and the age of the Universe are well approximated in terms of the equilibrium temperature T of the plasma as

$$H \simeq 0.3\,\text{sec}^{-1}\sqrt{g_*}\left(\frac{T}{1\text{ MeV}}\right)^2 , \qquad t = \frac{1}{2H} \simeq 1\text{ sec}\left(\frac{T}{1\text{ MeV}}\right)^{-2} g_*^{-1/2} , \tag{1.1}$$

where g_* is the total spin-degeneracy factor shown in Figure 1.1. Particles stay in thermal equilibrium with the plasma, as long as their interaction rate Γ with the plasma remains sufficiently high:

$$\Gamma = n\,\langle\sigma v\rangle \geq H , \tag{1.2}$$

where σ is the interaction cross section $[\sigma] = L^2$, v is the relative velocity of the particles in interaction, and n is the particle number density. Note that the interaction rate is averaged over the particle velocity distribution. A useful conversion relation is

$$1\text{ MeV} = 1.602 \times 10^{-6}\text{ erg} = 1.161 \times 10^{10}\text{ K} . \tag{1.3}$$

At $T < 10^{16}$ GeV, the dominant interaction among the relativistic particles is mediated by massless gauge bosons, and the cross section is $\sigma \sim \alpha^2/T^2$, such that the interaction rate is $\Gamma \propto n\sigma v \sim \alpha^2 T$, where the $SU(2)$ gauge coupling constant is $g = 1/\sqrt{4\pi\alpha}$ and we used $n \propto T^3$, $v \sim 1$. Therefore, the interaction is efficient to maintain the thermal equilibrium

$$\frac{\Gamma}{H} \sim \frac{10^{16}\text{GeV}}{T} \gg 1 \quad \text{for } T < 10^{16}\text{ GeV} . \tag{1.4}$$

At lower temperature $T < 300$ GeV, the interactions are now mediated by *massive* gauge bosons (e.g., $m_Z \simeq 100$ GeV), and the cross section is $\sigma \sim G_F^2 T^2$, such that the interaction rate is $\Gamma \propto G_F^2 T^5$. Therefore, the interaction is again efficient to maintain the thermal equilibrium up to $T > 1$ MeV,

$$\frac{\Gamma}{H} \sim G_F^2 T^3 \sim \left(\frac{T}{1\text{ MeV}}\right)^3 \gg 1 \quad \text{for } T > 1\text{ MeV} , \tag{1.5}$$

where the Fermi constant is $G_F = 1.15 \times 10^{-5}$ GeV^{-2}. Then, the question arises: what happens at $T > 10^{16}$ GeV? The Universe might not have been in thermal equilibrium at such early time.

A brief overview of the most important cosmological events are as follows (Mo et al., 2010):

- At $T \gg 1$ TeV, two important events must take place: inflationary expansion and baryogensis. An inflationary expansion for a very short period of time must have taken place to explain some of the key problems in observational cosmology, and some mechanism beyond the standard model must have been in operation to generate the asymmetry between baryons and anti-baryons we observe today. The former is highly constrained and relatively well understood, while the latter is very poorly understood. Beyond these two events that must have happened in the early Universe, there might have been other interesting events in other beyond-the-standard-model physics such as the grand unification. During this stage, quarks and gluons are not bound to hardronic states, such that there exist no protons, neutrons and so on. The Universe was made of fundamental elementary particles, forming a hot plasma (or soup).

- At T ~150 MeV ($t \sim 10^{-5}$ sec), the quark–hadron phase transition occurs, confining quarks into hadrons, and the chiral symmetry is broken. Lattice QCD calculations show that the electroweak and QCD phase transitions are smooth. Once the transition was complete, the Universe was filled with a hot plasma consisting of three types of relativistic pions $\pi^\pm$, π^0 ($m_{\pi^\pm} = 139.6$ MeV, $m_{\pi^0} = 135.0$ MeV), non-relativistic nucleons (p, n), relativistic leptons $e^\pm$, $\mu^\pm$ ($m_\mu = 105$ MeV), and their associated neutrinos ($\nu_e, \bar\nu_e, \nu_\mu, \bar\nu_\mu, \nu_\tau, \bar\nu_\tau$), and photons, all in thermal equilibrium. Heavier lepton τ ($m_\tau = 1.78$ GeV) have already annihilated, and only a trace amount due to lepton asymmetry must have remained.

- At $T \sim 100$ MeV ($t \sim 10^{-4}$ sec), pions become non-relativistic, and $\pi^\pm$-pairs annihilate each other, while the neutral pions π^0 decay into photons. From this point on, protons and neutrons are the only hadronic species left. At about the same time, muons $\mu^\pm$ start to annihilate.

- At $T \sim 1$ MeV ($t \sim 1$ sec), electrons and positrons become non-relativistic, annihilating each other. At about the same time, e-neutrinos ν_e also decouple from the hot plasma. μ- and τ-neutrinos decouple a bit earlier than e-neutrinos. The weak interactions become ineffective, and the ratio of neutrons to protons is frozen.

- At $T \sim 0.1$ MeV ($t \sim 3$ minutes), the Big Bang Nucleosynthesis (BBN) starts, synthesizing protons and neutrons to produce D, He and a few other heavy elements. This nuclear fusion is exactly the same as one at the core of stars, but it takes place everywhere in the Universe.

- At $T \sim 4000$ K ($t \sim 2 \times 10^5$ yr), free electrons and protons recombine to form neutral hydrogen atoms. The Universe then becomes transparent to photons, and these free-streaming photons are observed today as the cosmic microwave background (CMB) in a black-body distribution.

- dark age, first stars, cosmic reionization, habitable planets and life formation, dark energy domination

1.2 Thermal Equilibrium in the Early Universe

1.2.1 Chemical Potential

• ***Thermodynamic Quantities***.— Consider creating a system with internal energy U in an environment with temperature T. The Helmholtz free energy $F = U - TS$ is needed to create such system with the help from the environment, where S is the entropy of the final system. In a given environment, the system tends to minimize the internal energy or maximize the entropy, i.e., minimize the Helmholtz free energy. At the minimum of the Helmholtz free energy, the system reaches the thermal equilibrium. The Enthalpy $H = U + PV$ is similar, but such system is created from a small volume, that more energy for PV work is needed. Finally, the Gibbs free energy is the combination of all: $G = U - TS + PV$. They are all related by the Legendre transformation.

• ***Legendre Transformation***.— converts a function of a set of variables to another function of their conjugate variables. For example, consider a function $f(x, y)$. The conjugate variables of (x, y) are (U, W)

$$U := \left(\frac{\partial f}{\partial x}\right)_y , \qquad W := \left(\frac{\partial f}{\partial y}\right)_x , \qquad df = U\,dx + W\,dy . \tag{1.6}$$

Now consider a combination of two variables Wy and a new function $g := f - Wy$:

$$d(Wy) = y\,dW + W\,dy\,, \qquad dg = df - d(Wy) = U\,dx - y\,dW\,, \tag{1.7}$$

which implies that the function g has two independent variables x and W:

$$g = g(x, W)\,, \qquad U = \left(\frac{\partial g}{\partial x}\right)_W\,, \qquad y = -\left(\frac{\partial g}{\partial W}\right)_x\,. \tag{1.8}$$

In this way, three functions can be obtained by Legendre transforming $f(x, y)$ with two variables.

• ***Chemical Potential***.— Consider a thermodynamic system, in which particles are created and annihilated. The amount of energy needed to create a particular species is called the chemical potential (by definition):

$$dU =: TdS - PdV + \sum_{i=1}^{n} \mu_i dN_i\,, \qquad \mu_i = \left(\frac{\partial U}{\partial N_i}\right)_{S,V,N_{j\neq i}}\,, \tag{1.9}$$

when the entropy and the volume of the system are held fixed. While exact in the definition, it is in practice difficult to find a situation, where the volume and the entropy is held fixed. Instead, the other relation is more illuminating for the meaning of the chemical potential:

$$dG = -SdT + VdP + \sum_{i=1}^{n} \mu_i dN_i\,, \qquad \mu_i = \left(\frac{\partial G}{\partial N_i}\right)_{T,P,N_{j\neq i}}\,. \tag{1.10}$$

In thermodynamic equilibrium with constant temperature and pressure, the system exchange particles with its environments. Then we have

$$dG = 0\,, \qquad \sum_{i=1}^{n} \mu_i dN_i = 0\,. \tag{1.11}$$

The chemical potential μ is independent of its fundamental physical properties of particles, but determined by the interactions and the thermodynamic system (e.g., what is conserved). However, since photons are always created and absorbed by a black body, its chemical potential is always zero in equilibrium. Another example is the electron pair production process:

$$\text{env} + \gamma + \gamma \longleftrightarrow e + \bar{e} + \text{env}\,, \qquad 2\,\mu_\gamma = \mu_e + \mu_{\bar{e}}\,, \qquad \therefore\ \mu_e = -\mu_{\bar{e}}\,, \tag{1.12}$$

in which the chemical potential of a particle and its anti-particle has the opposite sign.

1.2.2 Equilibrium Distribution

As long as the scattering process or the interactions between particles are rapid, particles are in kinetic equilibrium, and their phase-space distribution function $f(x, p, t)$ is described by the thermal equilibrium distribution:

$$f(p,t)d^3\mathbf{p} = \frac{g}{(2\pi)^3}\frac{d^3\mathbf{p}}{\exp[(E-\mu)/T] \pm 1}\,, \qquad \begin{cases} + & : \text{Fermion} \\ - & : \text{Boson} \end{cases}\,, \tag{1.13}$$

where g is the spin-degeneracy factor for a given phase-space density and $(2\pi\hbar)^3$ is the unit phase-space volume. Mind that our convention assumes $\hbar = c = k = 1$. In a homogeneous and isotropic background universe, the position dependence and directional dependence vanish. The physical quantities of such particle distribution are

$$n(t) = \int d^3\mathbf{p}\, f(p,t) = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{E^2 - m^2}\, EdE}{\exp[(E-\mu)/T] \pm 1}\,, \tag{1.14}$$

$$\rho(t) = \int d^3\mathbf{p}\, Ef(p,t) = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{E^2 - m^2}\, E^2dE}{\exp[(E-\mu)/T] \pm 1}\,, \tag{1.15}$$

$$P(t) = \int d^3\mathbf{p}\, \frac{1}{3}\frac{p^2}{E} f(p,t) = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{3/2}\, dE}{\exp[(E-\mu)/T] \pm 1}\,, \tag{1.16}$$

where the isotropic pressure is derived from $P = \frac{1}{3}n\langle pv\rangle$ and $v = p/E$. Since the baryon to photon number ratio is so small,

$$\eta := \frac{n_b}{n_\gamma} \simeq 5\times 10^{-10} , \tag{1.17}$$

the chemical potential of all species may be approximated as zero for computing the thermodynamic quantities of the early Universe, where photons with $\mu_\gamma = 0$ are the dominant. The ratio of the lepton number density to the photon is also expected to be the same as η.

For non-relativistic particles ($m \gg T$, $E \simeq m$), the distinction between Fermionic and Bosonic particles disappear, and they all follow the classical Maxwell-Boltzmann distribution

$$f(p,t) = \frac{g}{(2\pi)^3}\exp\left(-\frac{m-\mu}{T}\right)\exp\left(-\frac{p^2}{2mT}\right) . \tag{1.18}$$

By integrating the distribution, we obtain

$$n(t) = g\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left[-\frac{m-\mu}{kT}\right] , \qquad \rho(t) = mn , \qquad P(t) = nkT \tag{1.19}$$

In contrast, for relativistic particles ($T \gg m$, $T \gg \mu$), the physical quantities are

$$n(t) = \begin{cases} \frac{g}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3 & : \text{Boson} \\ \frac{3g}{4\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3 & : \text{Fermion}\end{cases} , \qquad \rho(t) = \begin{cases} \frac{g\pi^2}{30}kT\left(\frac{kT}{\hbar c}\right)^3 & : \text{Boson} \\ \frac{7}{8}\frac{g\pi^2}{30}kT\left(\frac{kT}{\hbar c}\right)^3 & : \text{Fermion}\end{cases} , \qquad P(t) = \frac{1}{3}\rho(t) \tag{1.20}$$

where the Riemann-Zeta function is

$$\zeta(n) := \sum_{i=1}^{\infty}\frac{1}{i^n} , \qquad \zeta(3) \simeq 1.202 . \tag{1.21}$$

Since the number density of non-relativistic particles in thermal equilibrium is exponentially suppressed, only the relativistic components matter in determining the thermodynamic quantities of the system:

$$n_{\rm tot}(T) = \frac{\zeta(3)}{\pi^2}g_{*,n}T^3 , \qquad \rho_{\rm tot}(T) = \frac{\pi^2}{30}g_*T^4 , \qquad P_{\rm tot}(T) = \frac{1}{3}\rho(T) , \tag{1.22}$$

where we assumed $\mu_i \equiv 0$ for all species and defined

$$g_{*,n} := \sum_{i\in\text{Boson}} g_i\left(\frac{T_i}{T}\right)^3 + \left(\frac{3}{4}\right)\sum_{i\in\text{Ferm.}} g_i\left(\frac{T_i}{T}\right)^3 , \qquad g_* := \sum_{i\in\text{Boson}} g_i\left(\frac{T_i}{T}\right)^4 + \left(\frac{7}{8}\right)\sum_{i\in\text{Ferm.}} g_i\left(\frac{T_i}{T}\right)^4 . \tag{1.23}$$

1.2.3 Entropy Density

In the early Universe, particles are created and annihilated. As the Universe expands and cools, some particles annihilate and disappear. Hence the total number is *not* conserved. So, it is useful to have some quantity that is related to the conservation law, i.e., entropy density $s(T)$. Assuming $\mu \equiv 0$, the entropy density for relativistic particles is defined as

$$s := \frac{1}{T}(\rho + P)_{\rm tot} = g_{*,s}\left(\frac{2\pi^2}{45}\right)T^3 , \qquad g_{*,s} := \sum_{i\in\text{Boson}} g_i\left(\frac{T_i}{T}\right)^3 + \left(\frac{7}{8}\right)\sum_{i\in\text{Ferm.}} g_i\left(\frac{T_i}{T}\right)^3 . \tag{1.24}$$

We will show that the conservation of total entropy of the Universe states

$$S := sa^3 , \qquad \frac{d}{dt}S = 0 , \qquad g_{*,s}^{1/3}(T)\,T \propto \frac{1}{a} . \tag{1.25}$$

The thermodynamic laws $TdS = dU + PdV$ apply to the whole system, which is the Universe in our case. The total energy or entropy, etc of the Universe are ill-defined. Instead, we look for local densitites that represent the entropy, i.e., the entropy density s.

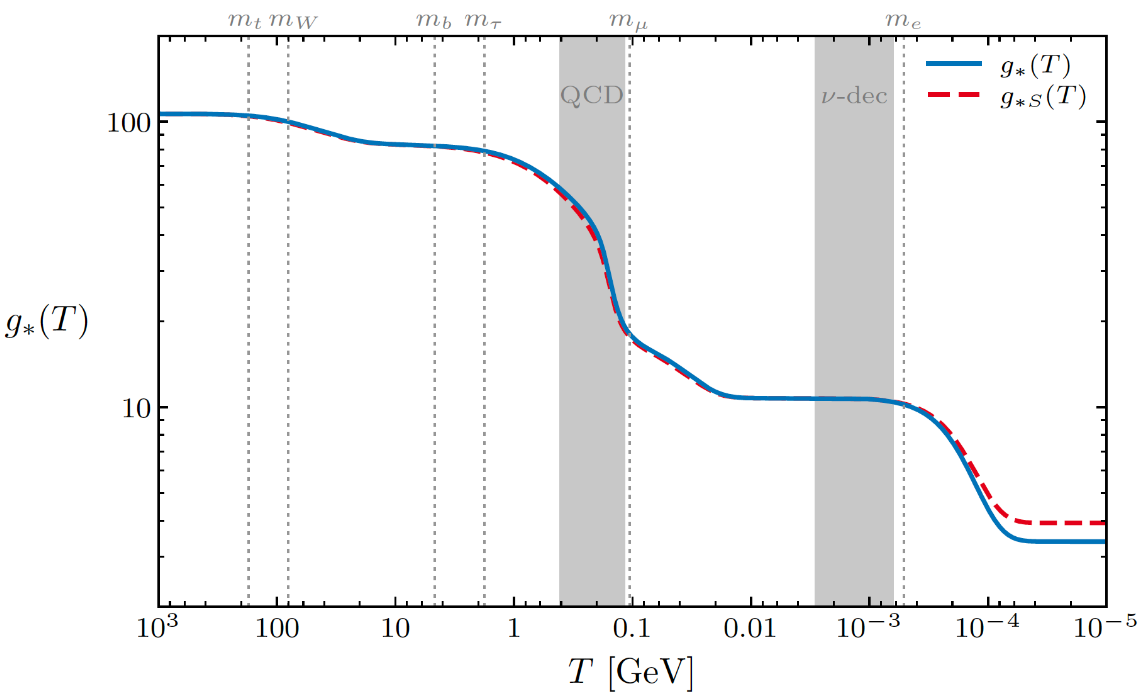

Figure 1.1: Effective number of relativistic dof. The gray bands represent the QCD phase transition and the neutrino decoupling. The difference around 1 TeV is due to the non-perturbative QCD effect. Taken from Baumann et al.

Taking the derivative of $P(t)$ with respect to T of a fluid and treating the chemical potential as a function of T, we obtain

$$\frac{dP}{dT} = -\frac{4\pi}{3}\int_0^\infty dp\,(p^3T)\left(\frac{df}{dp}\right)\left[\frac{E}{T^2}+\frac{d}{dT}\left(\frac{\mu}{T}\right)\right] , \qquad \frac{df}{dp} = -\frac{p}{ET}\,f^2(p,t)\exp\left(\frac{E-\mu}{T}\right) , \tag{1.26}$$

and with integration by part we re-write the derivative as

$$\frac{dP}{dT} = \frac{\rho+P}{T} + nT\frac{d}{dT}\left(\frac{\mu}{T}\right) \approx \frac{\rho+P}{T} . \tag{1.27}$$

Manipulating the conservation equation

$$\dot\rho + 3H(\rho+P) = 0 , \qquad d\left(\rho a^3\right) = -Pd\left(a^3\right) , \qquad \frac{d}{dT}\Big[(\rho+P)a^3\Big] = a^3\frac{dP}{dT} , \tag{1.28}$$

we can arrive at the conservation equation

$$d(sa^3) = -\left(\frac{\mu}{T}\right)d(na^3) \approx 0 , \qquad s := \frac{\rho+P}{T} - \frac{n\mu}{T} \approx \frac{\rho+P}{T} . \tag{1.29}$$

In most cases, the chemical potential is negligible ($\mu \ll T$) or the number density is conserved ($n \propto 1/a^3$), such that the combination (sa^3) is conserved throughout the evolution. Simplifying the relations for s and dP/dT by assuming $\mu \equiv 0$, we obtain the thermodynamic relation

$$\frac{dT}{T} = \frac{dP}{\rho+P} , \qquad S := sa^3 , \qquad TdS = d\left[(\rho+P)a^3\right] - (\rho+P)a^3\frac{dT}{T} = d\left(\rho a^3\right) + P\,d(a^3) , \tag{1.30}$$

with which we can identify s defined above as the entropy density and the total entropy S is conserved.

1.2.4 Spin degeneracy factors

The spin degeneracy factor accounts for the number of degenerate states at the same energy level. The photon has two polarization ($g_\gamma = 2$), while neutrinos are only left-handed ($g_\nu = 1$). Note that there exist three generations (ν, μ, τ) of neutrinos and their anti-particles ($\bar{\nu}$, $\bar{\mu}$, $\bar{\tau}$). Spin-1/2 fermions like electrons have $g_e = 2$, and there exist three generations and their anti-particles.

At $T > 300$ GeV, there exist 8 gluons ($g_g = 2$), 3 weak gauge bosons ($W^\pm$, Z), Higgs doublet ($m_H = 125$ GeV), three generations of quarks ($g_q = 2$; two quarks per generation per color) and leptons (g_e, g_ν) to yield[1]

$$g_* = g_\gamma + 8 \times g_g + 3 \times g_{W^\pm,Z} + g_H + \frac{7}{8} \times 3 \times (3 \times 2 \times 2 \times g_q + 2 \times g_\nu + 2 \times g_e) = 106.75 \,. \tag{1.31}$$

At 150 MeV gluons hadronize, and soon after most of the particles become non-relativistic, according to their mass ($m_H = 125$ GeV, $m_Z = 91$ GeV, $m_{W^\pm} = 80$ GeV, $m_\tau = 1.78$ GeV, $m_\mu = 105$ MeV). So at $T \sim 100$ MeV, there left only photons, electrons, and three generations of neutrinos:

$$g_* = g_\gamma + \frac{7}{8}\left(2 \times g_e + 2 \times 3 \times g_\nu\right) = 10.75 \,. \tag{1.32}$$

At a freeze-out temperature $T \sim 1$ MeV, all three generations of neutrinos decouple from the rest of the plasma ($T_\tau \simeq 3.7$ MeV, $T_\mu \simeq 2.4$ MeV, $T_\nu \simeq 1$ MeV), and its temperature strictly declines as $T_\nu \propto 1/a$, since its $g_{*,s}$ remains unchanged, after the decoupling. However, at about $T_\gamma \sim 0.51$ MeV, electrons and anti-electrons become non-relativistic, and they annihilate intro photons , transferring its entropy to the photon plasma, but not to the decoupled neutrinos, which slows the decline of T_γ. Assuming an instantaneous transfer of entropy, the change in the spin-degeneracy factor of the photon plasma can be computed as

$$g_{*,s}^{\rm before} = g_\gamma + \frac{7}{8}\left(g_e + g_{\bar{e}}\right) = \frac{11}{2} \quad \to \quad g_{*,s}^{\rm after} = g_\gamma \,. \tag{1.33}$$

Given the conservation of $g_{*,s}T^3$ throughout the annihilation, the neutrino temperature is slightly lower than the photon temperature, after the annihilation event

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \propto \frac{1}{a} \,, \tag{1.34}$$

and the spin degeneracy factor is then

$$g_* = g_\gamma + \frac{7}{8}\left(2 \times 3 \times g_\nu\right) \times \left(\frac{4}{11}\right)^{4/3} = 3.36 \,. \tag{1.35}$$

The total radiation density (γ, ν) is then

$$\rho_{\rm rad} = \left[1 + N_\nu \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]\rho_\gamma \,, \qquad N_\nu = 3 \,, \qquad \rho_\gamma = a_B T_\gamma^4 \,, \tag{1.36}$$

where the radiation constant a_B is related to the Stefan-Boltzmann constant σ_B as

$$a_B = \frac{4\sigma_B}{c} = 7.573 \times 10^{-15}\ {\rm erg\ cm^{-3}K^{-4}} \,. \tag{1.37}$$

In fact, at the annihilation of electrons and anti-electrons, the neutrino decoupling was incomplete, and some entropy is dumped into neurinos as well. Hence the the neutrino temperature relation above is not precise, and the correction is often rephrased as the effective relativistic degrees of freedom: $N_\nu = 3.04$. The evolution of the spin degeneracy factors is shown in Figure 1.1. Today the photon plasma cools down to

$$T_\gamma = 2.73\,{\rm K} \,, \qquad n_\gamma = 413\,{\rm cm}^{-3} \,, \qquad \rho_\gamma = 4.7\times10^{-34}\,g\,{\rm cm}^{-3} \,, \qquad \omega_\gamma = 2.5\times10^{-5} \,. \tag{1.38}$$

[1] After the spontaneous symmetry breaking, the weak gauge bosons are massive ($g_{W^\pm,Z} = 3$), and the Higgs boson is left with only one dof ($g_H = 1$), such that there exist 10 dof. Mind that at this energy scales, they are all non-relativistic. However, before the symmetry breaking, the gauge bosons are massless ($g_{W^\pm,Z} = 2$), and the Higgs boson doublet has full dof ($g_H = 4$; two per each component), such that the total dof remains the same.

The cosmic neutrino plasma is

$$T_\nu = 1.95\,\mathrm{K}\,, \qquad n_{\nu+\bar\nu} = 113\,\mathrm{cm}^{-3}\,, \qquad n_{\forall\nu} = 338\,\mathrm{cm}^{-3} \quad (\nu{+}\mu{+}\tau)\,, \qquad \omega_\nu = 1.7{\times}10^{-5}\,, \tag{1.39}$$

for massless neutrinos. Assuming they are relativistic at decoupling, the massive neutrinos

$$\rho_\nu = 113\; m_\nu\;\mathrm{cm}^{-3}\,, \qquad \omega_\nu = 0.1\left(\frac{m_\nu}{10\;\mathrm{eV}}\right)\,. \tag{1.40}$$

1.3 Distribution of Decoupled Species

As the Universe expands and cools down, the interaction rate Γ between species falls below the expansion rate $H(t)$, so that a particle species decouples from the plasma. This is called "freeze-out" because there exist no further interactions and its distribution remains frozen. Since the momentum of both massless and massive particles redshifts as $1/a$ in the background universe, the current phase-space distribution of a decoupled species can be expressed in terms of the equilibrium distribution at decoupling:

$$f(p,t) = f_{\rm eq}\left(p\frac{a}{a_{\rm dec}}, t_{\rm dec}\right)\,, \qquad t \geq t_{\rm dec}\,, \qquad p(t_{\rm dec}) = p\frac{a}{a_{\rm dec}}\,. \tag{1.41}$$

When a relativistic species is decoupled at $T_{\rm dec} \gg m$, the Fermi-Dirac or Bose-Einstein distribution is maintained, and hence the number density is as abundant as photons, but the freeze-out condition dictates its temperature declines as $1/a$:

$$f(p,t) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{p\,a}{a_{\rm dec}T_{\rm dec}}\right) \pm 1\right]^{-1}\,, \qquad T(t) = T_{\rm dec}\frac{a_{\rm dec}}{a}\,, \tag{1.42}$$

where $E \simeq p$ for relativistic particles. Note that the decoupled species evolves separately, so that the change in the spin-degeneracy factor in the other plasma is irrelevant here.

However, when particles are non-relativistic ($T \ll m$) at decoupling, the distribution function is the Maxwell-Boltzmann distribution, and according to the freeze-out condition, the temperature of the decoupled species declines faster than the relativistic particles

$$f(p,t) = \frac{g}{(2\pi)^3}\exp\left(-\frac{m}{T_{\rm dec}}\right)\exp\left(-\frac{p^2a^2}{2m\,a_{\rm dec}^2T_{\rm dec}}\right)\,, \qquad T(t) = T_{\rm dec}\left(\frac{a_{\rm dec}}{a}\right)^2\,, \tag{1.43}$$

where the exponential $\exp(-m/T_{\rm dec})$ is constant. Consequently, the number density of a decoupled species evolves as

$$n(t) = \left[\frac{a(t_{\rm dec})}{a(t)}\right]^3 n_{\rm eq}(t_{\rm dec})\,, \tag{1.44}$$

for both relativistic and non-relativistic particles.

Using the entropy conservation in Eq. (1.25), we obtain the temperature ratio and the number density ratio of a decoupled relativistic species to the photons as

$$\left[\frac{T_\gamma(t_{\rm dec})}{T_\gamma(t)}\right]^3 = \frac{g_{*,s}(t)}{g_{*,s}(t_{\rm dec})}\frac{a^3(t)}{a^3(t_{\rm dec})}\,, \qquad \frac{n(t)}{n_\gamma(t)} = \frac{g_{\rm eff}}{2}\left[\frac{T(t)}{T_\gamma(t)}\right]^3 = \frac{g_{\rm eff}}{2}\frac{g_{*,s}(T)}{g_{*,s}(T_{\rm dec})}\,, \tag{1.45}$$

where $g_{\rm eff} = g$ for bosons and $g_{\rm eff} = 3g/4$ for fermions and we used the temperature of the decoupled species $T(t_{\rm dec}) = T_\gamma(t_{\rm dec})$ at the time of decoupling.

1.3.1 Boltzmann Equation and Relic Number Density

The particle interactions involve multiple species, and they depend on the velocity distribution of the particles. Consequently, solving for their evolution requires coupled differential equations, called the Boltzmann equation. Consider an interaction $\psi + a + b + \cdots \leftrightarrow i + j + \cdots$ that involves many particles and their creation and annihilation. The Boltzmann equation for a species ψ (similarly for other particles) is

$$\frac{df_\psi}{dt} = C_\psi[f]\,, \tag{1.46}$$

where the right-hand side C is called the collision term that depends on the interaction and the distribution functions of the other particles in interaction. In the absence of collision, the Liouville theorem states that the phase-space distribution is conserved. In a homogeneous and isotropic universe, the phase-space distribution function cannot depend on a position or a direction, i.e., $f_\psi = f_\psi(p,t)$,

$$\frac{df_\psi}{dt} = \frac{\partial f_\psi}{\partial t} + \frac{\partial p}{\partial t}\frac{\partial f_\psi}{\partial p} \,, \qquad \frac{\partial p}{\partial t} = -Hp \,, \tag{1.47}$$

where we used $p \propto 1/a$ for any particles in the background universe. Integrating over the momentum, we derive that the number density evolves as

$$\frac{dn_\psi}{dt} + 3Hn_\psi = \int d^3p\, C_\psi[f] \,, \qquad n_\psi = \int d^3p\, f_\psi \,. \tag{1.48}$$

In the absence of collision, the number density decreases as $n_\psi \propto 1/a^3$, and the term $3Hn_\psi$ is called the Hubble drag (or friction) due to the expansion of the Universe.

The collision term depends on the interaction process, and formally it can be expressed as

$$\begin{aligned}\int d^3p\, C_\psi[f] \;\; &= \;\; \left(\prod_i \int d^4p_i\right)(2\pi)^4\delta^D(p_\psi + p_a + \cdots - p_i - p_j - \cdots) \\ &\quad\times\left[|\mathcal{M}|^2_\leftarrow f_i f_j \cdots (1\pm f_\psi)(1\pm f_a)\cdots - |\mathcal{M}|^2_\rightarrow f_a \cdots f_\psi(1\pm f_i)(1\pm f_j)\cdots\right] .\end{aligned} \tag{1.49}$$

The first line is just the energy-momentum conservation of the process, the second line shows the interaction of consideration. The invariant matrix element $\mathcal{M}$ can be derived from the QFT calculations, and with T-invariance (or CP-invariance) it is identical in both directions ($|\mathcal{M}|^2 := |\mathcal{M}|^2_\rightarrow = |\mathcal{M}|^2_\leftarrow$). The distribution functions $f_i f_j \cdots$ in the second line indicates that more particles $i, j, \cdots$ create more particles $\psi, a, b, \cdots$, and vice versa. The extra factors such as $(1 \pm f_\psi)$ are called the Pauli block $(-)$ or the Bose enhancement $(+)$.

For the moment, the collision term is macroscopically treated, and a significant simplification can be made, if most species but ψ are in thermal equilibrium and the temperature is low $T \ll E - \mu$. Consider a simplified interaction $\psi + \bar\psi \leftrightarrow X + \bar X$, in which ψ and $\bar\psi$ annihilate and a pair of X and $\bar X$ are created. At this low temperature $T \ll E - \mu$, the number densities of particles can be written as

$$n = \int d^3p\, f = e^{\mu/T} n_{\rm EQ} \,, \qquad n_{\rm EQ} := \int d^3p\, f_{\rm EQ} \,, \qquad f_{\rm EQ} := f(\mu \equiv 0) \simeq \frac{g}{(2\pi)^3} e^{-E/T} \,, \tag{1.50}$$

where we ignored ± 1 in the distribution function $f_{\rm EQ}$ Further ignoring the Pauli block or the Bose enhancement, The second line of the collision term is then greatly simplified as

$$f_X f_{\bar X} - f_\psi f_{\bar\psi} = e^{-(E_\psi + E_{\bar\psi})/T}\left[e^{(\mu_X + \mu_{\bar X})/T} - e^{(\mu_\psi + \mu_{\bar\psi})/T}\right] = e^{-(E_\psi + E_{\bar\psi})/T}\left[\frac{n_X n_{\bar X}}{n_X^{\rm EQ} n_{\bar X}^{\rm EQ}} - \frac{n_\psi n_{\bar\psi}}{n_\psi^{\rm EQ} n_{\bar\psi}^{\rm EQ}}\right] \,, \tag{1.51}$$

where we used the energy conservation $E_\psi + E_{\bar\psi} = E_X + E_{\bar X}$. Finally, we define the thermally-averaged velocity times cross-section $\langle\sigma v\rangle$ as

$$n_\psi^{\rm EQ} n_{\bar\psi}^{\rm EQ} \left\langle \sigma_{\psi\bar\psi\to X\bar X}|v|\right\rangle := \left(\prod_i \int d^3p_i\right)(2\pi)^4\delta^D(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})|\mathcal{M}|^2 e^{-(E_\psi + E_{\bar\psi})/T} \,, \tag{1.52}$$

and the Boltzmann equation (1.48) is now

$$\frac{dn_\psi}{dt} + 3Hn_\psi = n_\psi^{\rm EQ} n_{\bar\psi}^{\rm EQ} \left\langle \sigma_{\psi\bar\psi\leftrightarrow X\bar X}|v|\right\rangle \left[\frac{n_X n_{\bar X}}{n_X^{\rm EQ} n_{\bar X}^{\rm EQ}} - \frac{n_\psi n_{\bar\psi}}{n_\psi^{\rm EQ} n_{\bar\psi}^{\rm EQ}}\right] \,. \tag{1.53}$$

With near thermal equilibrium but ψ particles, we arrive at the final expression of the Boltzmann equation

$$\frac{dn_\psi}{dt} + 3Hn_\psi = \langle\sigma v\rangle\left(n_{\psi,\rm EQ}^2 - n_\psi^2\right) \,. \tag{1.54}$$

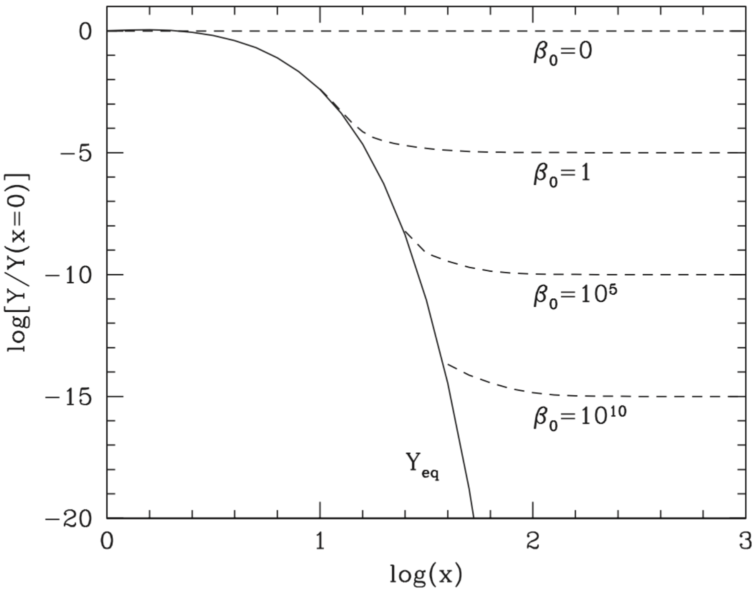

Figure 1.2: The relic abundance for a simple two-body process with a constant $\beta := \langle\sigma v\rangle$.

At thermal equilibrium, the number density n_ψ will be equivalent to $n_\psi^{\rm EQ}$, and no further net change (creation or annihilation) takes place. If $n_\psi > n_\psi^{\rm EQ}$, more decay of ψ and $\bar\psi$ will further reduce n_ψ and increase n_X, and this is reflected in the collision term, where the RHS is negative.

Given the entropy density scales as a^{-3}, it is convenient to define a scaled number density Y that does not change in time as long as $n \propto 1/a^3$,

$$Y_\psi := \frac{n_\psi}{s} , \qquad Y_\psi^{\rm eq} := \frac{n_\psi^{\rm eq}}{s} . \tag{1.55}$$

The Boltzmann equation is then manipulated in terms of Y as

$$\frac{dY_\psi}{dt} = s\,\langle\sigma v\rangle \left(Y_{\psi,\rm eq}^2 - Y_\psi^2\right) , \tag{1.56}$$

and by defining a scaled (inverse) temperature x

$$x := \frac{m_\psi}{T} , \qquad t = \frac{1}{2H} \propto \frac{1}{T^2} , \qquad \frac{d\ln t}{dx} = \frac{2}{x} , \tag{1.57}$$

the Boltzmann equation can be written as

$$\frac{x}{Y_\psi^{\rm eq}}\frac{dY_\psi}{dx} = -\frac{n_\psi^{\rm eq}\,\langle\sigma v\rangle}{H(x)}\left[\left(\frac{Y_\psi}{Y_\psi^{\rm eq}}\right)^2 - 1\right] . \tag{1.58}$$

The variable x determines if the particle species is relativistic ($x \ll 1$) or non-relativistic ($x \gg 1$), but it also determines the flow of time ($x \gg 1$ at late time). Given the interaction cross-section, the Boltzmann equation can be numerically solved with the initial condition of thermal equilibrium at early time $Y(x = 0) = Y_{\rm eq}$ for all species. Assuming a constant cross-section, several solutions to the Boltzmann equation are given in Figure 1.3, in which the equilibrium distribution decays in time as the particle species becomes non-relativistic ($x \simeq 1$) and its abundance is exponentially suppressed compared to the plasma. For a weak cross-section, the particle species decouples early when they are relativistic, and their final abundance is similar to those of photons, rather insensitive of its exact value of the cross-section. For a stronger cross-section, the particles stay in thermal equilibrium longer, and its final relic abundance is sensitively dependent on the value of the cross-section.

A simple analytic approximation can be made to solve the Boltzmann equation. First, the equilibrium abundance $Y^{\rm eq} = n^{\rm eq}/s$ is obtained by using $n^{\rm eq}$ in the relativistic and the non-relativistic cases as

$$Y_\psi^{\rm eq}(x) = \frac{45\zeta(3)}{2\pi^4}\frac{g_\psi^{\rm eff}}{g_{*,s}(x)} \quad \text{for } x \ll 1 , \qquad Y_\psi^{\rm eq}(x) = \frac{90}{(2\pi)^{7/2}}\frac{g_\psi}{g_{*,s}(x)}\,x^{3/2}e^{-x} \quad \text{for } x \gg 1 , \tag{1.59}$$

where $g_\psi^{\rm eff}=g_\psi$ for Boson and $g_\psi^{\rm eff}=3g_\psi/4$ for Fermion. Clearly, as the Universe evolves (x increases), $Y^{\rm eq}$ also evolves due to the change in $n^{\rm eq}$. Second, the freeze-out (or decoupling) is assumed to be instantaneous, if $\Gamma=H$ at x_f. Equating the interaction rate Γ at equilibrium with the Hubble parameter in RDE

$$H(x)=\frac{8\pi G}{3}\rho_{\rm tot}=\sqrt{\frac{\pi^2g_*}{90}}\frac{m_\psi^2}{x^2M_{\rm pl}}~,\qquad M_{\rm pl}^2=\frac{1}{8\pi G}~,\qquad \Gamma=Y_\psi^{\rm eq}s\left<\sigma v\right>~,\tag{1.60}$$

we obtain the freeze-out time

$$x_f=\sqrt{\frac{90}{\pi^6g_*}}\zeta(3)g_\psi^{\rm eff}\left<\sigma v\right>m_\psi M_{\rm pl}\quad\text{for }x\ll1~,\qquad x_f^{-1/2}e^{x_f}=\sqrt{\frac{45}{4\pi^5g_*}}g_\psi\left<\sigma v\right>m_\psi M_{\rm pl}\quad\text{for }x\gg1~.\tag{1.61}$$

The condition $x_f\ll1$ for the relativistic case constrains the strength of $\left<\sigma v\right>$ and also the mass m_ψ. The freeze-out for the non-relativistic case needs to be solved, but its value is due to the exponential factor highly sensitive to the values in the RHS.

Now we consider the relic densities today. First, they can be relativistic or non-relativistic *today*. The former is similar to the photon distribution, and hence negligible. The latter, non-relativistic relic species today, is often called as the WIMP (weakly interacting massive particles). WIMPS are non-relativistic today, and its energy density is dominated by their rest mass energy. However, it can be relativistic ($x_f\ll1$) or non-relativistic ($x_f\gg1$) at the freeze-out. The former is called the hot relic, and their abundance is as much as the photons today, while the latter is called the cold relic.

• ***Relativistic species today***.— The relic density of a relativistic species can then be obtained by using Eq. (1.45) as

$$\frac{\Omega_\psi h^2}{\Omega_\gamma h^2}=\frac{\rho_\psi}{\rho_\gamma}=\frac{g_\psi^{\rm eff}}{2}\left(\frac{T_\psi}{T_\gamma}\right)^4=\frac{g_\psi^{\rm eff}}{2}\left[\frac{g_{*,s}(x)}{g_{*,s}(x_f)}\right]^{4/3}~.\tag{1.62}$$

Given that $g_{*,s}$ always decreases in time and $\Omega_\gamma h^2=2.5\times10^{-5}$, the relic density of a relativistic species today is as negligible as the photon energy density.

• ***Hot relics***.— The relic density of a hot species is then

$$\rho_\psi=m_\psi Y_\psi^{\rm eq}(x_f)s(x_0)\quad\propto\quad m_\psi~,\qquad \Omega_\psi h^2=\frac{8\pi G}{3H_0^2}\rho_\psi h^2=7.64\times10^{-2}\left[\frac{g_\psi^{\rm eff}}{g_{*,s}(x_f)}\right]\left(\frac{m_\psi}{1\text{ eV}}\right)\propto m_\psi~,\tag{1.63}$$

where x_0 is today. For hot relics, their number density is as large as the photons, and it is rather insensitive to x_f. Note that the dependence of $\left<\sigma v\right>$ is included in the freeze-out x_f. For a large mass m_ψ, x_f becomes comparable to unity, and it cannot be hot-relic any more. Given the observational constraint $\Omega_{\rm tot}h^2\lesssim1$, we can derive that the mass of hot relics should be smaller than

$$m_\psi\leq13.1\text{ eV}\left[\frac{g_{*,s}(x_f)}{g_\psi^{\rm eff}}\right]~,\tag{1.64}$$

which corresponds to

$$m_\nu\leq93.8\text{ eV}~,\qquad g_{*,s}(x_f)=10.75~,\qquad g_\nu^{\rm eff}=\frac{3}{4}\times2\times g_\nu~,\tag{1.65}$$

for massive neutrinos (one species). This cosmological limit is called the Cowsik-McClelland bound. The Planck constraint is $\Sigma m_\nu<0.23$ eV, and some recent Lyα-forest constraint is <0.12 eV. The neutrino oscillation constraints give $0.0006<\omega_\nu<0.0025$.

• ***Cold relics***.— Similar calculations can be made for cold relics, but the abundance $Y_\psi^{\rm eq}$ is exponentially sensitive to x_f. The LHS for the freeze-out condition

$$x_f^{-1/2}e^{x_f}=\sqrt{\frac{45}{4\pi^5g_*}}g_\psi\left<\sigma v\right>m_\psi M_{\rm pl}~,\tag{1.66}$$

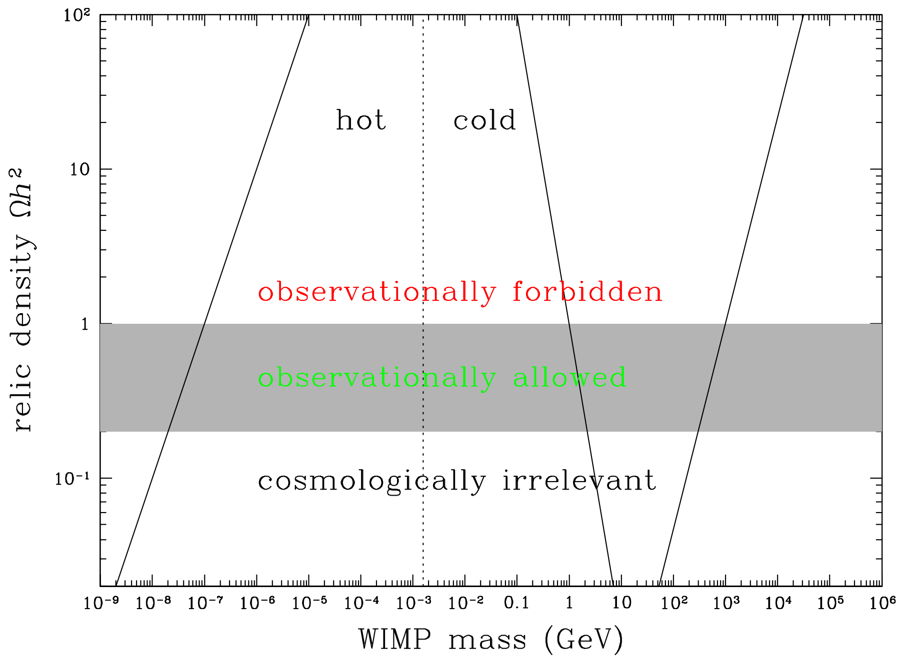

Figure 1.3: Cosmological constraints on the mass of WIMP $\Omega_\psi h^2$.

increases monotonically with x_f, i.e., the larger m_ψ or the stronger $\langle\sigma v\rangle$, the later the freeze-out becomes, suppressing the abundance exponentially. Keeping x_f in the equation, we express the abundance and the relic density

$$Y_\psi^{\rm eq}(x_f) = \sqrt{\frac{45}{8\pi^2}}\frac{x_f}{\sqrt{g_{*,s}(x_f)}}\frac{1}{\langle\sigma v\rangle\, m_\psi M_{\rm pl}}~, \qquad \Omega_\psi h^2 = 0.86\frac{x_f}{\sqrt{g_{*,s}(x_f)}}\left[\frac{\langle\sigma v\rangle}{10^{10}~{\rm GeV}^{-2}}\right]^{-1}~. \tag{1.67}$$

The relic abundance is sensitively dependent upon $\langle\sigma v\rangle$, and it goes down with m_ψ. While the relic energy density has no explicit dependence on m_ψ, the freeze-out time x_f increases with m_ψ. For example, stable neutrinos of mass between 1 MeV and $m_Z = 100$ GeV fall into this cold relic, and their weak interaction rate is

$$\langle\sigma v\rangle \approx \frac{c_2}{2\pi}G_F^2 m_\nu^2 x^{-b} \qquad \text{for}~~m_\psi < m_Z~, \tag{1.68}$$

where $c_2 \simeq 5$ for a Dirac neutrino, $b \sim 1$, and G_F is the Fermi constant. With this, the freeze-out time can be solved as

$$x_f \simeq 17.8 + 3\ln\left(\frac{m_\nu}{1~{\rm GeV}}\right)~, \tag{1.69}$$

and the relic energy density is

$$\Omega_\nu h^2 \simeq \frac{3.95}{c_2}\frac{x_f^{b+1}}{\sqrt{g_{*,s}(x_f)}}\left(\frac{m_\nu}{1~{\rm GeV}}\right)^{-2} = 1.82\left(\frac{m_\nu}{1~{\rm GeV}}\right)^{-2}\left[1+0.17\ln\left(\frac{m_\nu}{1~{\rm GeV}}\right)\right]~, \tag{1.70}$$

The observational constraint puts the mass of stable neutrinos

$$m_\nu \geq 1.4~{\rm GeV}~, \tag{1.71}$$

and the relic density is smaller with larger mass due to the larger cross-section and the suppression of the abundance. However, for particles of mass $m_\psi \gg m_Z$, the cross-section decreases with particle mass as m^{-2}, instead of increasing with m^2. This implies

$$\Omega_\psi h^2 \simeq \left(\frac{m_\psi}{1~{\rm TeV}}\right)^2~, \qquad m_Z \leq m_\psi \leq 3~{\rm TeV}~. \tag{1.72}$$

For the cold relics, the bound is stronger, because of the non-relativistic freeze-out, and this gives a lower limit for the massive cold relics, called the Lee-Weinberg bound. Figure 1.3 summarizes the cosmological bounds on viable models of WIMPs.

1.3.2 Relic Density of Decaying Particles

If a particle is unstable and decays into other particles, the Boltzmann equation can be supplemented by an extra term for such decay. The number density of decaying particles is always governed by the half-life τ, beyond which the number density is exponentially suppressed as

$$n(t) \propto \exp[-t/\tau] \ , \tag{1.73}$$

but below which the particles behave like stable particles.

If the decay product involves photons, such particles are subject to more stringent observational constraints. A particle decay into photons often involves strong Gamma rays, and these photons should be hidden from observations by preventing the particles decay with longer life-time or by themalizing them. It takes a while to thermalize Gamma ray photons with background radiation, such that the decays must happen early enough.

1.4 Big Bang Nucleosynthesis

Where do we come from? To this philosophical question, here we find some physical answers. The basic structural unit of life is cells that contain lots of molecules, and molecules are electrically neutral groups of atoms held by chemical bonds. Atoms form the smallest unit of the (ordinary) matter, and they are composed of one nucleus and several electrons bound to the nucleus. Where do they come from? Normal stars at the core fuse lighter elements like hydrogen and helium, and more massive stars synthesize carbon, oxygen, and silicon, yielding irons, beyond which no net energy is gained through nuclear fusion. Heavier elements are further generated by neutron captures in supernova explosions. However, observations show that hydrogen and helium are ubiquitous in the Universe with almost constant ratio 75% hydrogen and 24% helium by mass. Indeed, the origin of those elements are primordial and global, rather than localized stars.

1.4.1 Proton and Neutron Abundances

All nuclei are made of protons and neutrons, and they are characterized by its charge number Z (number of protons) and the atomic mass A (number of protons and neutrons). Given their mass $m_p \simeq m_n \simeq 940$ MeV, protons and neutrons are non-relativistic at $t \simeq 10^{-6}$ sec ($T \simeq m_p$) with their number densities

$$n_{n,p} = g_{n,p} \left(\frac{m_{n,p}T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_{n,p} - \mu_{n,p}}{T}\right] \ , \tag{1.74}$$

and they remain in thermal equilibrium until $T \sim 0.8$ MeV via low-energy weak interactions

$$p + e \leftrightarrow n + \nu_e \ , \qquad n + \bar{e} \leftrightarrow p + \bar{\nu}_e \ , \qquad n \leftrightarrow p + e + \bar{\nu}_e \ . \tag{1.75}$$

Hence the ratio of the number densities in thermal equilibrium is

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[-\frac{m_n - m_p}{T} + \frac{\mu_n - \mu_p}{T}\right] \simeq \exp\left[-\frac{Q}{T}\right] \ , \qquad Q := m_n - m_p = 1.294 \text{ MeV} \ , \tag{1.76}$$

where we ignored the difference in the chemical potential $\mu_n - \mu_p = \mu_e - \mu_\nu \simeq 0$ in the weak interactions. At temperature $T \gg Q$, the ratio of the number densities is unity, but it continuously decreases at lower temperature ($T < Q$), because neutrons are slightly heavier than protons. However, due to the neutrino decoupling at $T = 1$ MeV, the weak interactions become inefficient to keep protons and neutrons in thermal equilibrium, such that the ratio freezes out at $T \sim 0.8$ MeV

$$\frac{n_n}{n_p} \sim \exp\left[-\frac{1.294}{0.8}\right] \simeq \frac{1}{5} \ . \tag{1.77}$$

Free neutrons can further β-decay into protons at any time with its half-life $\tau = 887 \pm 2$ sec ($\simeq 15$ min), which could have exhausted neutrons in our Universe. However, before they decay into protons, most neutrons are indeed captured in deuterium and helium nuclei, where they are stable.[2] By the time the big bang nucleosynthesis is active, the ratio becomes

$$\frac{n_n}{n_p} \simeq \frac{1}{7} \qquad \text{at } t \simeq 300 \text{ sec} \ . \tag{1.78}$$

[2]Sometimes, Pauli's exclusion principle is invoked for such stability, but neutrons do decay in nuclei, when energetically favorable. In nuclei, all the neutrons and protons form a system, in which protons typically occupy higher energy state due to electromagnetic repulsion, such that nuclei with somewhat more neutrons than protons are stable, because converting one neutron into a proton would need more energy. Of course, if even more neutrons are present in nuclei, they inevitably occupy higher energy state than protons, and β-decay is then energetically favorable.

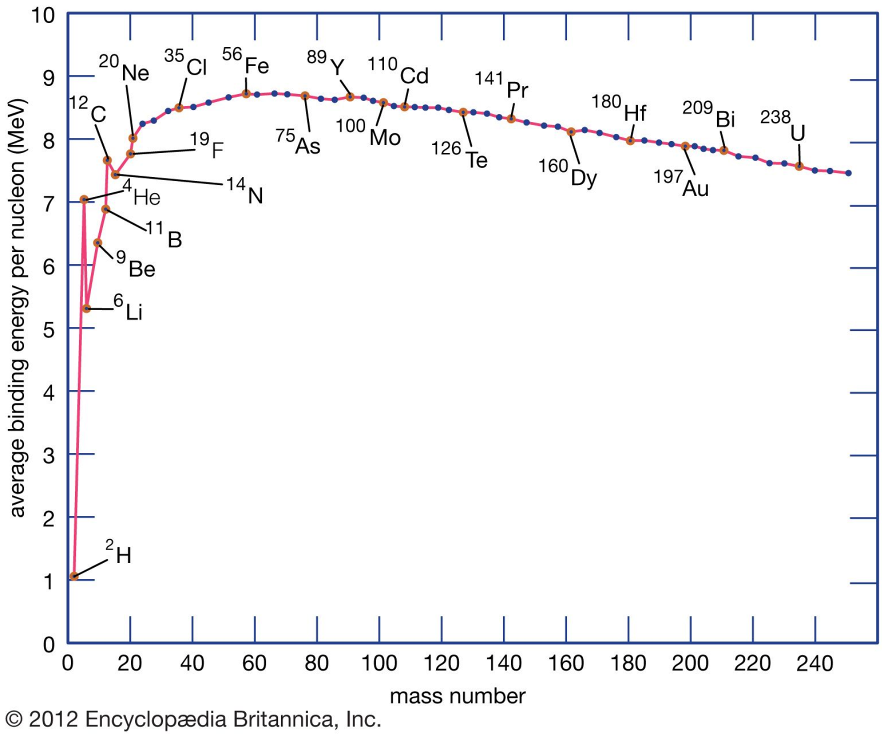

Figure 1.4: Average binding energy of nuclei per proton. With the steep increase in the binding energy, nuclear fusion is an efficient way to generate energy up to iron, beyond which the binding energy decreases. Nuclear fission can be used for elements heavier than irons to extract energy, though not as efficient as nuclear fusion.

1.4.2 Nuclear Synthesis of Heavier Elements

With numerous protons and neutrons, they can be forged to form heavier nuclei, but they are dissociated immediately by energetic photons, until the Universe cools below their binding energy (e.g., 2.22 MeV for deuterium). In thermal equilibrium, the abundance of nuclei with atomic mass A with charge Z is

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_A - \mu_A}{T}\right] = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_A}{T}\right] \left[\exp\left(\frac{\mu_p}{T}\right)\right]^Z \left[\exp\left(\frac{\mu_n}{T}\right)\right]^{(A-Z)} , \quad (1.79)$$

where we used the relation for the chemical potential

$$\mu_A = Z\mu_p + (A - Z)\mu_n \, . \quad (1.80)$$

With the same formulas for the proton and the neutron number densities in equilibrium, we can remove the chemical potentials μ_p and μ_n in favor of n_p and n_p to express

$$n_A = \frac{g_A A^{3/2}}{g_N^A} n_p^Z n_n^{A-Z} \left(\frac{m_N T}{2\pi}\right)^{\frac{3}{2}(1-A)} \exp\left(\frac{B_A}{T}\right) , \qquad g_N := g_p = g_n = 2 \, , \quad (1.81)$$

where we approximated $m_N := m_p \simeq m_n$ and $m_A \simeq A m_N$, and defined the binding energy of nucleus

$$B_A := Z m_p + (A - Z) m_n - m_A \, . \quad (1.82)$$

In the presence of heavier elements, the baryon number density is

$$n_b := n_p + n_n + \sum_i A_i n_{A,i} \, , \quad (1.83)$$

and the mass fraction of each nucleus A:

$$X_{A_i} := \frac{A_i n_{A,i}}{n_b} \, , \qquad 1 = \sum_i X_{A,i} \, . \quad (1.84)$$

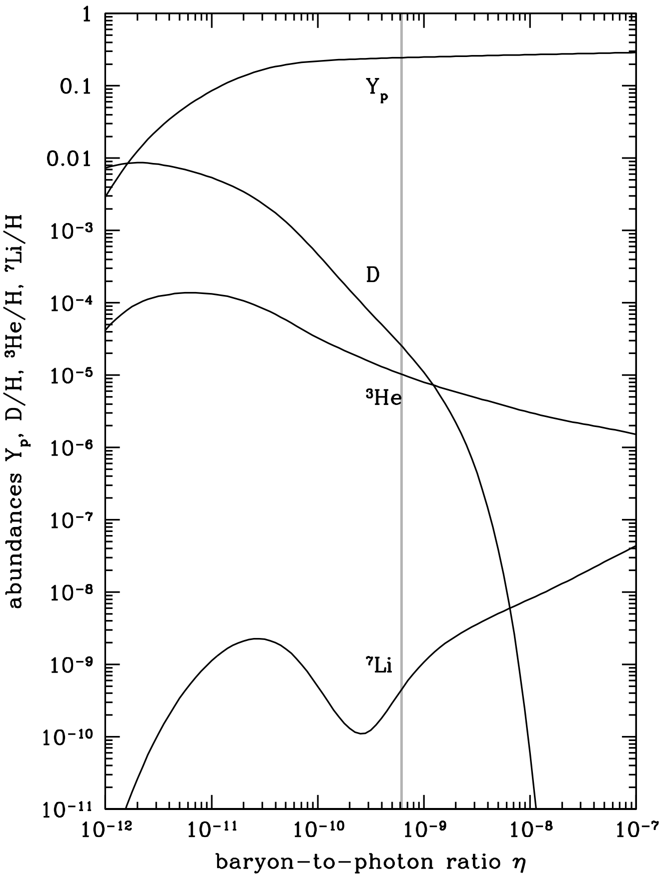

Figure 1.5: Primordial abundances of light elements as a function of the baryon-to-photon ratio.

The baryon number density includes free neutrons and protons, but also accounts for those inside nuclei with weight A_i, i.e., it is the total number densities of protons or neutrons, while the mass fraction shows how many nuclei are captured in the nucleus. The number density of nucleus A can then be re-expressed by using n_γ in Eq. (1.20) as

$$X_A = \frac{g_A}{2} A^{5/2} \left[\frac{4\zeta(3)}{\sqrt{2\pi}}\right]^{A-1} X_p^Z X_n^{A-Z} \eta^{A-1} \left(\frac{m_N}{T}\right)^{\frac{3}{2}(1-A)} \exp\left(\frac{B_A}{T}\right) , \tag{1.85}$$

where we defined the baryon-to-photon ratio:

$$\eta := \frac{n_b}{n_\gamma} = 2.72 \times 10^{-8} \omega_b \left(\frac{T_{\rm cmb}}{2.73\ {\rm K}}\right)^{-3} \approx 5 \times 10^{-10} . \tag{1.86}$$

As the temperature of the Universe cools below the binding energy, nuclei with atomic mass A can form, and for the mass fraction to be non-negligible ($X_A \simeq 1$), the temperature has to be below

$$\ln X_A \simeq 0 , \qquad T_A \approx \frac{|B_A|}{(A-1)\left[|\ln \eta| + \frac{3}{2}\ln(m_N/T_A)\right]} . \tag{1.87}$$

The deuterium ^{2}D has the lowest binding energy $B_D = 2.22$ MeV, but the formation of deuterium takes place only when the temperature of the Universe is an order-of-magnitude below $B_D \simeq 2 \times 10^{10}$ K due to large number of photons. The high-energy tail (Wien) of the photon distribution is sufficiently large enough to destroy deuterium nuclei, until it reaches $T_D \approx 10^9$ K ($t \sim 100$ sec). This is the beginning of the Big Bang Nucleosynthesis (BBN).

Once the nucleosynthesis begins, many channels of nuclear reaction take place. However, since the number densities of nuclei in the Universe are quite low at the time of BBN, only two-body interactions are possible, and the fact that there are no stable nuclei with atomic mass 5 or 8 implies that no elements heavier than lithium ^{7}Li (3 protons) can be produced. The next element in periodic table is ^{9}Be (4 protons). In contrast, at the core of massive stars, where the densities are even higher, many-body interaction channels are allowed, and even a short-lived ^{8}Be that formed through ^{4}He-^{4}He collision can quickly capture another ^{4}He to form a stable carbon ^{12}C, allowing further nuclear reactions to proceed.

Since the binding energy of deuterium is the lowest, the formation of deuterium nuclei acts as a bottleneck for nucleosynthesis, as heavier elements are already allowed to form by T_D. Consequently, almost all the deuterium nuclei (or free neutrons) are processed to form helium nuclei, and the mass fraction of helium is

$$Y := X_{^4\mathrm{He}} \simeq \frac{4(n_n/2)}{n_n + n_p} = \frac{2(n_n/n_p)_D}{1 + (n_n/n_p)_D} \approx \frac{1}{4}\,, \qquad \left(\frac{n_n}{n_p}\right)_D \approx \frac{1}{7}\,, \tag{1.88}$$

where the subscript D indicates the time of deuterium formation, when helium nuclei are yet to form, i.e., $n_b = n_n + n_p$. Observations of the helium mass fraction is about 24% everywhere, and the confirmation of this prediction for He is one of the success of the Big Bang model in the early days.

The predictions of primordial nucleosynthesis and their observational confirmation is of course important. In particular, it helps constrain η or the baryon density ω_b. However, it is in fact not easy to determine the primordial abundances from observations, because the observed abundances have been re-processed through stars and other astrophysical events. In the following we give a brief summary of the present observational situation (Mo et al., 2010):

- ^{4}He: With its large abundances, it is relatively easy to make observations, and the abundances are often estimated from ionized HII clouds by using the recombination lines. Since ^{4}He can be produced in stars, the estimates are the upper bound of the primordial abundances. In order to reduce this contamination, observers often target metal-poor gas clouds. In reality, observations are made as a function of metalicity, and the helium abundance is estimated by extrapolating it to zero-metalicity. The current estimate is $Y_p = 0.24 \pm 0.01$, but its abundance is relatively insensitive to η.

- ^{2}D: The deuterium abundance is estimated from UV absorption lines in the interstellar medium or in Lyα clouds at high redshifts. Since deuterium is rather weakly bound, it is easy to destroy them, but at the same time, it is hard to produce in stars. Therefore, the deuterium estimates serve as a lower bound. In particular, Lyα clouds at high redshifts are quite close to primordial. The local estimates give [D/H]$\simeq 1.6 \times 10^{-5}$, while the estimates from Lyα clouds yield $2.82 \pm 0.53 \times 10^{-5}$. Since the deuterium abundance sensitively changes with ω_b, its measurements are crucial in determining ω_b.

- ^{3}He: The abundance of ^{3}He can be measured by using meteorites and the solar wind in the solar system or by measuring the strength of the ^{3}He+ hyperfine transition line in HII regions. Old meteorites should contain material at the formation of the solar system. Since ^{2}D can be burned to ^{3}He in the Sun, the sum of (D+^{3}He) is a good measure of the pre-solar abundance from the solar wind. While ^{3}He can be destroyed at the core of stars, it is much harder than ^{2}D. The current measurements from the Solar system give an upper limit on $[(\mathrm{D} + {}^3\mathrm{He})/\mathrm{H}] < 10^{-4}$.

- ^{7}Li: Estimates of the ^{7}Li abundance come from stellar atmospheres. Since ^{7}Li is quite fragile, they are depleted if transported deeper into the centers of stars, which results in significant variations in observations. With weak convection, the estimates from metal-poor stars are believed to be more robust and close to the primordial abundances. The current observations yield $[{}^7\mathrm{Li}/\mathrm{H}] \simeq (1.5 \pm 0.4) \times 10^{-10}$.

With precise determination of T_γ and ω_b from CMB measurements, the predictions of BBN are completely fixed under the standard model of particle physics and cosmology, and they are used for consistency check with observations, in particular, of the abundances of ^{4}He and ^{2}D. On the other hand, the situation with ^{3}He is too complex for a meaningful comparison to be possible, and the results for ^{7}Li appear to disagree within uncertainties. This discrepancy reflects observational challenges in inferring the primordial abundances, but it might imply that the early Universe might have been different from what the standard model physics predicts.

1.5 Recombination and Matter-Radiation Decoupling

1.5.1 Recombination of Hydrogen Atoms

Once the nucleosynthesis is completed, the Universe consists of protons, helium nuclei, electrons, photons, decoupled neutrinos, and a trace amount of other elements such as ^{2}D, ^{3}He and so on. All particles except photons and neutrinos are already non-relativistic, and they stay in thermal equilibrium mainly through the electromagnetic interactions. As the Universe cools, the next cosmological event is to form neutral hydrogen atoms by combining free electrons and protons, which is called the cosmic recombination.

Assuming the thermal equilibrium and $\mu_H = \mu_p + \mu_e$, we can derive the hydrogen number density in the exactly same way to Eq. (1.81) as

$$n_H = \left(\frac{g_H}{g_p g_e}\right) n_p n_e \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right) , \qquad g_H = g_e = 2 , \quad g_p = 1 , \tag{1.89}$$

where the binding energy of hydrogen atoms is

$$B_H := m_p + m_e - m_H = 13.6 \text{ eV} . \tag{1.90}$$

Mind that the degeneracy factor for electrons in neutral hydrogen atoms is $g_H = \sum 2n^2 \simeq 2$ and that for ionized protons is $g_p = 1$. Ignoring helium or any other elements $n_b \simeq n_p + n_H$ and assuming $n_e = n_p$, the hydrogen number density can be re-expressed as

$$\frac{n_H}{n_b} = \left(\frac{n_e}{n_b}\right)^2 \eta \, n_\gamma \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right) . \tag{1.91}$$

By defining the ionization fraction (or how many free electrons), we arrive at the Saha equation for the ionization fraction in thermal equilibrium:

$$X_e := \frac{n_e}{n_b} = \frac{n_p}{n_b} \le 1 , \qquad \frac{1 - X_e}{X_e^2} = \sqrt{\frac{32}{\pi}} \zeta(3) \, \eta \left(\frac{m_e}{T}\right)^{-3/2} \exp\left(\frac{B_H}{T}\right) . \tag{1.92}$$

Once the Universe cools below the binding energy B_H, the hydrogen atoms can form, but again due to the large number of high-energy photons at a given temperature compared to baryons, the formation of neutral hydrogen atoms is further delayed. If we define the completion of the recombination process as $X_e = 10\%$, the Saha equation states

$$\theta_{\rm rec}^{-3/2} \exp\left(\frac{13.6}{\theta_{\rm rec}}\right) = \frac{0.9}{0.01} \left(\sqrt{\frac{32}{\pi}} \zeta(3) \, \eta\right)^{-1} \left(\frac{m_e}{1 \text{ eV}}\right)^{3/2} = 3.2 \times 10^{17} \, (\omega_b)^{-1} , \tag{1.93}$$

where we defined

$$\theta := \frac{T}{1 \text{ eV}} \simeq \frac{1 + z}{4250} . \tag{1.94}$$

A numerical computation yields that the recombination takes place at

$$1 + z_{\rm rec} \approx \frac{1367}{1 - 0.024 \ln \omega_b} \approx 1249 , \qquad T_{\rm rec} = 0.3 \text{ eV} \ll B_H , \tag{1.95}$$

a lot lower temperature than B_H.

There are a few subtleties in the cosmic recombination. In a typical gas cloud, the recombination process takes place by a direct capture of free electrons to the ground state (case A recombination) or cascades of electronic transition to the ground state (case B recombination). Both of which are inefficient in the cosmic recombination, because both processes result in high energy photons that ionize hydrogen atoms again. The main channel in the cosmic recombination is a forbidden transition with $\Gamma \approx 8.23 \text{ sec}^{-1}$, so called, the two-photon decay, in which two photons are emitted by an electronic transition $2s \to 1s$, splitting the energy of Lyα. The other process is the cosmological redshift of Lyα photons. The detailed numerical computation shows that the ionization fraction $X_e = 1$ at $z \ge 2000$ decreases as the Universe cools, and it freezes out to a value $X_e \simeq 10^{-3}$ at $z \le 200$.

1.5.2 Decoupling of CMB Photons and Baryons

• ***Decoupling of CMB photons***.— The baryon-photon plasma (including leptons) maintains the equilibrium via Coulomb interactions between photons and free electrons. At this low energy scales, the interaction is mainly elastic, and its cross-section is described by the Thompson scattering as

$$\sigma_T := \frac{8\pi}{3} r_e^2 \simeq 6.651 \times 10^{-25} \text{ cm}^2 , \qquad r_e := \frac{e^2}{m_e c^2} = 2.818 \times 10^{-13} \text{ cm} , \tag{1.96}$$

where the radius of an electron is defined in terms of the Coulomb potential. The Thompson scattering describes a classical collision of ionized electrons. With higher mass, the Thompson scattering cross section for protons is smaller by $(m_e/m_p)^2 = 10^6$ and negligible, but the strong Coulomb interactions between free electrons and protons also keep the protons in thermal equilibrium. As the Universe cools and free electrons recombine to form neutral hydrogen atoms, the interaction rate in the baryon-photon plasma goes down:

$$n_e = X_e \, \eta \, n_\gamma \, , \qquad \Gamma_\gamma = n_e \sigma_T c = 1.01\sqrt{\omega_b}\theta^{9/4} \exp\left[-\frac{6.8}{\theta}\right] \text{ sec}^{-1} \, , \tag{1.97}$$

and the photons are released (or decoupled) from the plasma, when the interaction rate becomes lower than the expansion rate:

$$H \simeq H_0\sqrt{\Omega_m}(1+z)^{3/2} = 8.98 \times 10^{-13}\sqrt{\omega_m}\,\theta^{3/2} \text{ sec}^{-1} \, , \tag{1.98}$$

where we assumed that the Universe is deep in the matter dominated era. The decoupling takes place at

$$\theta^{-1}_{\text{dec}} \approx 3.927 + 0.074 \, \ln\left(\frac{\omega_b}{\omega_m}\right) \, , \qquad T_{\text{dec}} = 0.26 \text{ eV} \, , \qquad 1 + z_{\text{dec}} \simeq 1100 \, , \tag{1.99}$$

soon after the recombination of neutral hydrogen atoms takes place. Another way of understanding the decoupling of photons is to compute the optical depth:

$$\tau(z) := \int_0^z dz \, \frac{c\,dt}{dz} n_e \sigma_T \approx 0.37 \left(\frac{z}{1000}\right)^{14.25} \, , \tag{1.100}$$

where the numerical values are approximations to the best-fit model prediction. The Universe is fairly transparent at low redshift, and it becomes quickly opaque around z_{dec}. A simple analytic calculation shows that the observed CMB photons are indeed emitted at the peak of the visibility function defined as

$$P(\tau) := \tau e^{-\tau(z)} \, . \tag{1.101}$$

which peaks sharply at $z \simeq 1067$ with a width $\Delta z \simeq 80$. In other words, before the decoupling, the CMB photons were in thermal equilibrium with baryons via Thompson scattering, and they are un-polarized and opaque. However, within a narrow redshift width, they are released from the baryon plasma, and they are weakly polarized via last scattering.

• ***Decoupling of baryons***.— Now we consider the decoupling of baryons from the baryon-photon plasma. While the photons are released at $z_{\text{dec}} \simeq 1100$, the baryons are kept in thermal equilibrium long after the decoupling of photons, due to large number of photons per baryons. In general, the matter components cool as $T_m \propto 1/a^2$, faster than the photons, but because of the tight coupling it goes as $T_m \sim T_\gamma \propto 1/a$ until it is released from the photon plasma, i.e., energy is transferred to the baryon plasma from the photon plasma by the Compton scattering of high-energy photons. For the decoupling of photons, the relevant interaction rate was Γ_γ, and no energy transfer was made. For the decoupling of baryons, however, we have to account for this energy transfer to compute the proper interaction rate Γ_e.

The typical average energy transfer due to one Compton scattering of high-energy photons is given by

$$\Delta E = \frac{4}{3}\left(\frac{v_e}{c}\right)^2 \bar{E}_\gamma = 4\left(\frac{kT_e}{m_e c^2}\right)\frac{u_\gamma}{n_\gamma} \, , \qquad \bar{E}_\gamma = h\bar{\nu} = \frac{u_\gamma}{n_\gamma} \, , \tag{1.102}$$

and with larger number of photons n_γ, the energy transfer rate per unit volume is then

$$\frac{d\epsilon}{dt} = \Delta E \, n_\gamma \Gamma_\gamma = 4 n_e \sigma_T u_\gamma \left(\frac{kT_e}{m_e c}\right) \, . \tag{1.103}$$

Since free electrons are tightly coupled with free protons, this energy transfer is quickly shared with protons of typical energy density

$$\epsilon_m = \frac{3}{2}(n_e + n_b)kT_e \, . \tag{1.104}$$

Therefore, the proper interaction rate for electrons to be compared to the expansion rate is then

$$\Gamma_e = \frac{1}{\epsilon_m}\frac{d\epsilon}{dt} = 8.9 \times 10^{-6}\left(\frac{X_e}{1+X_e}\right)\theta^4 \text{ sec}^{-1} \, , \tag{1.105}$$

and the baryon plasma decouples at

$$1 + z = 6.8 \left(\frac{X_e}{1 + X_e} \right)^{-\frac{2}{5}} \omega_m^{1/5} \approx 150 \, . \tag{1.106}$$

Note that the Compton scattering conserves the number of photons, such that it can lead to a spectral distortion. However, the small baryon-to-photon ratio makes it negligible for the photon plasma. The free-free emission and absorption (Bremsstrahlung) can create and destroy photons, such that it is needed to thermalize. However, this process is inefficient at $T \leq 10^4$ eV.

2 Newtonian Perturbation Theory

2.1 Standard Newtonian Perturbation Theory

2.1.1 Summary of the Governing Equations

In Newtonian dynamics, fully nonlinear equation pressureless fluid (CDM and baryons) can be written down:

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{v}=-\frac{1}{a}\nabla\cdot(\mathbf{v}\delta)\ ,\qquad \nabla\cdot\dot{\mathbf{v}}+H\nabla\cdot\mathbf{v}+\frac{3H^2}{2}a\Omega_m\delta=-\frac{1}{a}\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\right]\ ,\qquad \nabla^2\phi=4\pi G\bar{\rho}a^2\delta\ . \tag{2.1}$$

The Euler equation can be split into one for divergence and one for vorticity. The vorticity vector $\nabla\times\mathbf{v}$ decays at the linear order. At nonlinear level, if no anisotropic pressure and no initial vorticity, the vorticity vanishes at all orders. However, in reality, the anisotropic pressure arises from shell crossing, generating vorticity on small scales, even in the absence of the initial vorticity. Of course, baryons are not exactly pressureless; they form galaxies, and their feedback effects are also important up to fairly large scales. These all modify the SPT equation.

• regime of validity, measurement precision, analytic vs numerical simulations, galaxy surveys

2.1.2 Basic Formalism

We consider multi-component fluids in the presence of isotropic pressure. In case of n-fluids with the mass densities ϱ_i, the pressures p_i, the velocities $\mathbf{v}_i$ ($i=1,2,\ldots n$), and the gravitational potential Φ, we have

$$\dot{\varrho}_i+\nabla\cdot(\varrho_i\mathbf{v}_i)=0\ ,\qquad \dot{\mathbf{v}}_i+\mathbf{v}_i\cdot\nabla\mathbf{v}_i=-\frac{1}{\varrho_i}\nabla p_i-\nabla\Phi\ ,\qquad \nabla^2\Phi=4\pi G\sum_{j=1}^{n}\varrho_j\ . \tag{2.2}$$

Assuming the presence of spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

$$\varrho_i=\bar{\varrho}_i+\delta\varrho_i,\qquad p_i=\bar{p}_i+\delta p_i\ ,\qquad \mathbf{v}_i=H\mathbf{r}+\mathbf{u}_i\ ,\qquad \Phi=\bar{\Phi}+\delta\Phi\ , \tag{2.3}$$

where $H:=\dot{a}/a$, and $a(t)$ is a cosmic scale factor. We move to the comoving coordinate $\mathbf{x}$ where

$$\mathbf{r}:=a(t)\mathbf{x}\ ,\qquad \nabla=\nabla_{\mathbf{r}}=\frac{1}{a}\nabla_{\mathbf{x}}\ ,\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}\Big|_{\mathbf{r}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}+\left(\frac{\partial}{\partial t}\Big|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}-H\mathbf{x}\cdot\nabla_{\mathbf{x}}\ . \tag{2.4}$$

In the following we neglect the subindex $\mathbf{x}$. To the background order we derive

$$\dot{\bar{\varrho}}_i+3H\bar{\varrho}_i=0\ ,\qquad \dot{H}+H^2=\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\sum_j\bar{\varrho}_j\ ,\qquad H^2=\frac{8\pi G}{3}\sum_j\bar{\varrho}_j+\frac{2E}{a^2}\ , \tag{2.5}$$

where the second equation is derived by taking the divergence of the Euler equation and for the third equation we used

$$\left(a^2H^2\right)^{\cdot}=2a^2H\left(H^2+\dot{H}\right)\ ,\qquad \sum\left(a^2\bar{\varrho}\right)^{\cdot}=-a^2H\sum\bar{\varrho}\ . \tag{2.6}$$

The integration constant E can be interpreted as the specific total energy in Newton's gravity; in Einstein's gravity we have $2E=-Kc^2$ where K can be normalized to be the sign of spatial curvature. Note the difference in the background equation in Newtonian cosmology. The nonlinear governing equations can be expressed in terms of the perturbation variables as

$$\dot{\delta}_i+\frac{1}{a}\nabla\cdot\mathbf{u}_i=-\frac{1}{a}\nabla\cdot\left(\delta_i\mathbf{u}_i\right)\ ,\qquad \frac{1}{a^2}\nabla^2\delta\Phi=4\pi G\sum_j\bar{\varrho}_j\delta_j\ , \tag{2.7}$$

$$\dot{\mathbf{u}}_i+H\mathbf{u}_i+\frac{1}{a}\mathbf{u}_i\cdot\nabla\mathbf{u}_i=-\frac{1}{a\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i}-\frac{1}{a}\nabla\delta\Phi\ . \tag{2.8}$$

By introducing the expansion θ_i and the rotation $\overrightarrow{\omega}_i$ of each component as

$$\theta_i := -\frac{1}{a}\nabla\cdot\mathbf{u}_i\,,\qquad \overrightarrow{\omega}_i := \frac{1}{a}\nabla\times\mathbf{u}_i\,,\tag{2.9}$$

we derive

$$\dot\theta_i + 2H\theta_i - 4\pi G\sum_j \bar\varrho_j\delta_j = \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar\varrho_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right)\,,\tag{2.10}$$

$$\dot{\overrightarrow{\omega}}_i + 2H\overrightarrow{\omega}_i = -\frac{1}{a^2}\nabla\times(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar\varrho_i}\frac{(\nabla\delta_i)\times\nabla\delta p_i}{(1+\delta_i)^2}\,.\tag{2.11}$$

By introducing decomposition of perturbed velocity into the potential- and transverse parts as

$$\mathbf{u}_i := -\nabla U_i + \mathbf{u}_i^{(v)}\,,\qquad \nabla\cdot\mathbf{u}_i^{(v)}\equiv 0\,,\qquad \theta_i = \frac{\Delta}{a}U_i\,,\qquad \overrightarrow{\omega}_i = \frac{1}{a}\nabla\times\mathbf{u}_i^{(v)}\,,\tag{2.12}$$

we have

$$\dot{\mathbf{u}}_i^{(v)} + H\mathbf{u}_i^{(v)} = -\frac{1}{a}\left[\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar\varrho_i}\frac{\nabla\delta p_i}{1+\delta_i} - \nabla\Delta^{-1}\nabla\cdot\left(\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar\varrho_i}\frac{\nabla\delta p_i}{1+\delta_i}\right)\right]\,.\tag{2.13}$$

Combining equations above, we can derive

$$\ddot\delta_i + 2H\dot\delta_i - 4\pi G\sum_j\bar\varrho_j\delta_j = -\frac{1}{a^2}\left[a\nabla\cdot(\delta_i\mathbf{u}_i)\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar\varrho_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right)\,.\tag{2.14}$$

These equations are valid to fully nonlinear order. The density fluctuation grows against the Hubble friction. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.

• numerical simulations, baryons

2.1.3 Linear-Order and Second-Order Solutions

We will derive the solutions for a single pressureless medium (now we change notation $\boldsymbol{u}_i\to\mathbf{v}$; $\delta\Phi\to\phi$)

$$\dot\delta + \frac{1}{a}\nabla\cdot\mathbf{v} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{v})\,,\qquad \dot\theta + 2H\theta - 4\pi G\bar\varrho\delta = \frac{1}{a^2}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})\,,\tag{2.15}$$

where we now use $\mathbf{v}$ to represent the velocity perturbation. These are the governing equation for the cosmological N-body simulations. The calculations are greatly simplified in Fourier space, and our convention is

$$A(\mathbf{x})\equiv\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,A(\mathbf{k})\,,\qquad A(\mathbf{k})\equiv\int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,A(\mathbf{x})\,,\tag{2.16}$$

and we often use the identity:

$$\delta^D(\mathbf{k}) = \int\frac{d^3\mathbf{x}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,.\tag{2.17}$$

First, we derive the linear-order solution. The conservation equation yields

$$\dot\delta^{(1)}(t,\mathbf{k}) = \theta^{(1)}(t,\mathbf{k})\,.\tag{2.18}$$

At the linear order in perturbations, we can separate the time-dependence and the spatial-dependence, i.e., all different Fourier modes evolve at the same rate, and the growth rate D satisfies

$$\ddot D + 2H\dot D - 4\pi G\bar\rho_m D = 0\,,\qquad D(t)\equiv\frac{D_1(t)}{D_1(t_0)}\,,\tag{2.19}$$

where the (dimensionless) growth factor $D(t)$ is normalized to unity at some early epoch t_0 when the nonlinearities are ignored $\delta(t_0, \mathbf{k}) := \delta_1^{(1)}(t_0, \mathbf{k}) \equiv \hat{\delta}(\mathbf{k})$. The growth eqaution is scale-independent, i.e., the perturbations on all scales grow at the same rate. In fact, the growth factor can be derived in Eq. (4.176) as

$$D(t) \propto c_1(\boldsymbol{k})H(t)\int\frac{dt}{a^2H^2} + c_2(\boldsymbol{k})H(t) \,, \tag{2.20}$$

where the second term is the decaying mode we will ignore here. The linear-order solution for the density and the velocity divergence is then

$$\delta^{(1)}(t,\mathbf{k}) = D(t)\hat{\delta}(\mathbf{k}) \,, \qquad \theta^{(1)}(t,\mathbf{k}) = HfD(t)\hat{\delta}(\mathbf{k}) \,, \qquad f := \frac{d\ln D}{d\ln a} \,, \qquad \dot{D} \equiv HfD \,, \tag{2.21}$$

where the superscript indicates the perturbation order, the logarithmic growth rate f is approximately time-independent and it is unity $f = 1$ in the matter-dominated era.

To derive the second-order solution, we need to Fourier decompose the source terms in the right-hand side of the dynamical equation. At the second-order in perturbations, the quadratic terms represent the product of two linear-order terms. Furthermore, the quadratic product in configuration space becomes the convolution in Fourier space:

$$\left[-\frac{1}{a}\nabla\cdot(\delta\mathbf{v})\right]^{(2)} = HfD^2\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\left(1+\frac{\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{Q_1^2}\right)\hat{\delta}(\boldsymbol{Q}_1)\hat{\delta}(\boldsymbol{Q}_2) \,, \tag{2.22}$$

$$\left\{\frac{1}{a^2}\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right\}^{(2)} = H^2f^2D^2\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\frac{|\boldsymbol{Q}_1+\boldsymbol{Q}_2|^2\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{2Q_1^2Q_2^2}\hat{\delta}(\boldsymbol{Q}_1)\hat{\delta}(\boldsymbol{Q}_2) \,, \tag{2.23}$$

where we defined $\boldsymbol{Q}_{12} = \boldsymbol{Q}_1 + \boldsymbol{Q}_2$. Using the source functions in Fourier space, we can solve the governing equations for the density and the velocity divergence as

$$\frac{\delta^{(2)}(t,\mathbf{k})}{D^2} = \int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12})\hat{\delta}(\mathbf{q}_1)\hat{\delta}(\mathbf{q}_2)\left[\frac{5}{7}+\frac{2}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2q_2^2}+\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)\right] \,, \tag{2.24}$$

$$\frac{\theta^{(2)}(t,\mathbf{k})}{HfD^2} = \int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12})\hat{\delta}(\mathbf{q}_1)\hat{\delta}(\mathbf{q}_2)\left[\frac{3}{7}+\frac{4}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2q_2^2}+\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)\right] \,. \tag{2.25}$$

• HW: derive the second-order solutions

2.1.4 General Solution

Beyond the linear order, the density and the velocity divergence grows in a nonlinear fashion, i.e., different Fourier modes couple. By assuming the separability of the time and the spatial dependences, the standard perturbation theory (SPT) takes a perturbative approach to the nonlinear solution:

$$\delta(t,\mathbf{k}) := \sum_{n=1}^{\infty}D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat{\delta}(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})F_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n) \equiv \sum_{n=1}^{\infty}D^n(t)\delta^{(n)}(\mathbf{k}) \,, \tag{2.26}$$

$$\frac{\theta(t,\mathbf{k})}{Hf} := \sum_{n=1}^{\infty}D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat{\delta}(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})G_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n) \equiv \sum_{n=1}^{\infty}D^n(t)\theta^{(n)}(\mathbf{k}) \,, \tag{2.27}$$

where $\mathbf{q}_{12\cdots n} \equiv \mathbf{q}_1 + \cdots + \mathbf{q}_n$, $\delta^{(n)}(\mathbf{k})$ and $\theta^{(n)}(\mathbf{k})$ are time-independent n-th order perturbations, $F_n^{(s)}$ and $G_n^{(s)}$ are the SPT kernels symmetrized over its arguments. With these decompositions in Fourier space, the LHS of the Newtonian dynamical equations become

$$\dot{\delta}+\theta = Hf\sum_{n=1}^{\infty}D^n\left(n\delta^{(n)}-\theta^{(n)}\right) \,, \qquad \dot{\theta}+2H\theta-4\pi G\bar{\rho}_m\delta = H^2f^2\sum\frac{D^n}{2}\left[(1+2n)\theta^{(n)}-3\delta^{(n)}\right] \,, \tag{2.28}$$

where we utilized the relation between the growth factor and the growth rate $\dot{D} = HDf$. The RHS of the Newtonian dynamical equations are the convolution in Fourier space:

$$\left[-\frac{1}{a}\nabla\cdot(\delta\mathbf{v})\right](\mathbf{k}) = \int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\alpha_{12}\theta(\boldsymbol{Q}_1,t)\delta(\boldsymbol{Q}_2,t) \equiv Hf\sum_{n=1}^{\infty}D^nA_n(\mathbf{k}) \,, \tag{2.29}$$

$$\left\{\frac{1}{a^2}\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right\}(\mathbf{k}) = \int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\beta_{12}\theta(\boldsymbol{Q}_1,t)\theta(\boldsymbol{Q}_2,t) \equiv H^2f^2\sum_{n=1}^{\infty}D^nB_n(\mathbf{k}) \,, \tag{2.30}$$

where the vertex functions are defined as

$$\alpha_{12} := \alpha(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \equiv 1 + \frac{\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{Q_1^2} \,, \qquad \beta_{12} := \beta(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \equiv \frac{|\boldsymbol{Q}_1+\boldsymbol{Q}_2|^2\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{2Q_1^2Q_2^2} \,, \tag{2.31}$$

and the n-th order perturbation kernels $A_n(\mathbf{k})$ and $B_n(\mathbf{k})$ are

$$A_n(\mathbf{k}) = \left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat{\delta}(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})\sum_{i=1}^{n-1}\alpha_{12}G_i(\mathbf{q}_1,\cdots,\mathbf{q}_i)F_{n-i}(\mathbf{q}_{i+1},\cdots,\mathbf{q}_n) \,, \tag{2.32}$$

$$B_n(\mathbf{k}) = \left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat{\delta}(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})\sum_{i=1}^{n-1}\beta_{12}G_i(\mathbf{q}_1,\cdots,\mathbf{q}_i)G_{n-i}(\mathbf{q}_{i+1},\cdots,\mathbf{q}_n) \,, \tag{2.33}$$

with $\boldsymbol{Q}_1 = \mathbf{q}_{1\cdots i}$ and $\boldsymbol{Q}_1+\boldsymbol{Q}_2 = \mathbf{k}$.

Therefore, the two Newtonian dynamical equations become algebraic equations with the time-dependence removed:

$$n\delta^{(n)} - \theta^{(n)} = A_n \,, \qquad (1+2n)\theta^{(n)} - 3\delta^{(n)} = 2B_n \,, \tag{2.34}$$

and the well-known recurrence formulas for the solutions are

$$\delta^{(n)} = \frac{(1+2n)A_n + 2B_n}{(2n+3)(n-1)} \,, \qquad \theta^{(n)} = \frac{3A_n + 2nB_n}{(2n+3)(n-1)} \,, \tag{2.35}$$

and similarly so for the SPT kernels

$$F_n = \sum_{i=1}^{n-1}\frac{G_i}{(2n+3)(n-1)}\left[(1+2n)\alpha_{12}F_{n-i} + 2\beta_{12}\,G_{n-i}\right] \,, \tag{2.36}$$

$$G_n = \sum_{i=1}^{n-1}\frac{G_i}{(2n+3)(n-1)}\left[3\alpha_{12}F_{n-i} + 2n\beta_{12}G_{n-i}\right] \,, \qquad F_1 = G_1 = 1 \,. \tag{2.37}$$

3 Probes of Inhomogeneity

In cosmology, the initial condition is set in the early Universe with Gaussian random fluctuations in Fourier space, as the quantum fluctuations in vacuum are stretched beyond the horizon scales during the inflationary epoch. Since the Gaussian distribution is completely specified by the variance, the power spectrum contains all the information in the early Universe. However, the nonlinear growth in the late time complicates the interpretations. Here we focus on the linear theory and study various ways to measure the two-point statistics.

3.1 Basic Formalism

3.1.1 Two-Point Correlation Function and Power Spectrum

• ***3D information***.— Suppose that we use some cosmological probes such as galaxies and measure, say, the matter density fluctuation δ. Now imagine we have measurements of such probe over all positions $\mathbf{x}$. We can then measure the two-point correlation function $\xi(r)$ and its Fourier transform, the power spectrum $P(k)$:

$$\xi(\mathbf{r}) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}P(\mathbf{k}) \,, \qquad \big\langle\delta(\mathbf{k})\delta(\mathbf{k}')\big\rangle = (2\pi)^3\delta^{3D}(\mathbf{k}+\mathbf{k}')P(\mathbf{k}) \,, \tag{3.1}$$

and the variance is

$$\sigma^2 = \xi(0) = \int d\ln k\, \frac{k^3}{2\pi^2}P(k) \,, \qquad \Delta_k^2 := \frac{k^3}{2\pi^2}P(k) \,, \tag{3.2}$$

where Δ_k^2 is the dimensionless power spectrum and it is the contribution to the variance per each log k.

Note that different Fourier modes are not correlated in the initial condition and the power spectrum characterizes the Gaussian distribution at each Fourier mode.[1] Therefore, using cosmological probes, we need to measure the distribution map $\delta(\mathbf{x})$ and compute the two-point correlation function or the power spectrum.

• ***1D information***.— Spectroscopic measurements of distant quasars yield the density fluctuations of neutral hydrogens along the line-of-sight. In this case, we probe the density fluctuation, but only in terms of the line-of-sight separation, say, z-direction. Given the 1D map, we can measure the 1D correlation function, and it is related to the power spectrum as

$$\xi_{\rm 1D}(z) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{z})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{ik_z z}P_{\rm 3D}(\mathbf{k}) \,, \tag{3.3}$$

where the separation vector is $\mathbf{z} = z\hat{\mathbf{z}}$ along the line-of-sight direction. We can also define 1D power spectrum that is a Fourier counterpart of the 1D correlation function:

$$P_{\rm 1D}(k_z) \equiv \int dz\, e^{-ik_z z}\xi_{\rm 1D}(z) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3}P(\mathbf{k}')\delta^{1D}(k_z'-k_z) = \int_0^\infty \frac{dk_\parallel'}{2\pi}\, k_\parallel' P(k_\parallel', k_z) = \int_{k_z}^\infty \frac{dk}{2\pi}\, kP(k) \,, \tag{3.4}$$

where we again assumed that the 3D power spectrum is isotropic. The 1D power spectrum is the projection of the 3D power spectrum over 2D Fourier space. For sufficiently high k, it is largely one-to-one, though it has bias (called aliasing) on low k. This relation can be inverted as

$$P(k) = -\frac{2\pi}{k}\frac{d}{dk}P_{1D}(k) \,, \tag{3.5}$$

and the dimensionless power spectrum in 1D is

$$\sigma_{\rm 1D}^2 = \int d\ln k_z\, \frac{k_z}{\pi}P_{\rm 1D}(k_z) \,, \qquad \Delta_{k,\rm 1D}^2 := \frac{k_z}{\pi}\, P_{\rm 1D}(k_z) \,. \tag{3.6}$$

[1] However, as we studied in Section 2, the nonlinear evolution results in the mode coupling.

• ***2D information***.— Though the distance in cosmology is difficult to measure, it is easy to have 2D information on the sky. We define the 2D power spectrum in a similar way as the Fourier counterpart of the 2D correlation function:

$$P_{\rm 2D}(k_x,k_y)\equiv\int dx\int dy\, e^{-ik_xx}e^{-ik_yy}\xi_{\rm 2D}(x,y)=\int\frac{dk'_z}{2\pi}\,P(k_x,k_y,k'_z)=\frac{1}{\pi}\int_{k_\perp}^\infty dk'\,\frac{k'P(k')}{\sqrt{k'^2-k_\perp^2}}\,, \tag{3.7}$$

where $k_\perp^2=k_x^2+k_y^2$. The 2D power spectrum is the projection over 1D Fourier space, and its similar relation to the 3D power spectrum exists. This relation can be again inverted by using the (non-trivial) Abell integral as

$$P(k)=-\frac{2}{k}\int_k^\infty dk_\perp\frac{P_{\rm 2D}(k_\perp)}{\sqrt{k_\perp^2-k^2}}\,, \tag{3.8}$$

and the dimensionless power spectrum in 2D is then

$$\sigma^2_{\rm 2D}=\int d\ln k_\perp\,\frac{k_\perp^2}{2\pi}\,P_{\rm 2D}(k_\perp)\,,\qquad\qquad \Delta^2_{k,\rm 2D}=\frac{k_\perp^2}{2\pi}P_{\rm 2D}(k_\perp)\,. \tag{3.9}$$

The projection-slice theorem says Fourier transformation of the projection is the slice of its Fourier transformation. It means exactly what we derived here. A similar relation holds in configuration space. The projected correlation function is related as

$$w_p(r_p):=\int dz\,\xi(r_p,z)=2\int_{r_p}^\infty dr\,\frac{\xi(r)}{\sqrt{r^2-r_p^2}}\,,\qquad\qquad \xi(r)=-\frac{1}{\pi}\int_r^\infty dr_p\,\frac{w_p(r_p)}{\sqrt{r_p^2-r^2}}\,. \tag{3.10}$$

3.1.2 Angular Correlation and Angular Power Spectrum

We briefly covered the statistics in a flat space. However, the sky is round, and we can only make observations by measuring the light signals. The cosmic microwave background anisotropies, for example, are measured only as a function of the angular position on the sky at the Earth. In cosmology, we often have angular information, but no distance measurements. Since this measurement $\delta(\hat\theta)$ is defined on a unit sphere, we can decompose it in terms of spherical harmonics as

$$\delta(\hat\theta):=\sum_{lm}a_{lm}Y_{lm}(\hat\theta)\,,\qquad\qquad a_{lm}\equiv\int d^2\hat\theta\,Y^*_{lm}(\hat\theta)\delta(\hat\theta)\,, \tag{3.11}$$

where we have discrete sum, instead of integral in Fourier space. The reality condition for δ imposes

$$a^*_{lm}=(-1)^m a_{l,-m}\,. \tag{3.12}$$

Similar to the case in 3D, we can define the angular correlation function and its Fourier counterpart:

$$w(\hat\gamma)=\Big\langle\delta(\hat\theta)\delta(\hat\theta+\hat\gamma)\Big\rangle=\sum_{lm}C_lY_{lm}(\hat\theta+\hat\gamma)Y^*_{lm}(\hat\theta)=\sum_l\frac{2l+1}{4\pi}C_lL_l(\cos\gamma)\,, \tag{3.13}$$

where we used the relation

$$\langle a_{lm}a^*_{l'm'}\rangle=\delta_{ll'}\delta_{mm'}C_l=\sum_m\frac{|a_{lm}|^2}{2l+1}\delta_{ll'}\delta_{mm'}\,, \tag{3.14}$$

and the Legendre polynomial is related to the spherical harmonics as

$$L_l(\mu)=\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,Y_{lm}(\hat\theta_1)Y^*_{lm}(\hat\theta_2)\,,\qquad\qquad \mu=\hat\theta_1\cdot\hat\theta_2\,. \tag{3.15}$$

The angular power spectrum can be obtained as

$$C_l=2\pi\int_{-1}^1 d\mu\,L_l(\mu)w(\theta)\,. \tag{3.16}$$

3.1.3 Flat-Sky Approximation

When the area of interest is relatively small in the sky, we can use the flat-sky approximation, and it often overlaps with the distant-observer approximation, in which the observer is so far away that the position angle is virtually constant, compared to their relative positions. In this case, the angular correlation and its power spectrum are closely related to those in flat space.

Now consider the 2D correlation function ξ_{2D} and 2D power spectrum $P_{2D}(k)$:

$$\xi_{2D}(x,y) = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} P_{2D}(k_\perp) = \int \frac{d^2l}{(2\pi)^2}\, e^{i\mathbf{l}\cdot\boldsymbol{\theta}} P_l \,, \qquad P_l \equiv \frac{1}{r^2} P_{2D}\left(k_\perp = \frac{l}{r}\right) \,, \tag{3.17}$$

where we used $\mathbf{x}_\perp = r\boldsymbol{\theta}$ and defined the (flat-sky) angular power spectrum P_l. Note that the 2D power spectrum is dimensionful, but the angular power spectrum is dimensionless. Given the radial distance r, the 2D correlation function ξ_{2D} can be considered as the angular correlation function, and assuming that the angular power spectrum is independent of its direction, we can further simply the relation:

$$w(\theta) = \xi_{2D}(r\boldsymbol{\theta}) = \int \frac{dl}{2\pi}\, lP_l\, J_0(l\theta) \,, \tag{3.18}$$

where J_0 is the Bessel function. The (full-sky) angular power spectrum is then obtained as

$$C_l = 2\pi \int d\mu\, L_l(\mu) w(\theta) \simeq \sum_{l'} l' P_{l'} \frac{2\delta_{ll'}}{2l+1} \simeq P_l \,, \tag{3.19}$$

where we manipulated the Bessel function for $l \gg 1$ and $\theta \ll 1$

$$J_0(l\theta) = \frac{1}{\pi}\int_0^\pi d\phi\, e^{il\theta\cos\phi} \simeq \frac{1}{\pi}\int_0^\pi d\phi \left(1 + \frac{il\theta\cos\phi}{l}\right)^l \simeq \frac{1}{\pi}\int_0^\pi d\phi\, (\cos\theta + i\sin\theta\cos\phi)^l = L_l(\cos\theta) \,. \tag{3.20}$$

The angular quantities such as $w(\theta)$ and C_l are defined on a unit sphere, whereas the 2D quantities such as ξ_{2D} and P_{2D} are defined on a 2D flat space. Hence, the former is related to each other via spherical harmonics, and the latter via Fourier transformation. But they are defined in a way that the angular power spectrum C_l and its flat-sky counterpart P_l are equivalent in the limit of small sky.

3.1.4 Projection and Limber Approximation

We often measure some angular quantities in cosmology, but they are often the projection of the 3D quantities. For example, one can measure the angular map in a given galaxy survey, but the angular quantity $\delta_2(\theta)$ we measure indeed derives from the 3D quantity $\delta(\mathbf{x})$, but projected along the line-of-sight direction with some weighting $W(r)$:

$$\delta_2(\theta) = \int dr\, W(r)\delta(\mathbf{x}) \,, \qquad \mathbf{x} = (r\boldsymbol{\theta}, r) \,. \tag{3.21}$$

The weight function is normalized to unity and it is often parametrized in terms of redshift as

$$1 = \int dr\, W_r(r) = \int dz\, W_z(z) \,, \tag{3.22}$$

where the weight function can be dimensionful, depending on its parametrization. The angular correlation is then

$$w(\theta) = \langle\delta_2(0)\delta_2(\theta)\rangle = \int dr_1\, W(r_1) \int dr_2\, W(r_2)\, \xi_{3D}(\mathbf{r}) \,, \qquad \mathbf{r} \simeq (r_1\boldsymbol{\theta}, r_2 - r_1) \,, \tag{3.23}$$

where we assumed the flat-sky approximation. The angular power spectrum is

$$P_l = \int \frac{d^2\theta}{(2\pi)^2} e^{-i\mathbf{l}\cdot\boldsymbol{\theta}} w(\theta) = \int dr_1\, W(r_1) \int dr_2\, W(r_2) \int \frac{dk_z}{2\pi}\, e^{-ik_z(r_2-r_1)} \frac{1}{r_1^2} P\left[\left(k_\perp = \frac{l}{r_1}, k_z\right)\right] \,. \tag{3.24}$$

Since we work in the flat-sky regime (or the distant observer), the radial distance is far larger than the transverse separation $r \gg r\boldsymbol{\theta}$. Hence, we have the separation of scale in Fourier space

$$k_\perp = \frac{l}{r} \simeq \frac{1}{r\theta} \gg \frac{1}{r} \simeq k_z \,, \tag{3.25}$$

and the power spectrum can be approximated as

$$P(k) \simeq P(k = k_\perp) + \frac{dP}{dk_z}k_z + \mathcal{O}(k_z)^2 \,, \qquad \frac{dP}{dk_z}k_z = \frac{dP}{dk}\frac{k_z^2}{k} \simeq \frac{P}{k^2r^2} \simeq \frac{P}{l^2} \ll P \,. \tag{3.26}$$

Keeping the leading term in the power spectrum, we can integrate over k_z and approximate the angular power spectrum as

$$P_l \simeq \int dr \frac{W^2(r)}{r^2}\, P\left[\left(k_\perp = \frac{l}{r_1}, k_z\right)\right] \,. \tag{3.27}$$

This is sometimes called the Limber approximation. When the window function is sufficiently broad compared to the coherent length scale of the correlation, the Limber approximation is very accurate and useful. Its relation to the angular correlation is

$$w(\theta) = \int \frac{dl}{2\pi}\, lP_l\, J_0(l\theta) \equiv \int dk\, kP(k)F(k,\theta) \,, \tag{3.28}$$

where we defined the kernel

$$F(k,\theta) := \int \frac{dr}{2\pi} W^2(r) J_0(kr\theta) = \frac{1}{k}\int \frac{dl}{2\pi} W^2\left(\frac{l}{k}\right) J_0(kr\theta) \,. \tag{3.29}$$

3.2 Matter Power Spectrum

The evolution equation (2.19) for the matter density growth yields simple solutions for the matter-dominated era (MDE) and the radiation-dominated era (RDE):

$$D_{\rm mde} \propto a \,, \qquad D_{\rm rde} \propto a^2 \,, \tag{3.30}$$

where the solution can be verified by direct substitutions. The growth in MDE is scale-independent, such that the perturbations on all scales grow equally in proportion to a. However, the growh in RDE is a bit different. In fact, the evolution equation is not valid in RDE, as we derived the equation by assuming the presureless medium, whereas the Universe in RDE is dominated by radiation (with large pressure). On small scales, the matter density cannot grow due to the radiation pressure, so no growth during the RDE, but on large scales (larger than the horizon scale in RDE) the evolution equation is valid, as the effect of pressure is negligible.[2]

Therefore, the matter density fluctuations on large scales can continuously growh throughout the periods of RDE and MDE, while those on small scales cannot grow, once they enter the horizon during RDE (remember that all modes were outside the horizon after inflation). So the scale of comparison is naturally the equality scale $k_{\rm EQ}$, where the epoch of equality is defined as $\bar\rho_m = \bar\rho_r$ at $t_{\rm EQ}$ (or $z_{\rm EQ} \simeq 3000$). The modes larger than the equality scale $k_A < k_{\rm EQ}$ stay outside the horizon during RDE, so that they continue to grow until today:

$$\delta(k_A;t_0) = \delta(k_A;t_i)\left(\frac{a_{\rm EQ}}{a_i}\right)^2\left(\frac{a_0}{a_{\rm EQ}}\right) \,, \tag{3.31}$$

where t_i is the initial time after inflation and t_0 is the present time. Similarly for the mode $k_{\rm EQ}$, and hence the ratio of the power spectra at k_A and $k_{\rm EQ}$ is

$$\frac{P(k_A;t_0)}{P(k_{\rm EQ};t_0)} = \left[\frac{P(k_A)}{P(k_{\rm EQ})}\right]_{t_i} = \left(\frac{k_A}{k_{\rm EQ}}\right)^{n_s} \,. \tag{3.32}$$

For a mode $k_A < k_{\rm EQ}$, the power spectrum is essentially primordial, up to the amplitude.

[2] For calculations outside the horizon, we need relativistic equations, so the validity of our Newtonian equation in this regime is a bit of coincident.

The modes smaller than the equality scale $k_B > k_{\rm EQ}$ start outside the horizon during RDE and grow for some time. However, after they enter the horizon during RDE, their growth freezes, until the Universe becomes MDE, so that their growth is

$$\delta(k_B;t_0)=\delta(k_B;t_i)\left(\frac{a_\star}{a_i}\right)^2\left(\frac{a_0}{a_{\rm EQ}}\right)\,, \tag{3.33}$$

where $a_\star$ is the time at which the modes k_B enter the horizon, i.e., $k_B=\mathcal{H}(t_\star)$. During RDE, the Hubble parameter H is proportional to a^{-2}, and the conformal Hubble parameter $\mathcal{H}:=aH$. Therefore, the scale factor at the horion crossing is $a_\star\propto 1/k_B$, and the ratio of the power spectra at k_B and $k_{\rm EQ}$ is then

$$\frac{P(k_B;t_0)}{P(k_{\rm EQ};t_0)}=\left[\frac{P(k_B)}{P(k_{\rm EQ})}\right]_{t_i}\left(\frac{a_\star}{a_{\rm EQ}}\right)^4=\left(\frac{k_B}{k_{\rm EQ}}\right)^{n_s-4}\,. \tag{3.34}$$

Compared to the initial condition, the growth in the matter power spectrum is suppressed on small scales due to the radiation pressure.

In fact, the dark matter density can still grows in RDE on small scales, as they do not feel the radiation pressure. However, the growth is indeed slowed due to the rapid Hubble expansion in RDE, so that the growth is only logarithmic, and the suppression on small scales is in fact $(\ln k/k^2)^2$, instead of $(1/k^2)^2$. This growth of dark matter density during RDE is important for structure formation today. CMB observations show that $\delta T/\bar T\sim\delta_b\sim 10^{-5}$ at $z=1100$. According to linear theory, this small matter density fluctuation can only grow by $D(z=1000)/D(z=0)\approx 1000$ to $\delta_b\sim 0.01$, which is not enough to form any nonlinear structure today. With dark matter already growing for a while, baryons can catch up quickly, once released from CMB.

3.3 Peculiar Velocity

3.3.1 Observations of Peculiar Velocities

The distant objects such as galaxies are receding from us due to the Hubble expansion, and this expansion (or the receding velocity v) is measured by the redshift z of the known line-emissions from the distant objects:

$$1+z=\frac{\lambda_{\rm obs}}{\lambda_{\rm rest}}\,. \tag{3.35}$$

If we interpret this measurement as the Doppler effect, we obtain the receding velocity

$$1+z\approx 1\pm\frac{v}{c}\,,\qquad\qquad v\equiv cz\,. \tag{3.36}$$

What happen to the objects at $z>1$? We can use the relativistic Doppler effect to obtain the receding velocity less than the speed of light, but this velocity is not really the physical velocity of the objects. The dominant contribution to the redshift is indeed the expansion of the Universe.

However, in addition to the Hubble expansion v_H, these objects are also moving, and this motion is referred to as the peculiar motion v_p. Due to the peculiar motion, the Doppler effect also contributes to the receding velocity, and the receding velocity can be written as

$$v=v_H+v_p\,,\qquad\qquad v_H=Hd=\mathcal{H}r\,, \tag{3.37}$$

where the object is assumed to be at the physical distance d (or comoving distance r). The redshift measurements (or the receding velocity) yield only the radial component of the receding velocity. The tangential peculiar motion can be measured. However, since this requires measurements of the angular motion of the distant objects over a long time, it is practically limited to the nearby objects such as stars in our own Galaxy. The measurements of the radial peculiar velocity also requires precise measurements of the distance d, which is very difficult in cosmology. For example, 10% error in the distance measurements at $d=50\ h^{-1}{\rm Mpc}$ yields the error of $500\ {\rm km\,s^{-1}}$ in the peculiar velocity measurement. Therefore, the peculiar velocity measurements are also limited to the low-redshift objects.

- receding velocity at $z>1$, gauge ambiguity, SN Ia or SZ measurements
- HW: derive Eq. (3.36) from Eq. (3.37)

3.3.2 Linear Theory

In Chapter 2, we learned that the velocity divergence is related to the density fluctuation:

$$\theta \equiv -\frac{1}{a}\nabla\cdot\boldsymbol{v} = Hf\delta\,. \tag{3.38}$$

Ignoring the vector perturbation, the velocity can be expressed in terms of the velocity potential U as

$$\boldsymbol{v} = -\nabla U\,,\qquad \theta = \frac{1}{a}\Delta U\,,\qquad U = \mathcal{H}f\Delta^{-1}\delta\,,\qquad \boldsymbol{v} = -\mathcal{H}f\nabla\Delta^{-1}\delta\,. \tag{3.39}$$

In Fourier space, the inverse Laplacian can be readily manipulated, and the velocity vector becomes

$$U(\mathbf{k}) = -\frac{\mathcal{H}f}{k^2}\delta(\mathbf{k})\,,\qquad \boldsymbol{v}(\mathbf{k}) = i\mathbf{k}\frac{\mathcal{H}f}{k^2}\delta(\mathbf{k})\,, \tag{3.40}$$

where we suppressed the time-dependence, for example,

$$\delta(\mathbf{k}) = D(t)\hat{\delta}(\mathbf{k},t_o)\,. \tag{3.41}$$

3.3.3 Two-Point Correlation of the Peculiar Velocities

Given the peculiar velocity (vector) field, we can compute the two-point correlation function of the peculiar velocities at two different points:

$$\Psi_{ij}(\mathbf{r}) = \langle v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})\rangle = \int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathcal{H}^2f^2P_m(k)\,\frac{k_ik_j}{k^4} \equiv \Psi_\perp(r)(\delta_{ij}-\hat{r}_i\hat{r}_j)+\hat{r}_i\hat{r}_j\Psi_\parallel(r)\,, \tag{3.42}$$

where $\hat{r}_i = \mathbf{r}_i/|\mathbf{r}|$, the matter density power spectrum is

$$\big\langle\delta(\mathbf{k})\delta(\mathbf{k}')\big\rangle = (2\pi)^3\delta^D(\mathbf{k}+\mathbf{k}')P_m(\mathbf{k})\,, \tag{3.43}$$

and we defined two velocity correlation functions, $\Psi_\parallel$ along the connecting direction and $\Psi_\perp$ perpendicular to it:

$$\Psi_\perp := \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\,\frac{j_1(kr)}{kr}\,,\qquad \Psi_\parallel := \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\left[j_0(kr)-\frac{2j_1(kr)}{kr}\right] = \frac{d}{dr}\left[r\Psi_\perp(r)\right]\,, \tag{3.44}$$

where we used

$$\int d\mu\,e^{\pm i\mu x} = 2j_0(x)\,,\qquad \int d\mu\,\mu^2\,e^{\pm i\mu x} = 2j_0(x)-\frac{4j_1(x)}{x}\,. \tag{3.45}$$

If we define the multipole correlation function of the matter as

$$\xi_l^n(x) := \int\frac{dk}{2\pi^2}\,k^nj_l(kx)P_m(k)\,, \tag{3.46}$$

we can show that the velocity correlation functions are

$$\Psi_\parallel \propto \frac{1}{3}\left(\xi_0^0-2\xi_2^0\right)\,,\qquad \Psi_\perp \propto \frac{1}{3}\left(\xi_0^0+\xi_2^0\right)\,. \tag{3.47}$$

The two-point correlation function of the velocity inner product is then

$$\langle\boldsymbol{v}(x)\cdot\boldsymbol{v}(x+r)\rangle = \Psi_\parallel(r)+2\Psi_\perp(r)\,, \tag{3.48}$$

and its variance is

$$\sigma^2_{\rm 3D} \equiv \langle\boldsymbol{v}(x)\cdot\boldsymbol{v}(x)\rangle = \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\,. \tag{3.49}$$

Since the peculiar velocity is often measured along the line-of-sight direction only, one-dimensional variance is often used in literature:

$$\sigma^2_{\rm 1D} = \frac{1}{3}\sigma^2_{\rm 3D}\,. \tag{3.50}$$

For the same reason, the two-point correlation function of the line-of-sight velocities is often measured, and it is related to the velocity correlation Ψ_{ij} as

$$\langle V_1V_2\rangle = \hat{n}_{1i}\hat{n}_{2j}\Psi_{ij}\,,\qquad V_1 := \hat{n}_1^iv_i(x_1)\,,\qquad \hat{n}_1 = \mathbf{x}_1/|\mathbf{x}_1|\,, \tag{3.51}$$

where $\hat{n}_1$ is the line-of-sight direction for the position $\mathbf{x}_1$.

3.4 Redshift-Space Distortion

3.4.1 Redshift-Space Power Spectrum

In cosmology, we rarely know the physical distance to any of the cosmological objects, but we can measure their redshift z with relative ease. The redshift-space distance s is then assigned to the object as

$$s = \int_0^z \frac{dz'}{H} \,. \tag{3.52}$$

As we discussed in Section 3.3.1, the observed redshift is the sum of the Hubble expansion and the peculiar velocity. However, since it is measured in terms of wavelength, it is more convenient to express it as

$$1 + z \equiv (1 + \bar{z})\,(1 + \delta z) \,, \qquad z = \bar{z} + (1 + \bar{z})\delta z \,, \tag{3.53}$$

where the redshift $\bar{z}$ in the background would represent the comoving distance to the object in the background

$$r = \int_0^{\bar{z}} \frac{dz'}{H} \,, \qquad d = \frac{r}{1 + \bar{z}} \,, \tag{3.54}$$

and the peculiar velocity or any contributions to the observed redshift other than the Hubble expansion is described by the perturbation δz:

$$\delta z = v_p + \cdots \,. \tag{3.55}$$

To the linear order in perturbations, we can expand the redshift-space distance as

$$s \simeq r + \frac{1+z}{H}\delta z = r + \mathcal{V} \,, \qquad \mathcal{V} := \frac{v_p}{\mathcal{H}} = -f\frac{\partial}{\partial r}\Delta^{-1}\delta \,, \tag{3.56}$$

where we replaced $\bar{z}$ with z at the linear order. Despite the distortion in the radial distance, the number of galaxies we measure in a given area of the sky remains unaffected: $n_g(s)d^3s = n_g(r)d^3r$. Therefore, the observed galaxy fluctuation δ_s in redshift-space is related to the real-space fluctuation δ_g as

$$1 + \delta_s = \frac{n_g(r)}{n_g(s)} \left| \frac{d^3s}{d^3r} \right|^{-1} = \frac{r^2 \bar{n}_g(r)}{s^2 \bar{n}_g(s)} \left(1 + \frac{d\mathcal{V}}{dr} \right)^{-1} (1 + \delta_g) \,. \tag{3.57}$$

This relation is exact but assumes that the redshift-space distortion is purely radial, ignoring angular displacements.

One can make a progress by expanding equation (3.57) to the linear order in perturbations, and the redshift-space galaxy fluctuation is then

$$\delta_s = \delta_g - \left(\frac{d}{dr} + \frac{\alpha}{r} \right) \mathcal{V} \,, \tag{3.58}$$

where the selection function α is defined in terms of the (comoving) mean number density $\bar{n}_g$ of the galaxy sample as

$$\alpha := \frac{d\ln r^2 \bar{n}_g}{d\ln r} = 2 + \frac{rH}{1+z} \frac{d\ln \bar{n}_g}{d\ln(1+z)} \,. \tag{3.59}$$

By adopting the distant-observer approximation ($r \to \infty$) and ignoring the velocity contributions, a further simplification can be made:

$$\delta_s \simeq \delta_g - \frac{d\mathcal{V}}{dr} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{s}}\, (b + f\mu_k^2)\, \delta_m(\mathbf{k}) \,, \tag{3.60}$$

where we used the linear bias approximation $\delta_g = b\,\delta_m$ and the cosine angle between the Fourier mode and the line-of-sight direction is $\mu_k = \hat{\mathbf{s}} \cdot \hat{\boldsymbol{k}}$. The galaxy power spectrum in redshift-space is then readily computed as

$$P_s(k, \mu_k) = (b + f\mu_k^2)^2 P_m(k) \,. \tag{3.61}$$

This redshift-space distortion effect was first derived by Nick Kaiser in 1987. Due to our redshift measurements as the radial distance, the Doppler effect affects our observation of the number density in redshift-space, such that the galaxy power spectrum becomes enhanced along the line-of-sight direction, representing the infall toward the overdense region.

• random motion on small scales, growth rate of structure

3.4.2 Multipole Expansion

The Kaiser formula for the redshift-space power spectrum indicates that the power spectrum is anisotropic, i.e., it depends not only a Fourier mode k, but also its direction. So, it is often convenient to expand $P_s(k,\mu_k)$ in terms of Legendre polynomials $L_l(x)$ as

$$P_s(k,\mu_k) = \sum_{l=0,2,4} L_l(\mu_k) P_l^s(k) , \tag{3.62}$$

and the corresponding multipole power spectra are

$$P_l^s(k) = \frac{2l+1}{2} \int_{-1}^{1} d\mu_k \; L_l(\mu_k) P_s(k,\mu_k) . \tag{3.63}$$

With its simple angular structure, the simple Kaiser formula in equation (3.61) is completely described by three multipole power spectra

$$P_0^s(k) = \left(b^2 + \frac{2fb}{3} + \frac{f^2}{5}\right) P_m(k) , \qquad P_2^s(k) = \left(\frac{4bf}{3} + \frac{4f^2}{7}\right) P_m(k) , \qquad P_4^s(k) = \frac{8}{35} f^2 P_m(k) , \tag{3.64}$$

while any deviation from the linearity or the distant-observer approximation can give rise to higher-order even multipoles ($l > 4$) and deviations of the lowest multipoles from the above equations.

The correlation function in redshift-space is the Fourier transform of the redshift-space power spectrum $P_s(k,\mu_k)$. With the distant-observer approximation the redshift-space correlation function can be computed and decomposed in terms of Legendre polynomials as

$$\xi_s(s,\mu) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{s}} \, P_s(k,\mu_k) = \sum_{l=0,2,4} L_l(\mu) \, \xi_l^s(s) , \tag{3.65}$$

and the multipole correlation functions are related to the multipole power spectra as

$$\xi_l^s(s) = i^l \int \frac{dk\,k^2}{2\pi^2} \, P_l^s(k) j_l(ks) , \tag{3.66}$$

$$P_l^s(k) = 4\pi(-i)^l \int dx \; x^2 \xi_l^s(x) j_l(kx) , \tag{3.67}$$

where $j_l(x)$ denotes the spherical Bessel functions and the cosine angle between the line-of-sight direction $\hat{\mathbf{n}}$ and the pair separation vector $\mathbf{s}$ is $\mu = \hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$. With the distant-observer approximation, there are no ambiguities associated with how to define the line-of-sight direction of the galaxy pair, as all angular directions are identical.

3.5 Galaxy Clusters

So far, we discussed the two-point statistics of some cosmological probes. One-point statistics such as the number density has also important cosmological information.

3.5.1 Spherical Collapse Model

A simple spherical collapse model was developed long time ago to serve as a toy model for dark matter halo formation. The idea is that a slightly overdense region in a flat universe evolves as if the region were a closed universe, such that it expands almost together with the background universe but eventually turns around and collapses. The overdense region described by the closed universe would collapse to a singularity, but in reality it virializes and stops contracting. By using the analytical solutions for the two universes, we can readily derive many useful relations about the evolution of such overdense regions.

Einstein-de Sitter Universe

A flat homogeneous universe dominated by pressureless matter is called the Einstein-de Sitter Universe:

$$H^2 = \frac{8\pi G}{3}\rho_m , \qquad \rho_m := \frac{\rho_0}{a^3} , \tag{3.68}$$

where the reference point t_0 satisfies $a(t_0) = 1$. This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations and the solution are

$$a := \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{\eta_0}\right)^2 , \qquad \frac{t}{t_0} = \left(\frac{\eta}{\eta_0}\right)^3 , \qquad \eta_0 = 3t_0 , \tag{3.69}$$

$$H = \frac{2}{3t} , \qquad \mathcal{H} = \frac{2}{\eta} , \qquad \rho_m = \frac{1}{6\pi G t^2} , \qquad r = \eta_0 - \eta = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) , \tag{3.70}$$

where the reference time can be any time $t_0 \in (0, \infty)$. At a given epoch t_0, one can define a mass scale

$$M := \frac{4\pi}{3}\rho_0 = \frac{H_0^2}{2G} = \frac{2}{9Gt_0^2} , \qquad \rho_0 = \frac{1}{6\pi G t_0^2} , \qquad H_0 = \frac{2}{3t_0} . \tag{3.71}$$

Closed Homogeneous Universe

An analytic solution can be derived for a closed universe with again pressureless matter. The evolution equations for a closed universe and their solution are parametrized in terms of θ as

$$\tilde a := \tilde a_t \frac{1-\cos\theta}{2} , \qquad t := t_t \frac{\theta - \sin\theta}{\pi} , \qquad d\eta = \frac{dt}{\tilde a} = \frac{2t_t}{\pi \tilde a_t} d\theta , \qquad \theta \in [0, 2\pi] , \tag{3.72}$$

$$\tilde H = \frac{1}{\tilde a}\frac{d\tilde a}{d\theta}\left(\frac{dt}{d\theta}\right)^{-1} = \frac{\pi}{t_t}\frac{\sin\theta}{(1-\cos\theta)^2} , \qquad \tilde H^2 = \frac{8\pi G}{3}\tilde\rho_m - \frac{K}{\tilde a^2} = \frac{K}{\tilde a^2}\left(\frac{\tilde a_t}{\tilde a} - 1\right) , \tag{3.73}$$

$$\therefore \quad K = \frac{\pi^2 \tilde a_t^2}{4t_t^2} , \qquad \tilde\rho_m = \frac{3K\tilde a_t}{8\pi G}\frac{1}{\tilde a^3} =: \frac{\rho_0}{\tilde a^3} , \qquad t = \frac{\tilde a_t(\theta - \sin\theta)}{2\sqrt{K}} , \qquad d\eta = \frac{\tilde a_t}{\sqrt{K}} d\theta , \tag{3.74}$$

where we used tilde to distinguish quantities in the closed universe from the flat universe and the maximum expansion (or turn-around $\tilde a_t$) is reached at $\theta = \pi$ ($\tilde H_t = 0$). The density parameters are related to the curvature K of the universe as

$$K = \frac{8\pi G}{3}\frac{\rho_0}{\tilde a_t} = \frac{H_0^2}{\tilde a_t} = \frac{4}{9t_0^2 \tilde a_t} , \qquad \therefore \quad \frac{16}{9}\left(\frac{t_t}{\pi t_0}\right)^2 = \tilde a_t^3 . \tag{3.75}$$

Note that we used the same ρ_0 for $\tilde\rho_m$ to simplify the calculation, but it retains the full generality (mind that $\tilde a \neq 1$ at t_0).

Spherical Collapse Model

The time evolution of the overdense region can be derived in a non-perturbative way as

$$1 + \delta = \frac{\tilde\rho_m}{\rho_m} = \left(\frac{a}{\tilde a}\right)^3 = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} \quad \to \quad 1 \ \text{ at } \theta = 0 , \tag{3.76}$$

where we used

$$a^3 = \left(\frac{t}{t_0}\right)^2 = \left(\frac{t_t}{\pi t_0}\right)^2 (\theta - \sin\theta)^2 , \qquad \tilde a^3 = \left(\frac{\tilde a_t}{2}\right)^3 (1-\cos\theta)^3 = \frac{2}{9}\left(\frac{t_t}{\pi t_0}\right)^2 (1-\cos\theta)^3 . \tag{3.77}$$

The density contrast vanishes at $\theta \to 0$, which implies that $\rho_m = \tilde\rho_m$ only at $\theta \to 0$ (or $t \to 0$). The density contrast δ_t at its maximum expansion

$$1 + \delta_t = \frac{9\pi^2}{16} \simeq 5.6 , \tag{3.78}$$

is about a few, while the density contrast δ_v at its virialization

$$1 + \delta_v = 18\pi^2 \simeq 177.7 , \tag{3.79}$$

is a few hundreds, under the assumption that the overdensity region virialized at the half of its maximum expansion. Note that the universe further expands and the background density is reduced by factor 4, until it collapses at $t_v = 2t_t$ (or $\theta = 2\pi$).

Finally, expanding the expressions to the linear order

$$a = \frac{1}{36^{1/3}}\left(\frac{t_t}{\pi t_0}\right)^{2/3}\theta^2 + \cdots , \qquad \delta = \frac{3}{20}\theta^2 + \cdots , \tag{3.80}$$

and evaluating the linear order expressions at θ_i for a_i and δ_i, we first compute

$$\frac{\delta_i}{a_i} = \frac{3}{20}36^{1/3}\left(\frac{t_t}{\pi t_0}\right)^{-2/3} + \cdots , \tag{3.81}$$

and the density contrast linearly extrapolated to late time and its value at virialization are then derived as

$$\delta_L = \frac{D}{D_i}\delta_i = \frac{a}{a_i}\delta_i = \frac{3}{10}\left(\frac{9}{2}\right)^{1/3}(\theta - \sin\theta)^{2/3} , \qquad D \propto a . \tag{3.82}$$

This equation implies that at the time of collapse the density contrast δ_L is

$$\delta_v \simeq 1.686 . \tag{3.83}$$

For $|\delta| \ll 1$, we derive the relation

$$\delta = \delta_L + \frac{17}{21}\delta_L^2 + \frac{341}{567}\delta_L^3 + \frac{55805}{130977}\delta_L^4 + \cdots , \qquad \delta_L = \delta - \frac{17}{21}\delta^2 + \frac{2815}{3969}\delta^3 - \frac{590725}{916839}\delta^4 + \cdots . \tag{3.84}$$

Biased Tracer

For any biased tracer δ_X, the Eulerian and the Lagrangian bias parameters can be written in a series

$$\delta_X = \sum_{n=1}^{\infty}\frac{b_n}{n!}\delta^n , \qquad \delta_X^L = \sum_{n=1}^{\infty}\frac{b_n^L}{n!}\delta_L^n , \tag{3.85}$$

where the superscript L represents quantities in the Lagrangian space. If the number density of the objects X is conserved

$$\rho\, d^3x = \bar\rho\, d^3q , \qquad \rho_X\, d^3x = \rho_X^L\, d^3q , \qquad \therefore\ 1 + \delta_X = (1+\delta)(1+\delta_X^L) , \tag{3.86}$$

the bias parameters are related as

$$b_1 = b_1^{\rm L} + 1 , \qquad b_2 = b_2^{\rm L} + \frac{8}{21}b_1^{\rm L} , \qquad b_3 = b_3^{\rm L} - \frac{13}{7}b_2^{\rm L} - \frac{796}{1323}b_1^{\rm L} , \qquad b_4 = b_4^{\rm L} - \frac{40}{7}b_3^{\rm L} + \frac{7220}{1323}b_2^{\rm L} + \frac{476320}{305613}b_1^{\rm L} . \tag{3.87}$$

This simple relation owes to the fact that the spherical collapse model is local in both Eulerian and Lagrangian spaces.

3.5.2 Dark Matter Halo Mass Function

Given the simple spherical collapse model, we would like to associate the collapsed region with some virialized objects like massive galaxy clusters or dark matter halos. Of our main interest is then the number density of such objects in a mass range $M \sim M + dM$, and this is called the mass function.

A simple model called, the excursion set approach, was developed: One starts with a smoothing scale R and its associated mass M. The density fluctuation δ_R after smoothing with R is very small ($\delta_R = 0$, if $R = \infty$), and this region has never reached the critical density threshold δ_c in its entire history. This implies that there is no virialized object associated with such mass. One then decreases the smoothing scale (or mass), and looks for the collapsed probability: Some overdense regions have at some point in the past reached the critical density, while some underdense regions have not. Therefore, the total fraction F_c of collapse can be obtained by using the survival probability P_s of a given scale, and it is related to the mass function as

$$F_c = 1 - \int_{-\infty}^{\delta_c} d\delta\ P_s = \int_M^{\infty} dM\ \frac{dn}{dM}\ \frac{M}{\bar\rho_m} , \qquad \therefore\ \frac{dn}{dM} = \frac{\bar\rho_m}{M}\left(-\frac{\partial F_c}{\partial M}\right) \equiv \frac{\bar\rho_m}{M}\ f(\nu)\ \frac{d\ln\nu}{dM} , \tag{3.88}$$

where it is assumed that the mass function only depends on mass and we defined the multiplicity function f through the relation

$$\nu \equiv \frac{\delta_c(z)}{\sigma(M)} , \qquad \int_0^\infty \frac{d\nu}{\nu} \; f = 1 . \tag{3.89}$$

The task of obtaining the mass function boils down to computing the survival probability and expressing it in terms of the multiplicity function. The way to find the survival probability at a given mass scale M is to derive the evolution of the density fluctuation as we decrease the smoothing scale R. The reason is that the region may have already collapsed at a larger mass scale or smoothing scale, and this contribution should be removed in computing the survival probability at a lower mass scale. The survival probability at n-th step depends on the entire history of the trajectory (non-Markovian process) as

$$P_s(\delta_n, \sigma_n) d\delta_n = d\delta_n \int_{-\infty}^{\delta_c} d\delta_{n-1} \cdots \int_{-\infty}^{\delta_c} d\delta_1 \; P_s(\delta_1, \cdots \delta_n, \sigma_1, \cdots , \sigma_n) , \tag{3.90}$$

it is notoriously difficult to solve, even numerically. However, once we assume that the fluctuations are independent at each smoothing and are Gaussian distributed (true only in Fourier space at linear order), the trajectory only depends on the previous step (Markovian process) and the survival probability becomes

$$P_s(\delta_n, \sigma_n) = \int_{-\infty}^{\delta_c} d\delta_{n-1} P_t(\delta_n, \sigma_n | \delta_{n-1}, \sigma_{n-1}) \; P_s(\delta_{n-1}, \sigma_{n-1}) , \tag{3.91}$$

where the transition probability P_t is nothing but a conditional probability. With the boundary condition $P_s = 0$ at $\delta = \delta_c$, the solution is (derived by Chandrasekhar for other purposes)

$$P_s = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2} \right) - \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(2\delta_c - \delta)^2}{2\sigma^2} \right] . \tag{3.92}$$

The survival probability for its simplest case is described by a Gaussian distribution, but the second term reflects that there exist equally likely trajectories around the threshold that have reached the threshold in the past. The collapsed fraction is

$$F_c = 1 - \frac{1}{2} \text{erf} \left(\frac{\nu_c}{\sqrt{2}} \right) - \frac{1}{2} \text{erf} \left(\frac{\nu_c}{\sqrt{2}} \right) = \text{erfc} \left(\frac{\nu_c}{\sqrt{2}} \right) , \tag{3.93}$$

and the multiplicity function is

$$f(\nu) = \sqrt{\frac{2}{\pi}} \; \nu \; e^{-\nu^2/2} . \tag{3.94}$$

Of course, this model relies on many approximations, and it is not accurate. However, it provides physical intuitions, connecting the complicated formation of galaxy clusters and the dynamical evolution of the matter density fluctuations. In general, numerical N-body simulations are run, and dark matter halos are identified by using some algorithm such as the friends-of-friends method or its variants to derive the mass function from the simulations.

4 Relativistic Perturbation Theory

4.1 Metric Decomposition and Gauge Transformation

4.1.1 FRW Metric and its Perturbations

We describe the background for a spatially homogeneous and isotropic universe with the FRW metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -a^2(\eta)\, d\eta^2 + a^2(\eta)\, \bar{g}_{\alpha\beta}\, dx^\alpha dx^\beta\ , \tag{4.1}$$

where $a(\eta)$ is the scale factor and $\bar{g}_{\alpha\beta}$ is the metric tensor for a three-space with a constant spatial curvature $K = -H_0^2\,(1-\Omega_{\rm tot})$. We use the Greek indices $\alpha, \beta, \cdots$ for 3D spatial components and $\mu, \nu, \cdots$ for 4D spacetime components, respectively. To describe the real (inhomogeneous) universe, we parametrize the perturbations to the homogeneous background metric as

$$g_{00} := -a^2\,(1+2A)\,, \qquad g_{0\alpha} := -a^2 B_\alpha, \qquad g_{\alpha\beta} := a^2\,(\bar{g}_{\alpha\beta}+2C_{\alpha\beta})\,, \tag{4.2}$$

where 3-tensor A, B_α and $C_{\alpha\beta}$ are perturbation variables and they are based on the 3-metric $\bar{g}_{\alpha\beta}$. Due to the symmetry of the metric tensor, we have ten components, capturing the deviation from the background. The inverse metric tensor can be obtained by using $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ and expanding to the linear order as

$$g^{00} = \frac{1}{a^2}\,(-1+2A)\,, \qquad g^{0\alpha} = -\frac{1}{a^2}B^\alpha\ , \qquad g^{\alpha\beta} = \frac{1}{a^2}\left(\bar{g}^{\alpha\beta}-2C^{\alpha\beta}\right). \tag{4.3}$$

For later convenience, we also introduce a time-like four-vector, describing the motion of an observer ($-1 = u_\mu u^\mu$):

$$u^0 = \frac{1}{a}\,(1-A)\,, \qquad u^\alpha := \frac{1}{a}U^\alpha\ , \qquad u_0 = -a\,(1+A)\,, \tag{4.4}$$

$$u_\alpha = a\,(U_\alpha - B_\alpha) := a\, v_\alpha := a(-v_{,\alpha}+v^{(v)}_\alpha)\ , \qquad v := U+\beta\ , \qquad v^{(v)}_\alpha = U^{(v)}_\alpha - B^{(v)}_\alpha\ , \tag{4.5}$$

where U^α is again based on $\bar{g}_{\alpha\beta}$.

4.1.2 Scalar-Vector-Tensor Decomposition

Given the splitting of the spatial hypersurface and the symmetry associated with it, we decompose the perturbation variables (*to all orders*) as

$$A := \alpha\ , \qquad B_\alpha := \beta_{,\alpha} + B^{(v)}_\alpha\ , \qquad C_{\alpha\beta} := \varphi\bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}\ , \qquad U^\alpha := -U^{,\alpha} + U^{(v)\alpha}\ , \tag{4.6}$$

subject to the transverse and traceless conditions:

$$B^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \quad C^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \quad v^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \quad C^{(t)\alpha}{}_{\alpha} \equiv 0\,, \quad C^{(t)\beta}{}_{\alpha|\beta} \equiv 0\,, \quad U^{(v)\alpha}{}_{|\alpha} = 0\,, \tag{4.7}$$

where the vertical bar represents the covariant derivative with respect to the 3-metric $\bar{g}_{\alpha\beta}$:

$$X^\alpha{}_{|\beta} = X^\alpha{}_{,\beta} + \bar{\Gamma}^\alpha_{\beta\gamma}X^\gamma\ , \qquad X_{\alpha|\beta} = X_{\alpha,\beta} - \bar{\Gamma}^\gamma_{\alpha\beta}X_\gamma\ . \tag{4.8}$$

This simply implies that the scalar perturbations describe the longitudinal modes and the vector (v) and the tensor (t) perturbations describe the transverse modes. Furthermore, the tensor perturbation is traceless. The decomposed scalar perturbations can be obtained as

$$\beta = \Delta^{-1}\nabla^\alpha B_\alpha\ , \qquad \gamma = \frac{1}{2}\left(\Delta+\frac{1}{2}\bar{R}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta C_{\alpha\beta} - C^\alpha_\alpha\right)\ , \tag{4.9}$$

$$\varphi = \frac{1}{3}C^\alpha_\alpha - \frac{1}{6}\Delta\left(\Delta+\frac{1}{2}\bar{R}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta C_{\alpha\beta} - C^\alpha_\alpha\right)\ ,$$

where ∇_α is the covariant derivative based on $\bar{g}_{\alpha\beta}$ (i.e., vertical bar) and $\Delta = \nabla^\alpha\nabla_\alpha$ is the Laplacian operator. The presence of the Ricci scalar ($\bar{R} = 6K$) for the three-space indicates that covariant derivatives are non-commutative.

$$\bar{R}_{\alpha\beta\gamma\delta} = 2K\ \bar{g}_{\alpha[\gamma}\bar{g}_{\delta]\beta}\ . \tag{4.10}$$

The decomposed vector and tensor components are computed in a similar manner as

$$B^{(v)}_\alpha = B_\alpha - \nabla_\alpha\Delta^{-1}\nabla^\beta B_\beta\ , \qquad C^{(v)}_\alpha = 2\left(\Delta + \frac{1}{3}\bar{R}\right)^{-1}\left[\nabla^\beta C_{\alpha\beta} - \nabla_\alpha\Delta^{-1}\nabla^\beta\nabla^\gamma C_{\beta\gamma}\right]\ , \tag{4.11}$$

$$\begin{aligned} C^{(t)}_{\alpha\beta} &= C_{\alpha\beta} - \frac{1}{3}C^\gamma_\gamma\bar{g}_{\alpha\beta} - \frac{1}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}\bar{g}_{\alpha\beta}\Delta\right)\left(\Delta + \frac{1}{2}\bar{R}\right)^{-1}\left[3\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\gamma\delta} - C^\gamma_\gamma\right] \\ &\quad -2\nabla_{(\alpha}\left(\Delta + \frac{1}{3}\bar{R}\right)^{-1}\left[\nabla^\gamma C_{\beta)\gamma} - \nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\gamma\delta}\right]\ , \end{aligned}$$

and they satisfy the transverse condition $B^{(v)\alpha}_{\ \ |\alpha} = C^{(v)\alpha}_{\ \ |\alpha} = C^{(t)\beta}_{\alpha|\beta} = 0$ and the traceless condition $C^{(t)\alpha}_\alpha = 0$.

4.1.3 Comparison in Notation Convention

Bardeen convention:

$$A \to \alpha\ , \qquad B^{(0)} \to -k\beta\ , \qquad H_L \to \varphi + \frac{1}{3}\Delta\gamma\ , \qquad H_T \to -\Delta\gamma\ , \tag{4.12}$$

$$B^{(1)}Q^{(1)}_\alpha \to B_\alpha\ , \qquad H^{(1)}_T Q_\alpha \to -kC_\alpha\ , \qquad H^{(2)}_T Q_{\alpha\beta} \to C_{\alpha\beta}\ . \tag{4.13}$$

Weinberg convention:

$$\Phi \to \alpha_\chi\ , \qquad \Psi \to -\varphi_\chi\ , \qquad \delta u \to -av_\chi \qquad \mathcal{R} \to \varphi_v\ , \qquad \zeta \to \varphi_\delta\ , \tag{4.14}$$

$$\delta p \to \delta p - \frac{1}{3a^2}\Delta\Pi\ , \qquad \pi^S := \delta\sigma \to \frac{\Pi}{a^2}\ , \qquad \pi^V_i \to \frac{1}{2a}\Pi_\alpha\ , \qquad \pi^T_{ij} \to \Pi^{(t)}_{\alpha\beta}\ . \tag{4.15}$$

Dodelson convention:

$$\psi \to \alpha_\chi\ , \qquad \phi \to \varphi_\chi\ , \qquad ikv \to k^2 v_\chi\ , \qquad v \to -ikv_\chi\ . \tag{4.16}$$

Ma & Bertschinger:

$$\psi \to \alpha_\chi\ , \qquad \phi \to -\varphi_\chi\ , \qquad h \to 6\varphi_v + 2\Delta\gamma\ , \qquad \eta \to -\varphi_v\ , \qquad \theta \to k^2 v_\chi\ . \tag{4.17}$$

CLASS Boltzmann code:

$$\psi \to \alpha_\chi\ , \qquad \phi \to -\varphi_\chi\ , \qquad \theta \to k^2 v\ , \tag{4.18}$$

where θ_i and δ_i depend on the choice of gauge condition.

4.1.4 Gauge Transformation

The general covariance of general relativity guarantees that any coordinate system can be used to describe the physics and it has to be independent of coordinate systems. This is known as the diffeomorphism symmetry in general relativity. However, when we split the metric into the background and the perturbations around it by choosing a coordinate system, we explicitly change the correspondence of the physical Universe to the background homogeneous and isotropic Universe. Hence, the metric perturbations transform non-trivially (or gauge transform), and the diffeomorphism invariance implies that the physics should be gauge-invariant.

The gauge group of general relativity is the group of diffeomorphisms. A diffeomorphism corresponds to a differentiable coordinate transformation. The coordinate transformation on the manifold $\mathcal{M}$ can be considered as one generated by a smooth vector field ζ^μ. Given the vector field ζ^μ, consider the solution of the differential equation

$$\left.\frac{d\chi^\mu(\lambda)}{d\lambda}\right|_P = \zeta^\mu\left[\chi^\nu_P(\lambda)\right]\ , \qquad \chi^\mu_P(\lambda = 0) = x^\mu_P\ , \qquad \frac{d}{d\lambda} = \zeta^\mu\partial_\mu\ , \tag{4.19}$$

defines the parametrized integral curve $x^\mu(\lambda) = \chi^\mu_P(\lambda)$ with the tangent vector $\zeta^\mu(x_P)$ at P. Therefore, given the vector field ζ^μ on $\mathcal{M}$ we can define an associated coordinate transformation on $\mathcal{M}$ as $x^\mu_P \to \tilde{x}^\mu_P = \chi^\mu_P(\lambda = 1)$ for any given P. Assuming that ζ^μ is small one can use the perturbative expansion for the solution of equation to obtain

$$\tilde{x}^\mu_P = \chi^\mu_P(\lambda=1) = \chi^\mu_P(\lambda=0) + \frac{d}{d\lambda}\chi^\mu_P\bigg|_{\lambda=0} + \frac{1}{2}\frac{d^2}{d\lambda^2}\chi^\mu_P\bigg|_{\lambda=0} + \cdots = x^\mu_P + \zeta^\mu(x_P) + \frac{1}{2}\zeta^\mu{}_{,\nu}\zeta^\nu + \mathcal{O}(\zeta^3) = e^{\zeta^\nu\partial_\nu}x^\mu \,. \tag{4.20}$$

This parametrization corresponds to the gauge-transformation with ζ^μ.

In general, any gauge-transformation of tensor $\mathbf{T}$ for an infinitesimal change ζ can be expressed in terms of the Lie derivative (valid to all orders of $\mathbf{T}$)

$$\delta_\zeta\mathbf{T} := \tilde{\mathbf{T}} - \mathbf{T} = -\pounds_\zeta\mathbf{T} + \mathcal{O}(\zeta^2)\,, \qquad \pounds_\zeta A^\mu = A^\mu{}_{,\nu}\zeta^\nu - \zeta^\mu{}_{,\nu}A^\nu\,, \qquad \pounds_\zeta T_{\mu\nu} = T_{\mu\nu,\rho}\zeta^\rho + T_{\rho\nu}\zeta^\rho{}_{,\mu} + T_{\mu\rho}\zeta^\rho{}_{,\nu}\,, \tag{4.21}$$

where they are all evaluated at the same coordinate and the derivatives in the Lie derivatives can be replaced with covariant derivatives (Lie derivatives are tensorial). To all orders in ζ, we have

$$\tilde{\mathbf{T}}(x) = \mathbf{T}(x) - \pounds_\zeta\mathbf{T} + \frac{1}{2}\pounds^2_\zeta\mathbf{T} + \cdots = \exp\left[-\pounds_\zeta\right]\mathbf{T}\,. \tag{4.22}$$

Therefore, the gauge-transformation in perturbation theory is simply

$$\delta_\zeta\bar{\mathbf{T}} = 0\,, \qquad \delta_\zeta\mathbf{T}^{(1)} = -\pounds_\zeta\bar{\mathbf{T}}\,, \qquad \delta_\zeta\mathbf{T}^{(n)} = -\pounds_\zeta\mathbf{T}^{(n-1)}\,, \tag{4.23}$$

where we used that ζ is also a perturbation.

In fact, there are two ways of looking at the transformation in perturbation theory. For example, the metric tensor has to transform as a tensor. But once we split it into the background and the perturbations, there exist two ways

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad \delta_\zeta g = -\pounds_\zeta g \;:\; (1)\;\delta_\zeta\bar{g} = -\pounds_\zeta\bar{g}\,, \quad \delta_\zeta h = -\pounds_\zeta h\,, \qquad (2)\;\delta_\zeta\bar{g} = 0\,, \quad \delta_\zeta h = -\pounds_\zeta h - \pounds_\zeta\bar{g}\,, \tag{4.24}$$

where we suppressed the tensor indices. In (1), the background and the perturbation transform altogether like tensors (at the same coordinates), such that the sum transforms like a tensor. In perturbation theory, we do not use this, because the infinitesimal transformation ζ is always considered as a perturbation. However, for example we can consider some general spatial rotation ζ, such that the background metric also changes.[1]

4.1.5 Linear-Order Gauge Transformation

At the linear order, the Lie derivative is trivial, and the the most general coordinate transformation in Eq. (4.20) becomes

$$\tilde{x}^\mu = x^\mu + \xi^\mu\,, \qquad \xi^\mu := (T, \mathcal{L}^\alpha)\,, \qquad \mathcal{L}^\alpha := L^{,\alpha} + L^{(v)\alpha}\,, \tag{4.25}$$

where we now use $\xi^\mu = \zeta^\mu$. The transformation of the metric tensor at the leading order in ξ is then

$$\delta_\xi g_{\mu\nu}(x) := \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\pounds_\xi g_{\mu\nu} = -\left(\xi_{\mu;\nu} + \xi_{\nu;\mu}\right)\,, \qquad \xi_\mu = g_{\mu\nu}\xi^\nu = a^2(-T,\ \mathcal{L}_\alpha)\,, \tag{4.26}$$

where the semi-colon represents the covariant derivative with respect to the full metric $g_{\mu\nu}$. The transformation equations are explicitly

$$\delta_\xi g_{00} = -2a^2\,\delta_\xi A = 2a^2\left[\frac{1}{a}\left(aT\right)'\right]\,, \qquad \delta_\xi g_{0\alpha} = -a^2\,\delta_\xi B_\alpha = a^2\left(T_{,\alpha} - \mathcal{L}'_\alpha\right)\,, \tag{4.27}$$

$$\delta_\xi g_{\alpha\beta} = 2a^2\,\delta_\xi C_{\alpha\beta} = -2a^2\left[\mathcal{H}T\bar{g}_{\alpha\beta} + \mathcal{L}_{(\alpha|\beta)}\right]\,. \tag{4.28}$$

Using the scalar-vector-tensor decomposition, we derive that the scalar quantities gauge-transform as

$$\tilde{\alpha} = \alpha - \frac{1}{a}(aT)'\,, \qquad \tilde{\beta} = \beta - T + L'\,, \qquad \tilde{\varphi} = \varphi - \mathcal{H}T\,, \qquad \tilde{\gamma} = \gamma - L\,, \tag{4.29}$$

$$\tilde{U} = U - L'\,, \qquad \tilde{v} = v - T\,, \qquad \tilde{\chi} = \chi - aT\,, \qquad \tilde{\kappa} = \kappa + 3\dot{H}aT + \frac{\Delta}{a}T\,, \tag{4.30}$$

[1] In FRW, we use the spatial metric $\bar{g}_{\alpha\beta}$ unspecified, implying we can do a further spatial transformation to e.g., spherical coordinate and so on and change the background metric (while it remains covariant). The time component is fixed, otherwise it ruins the FRW symmetry ($g_{0\alpha}$ component in the background or different coefficient in time component for example.

the vector metric perturbations gauge-transform as

$$\tilde{B}^{(v)}_\alpha = B^{(v)}_\alpha + L'_\alpha \,, \qquad \tilde{C}^{(v)}_\alpha = C^{(v)}_\alpha - L_\alpha \,, \qquad \tilde{U}^{(v)}_\alpha = U^{(v)}_\alpha + L^{(v)\prime}_\alpha \,, \tag{4.31}$$

and the tensor perturbations are gauge-invariant at the linear order, where we defined the scalar shear $\chi := a\,(\beta + \gamma')$ of the normal observer ($n_\alpha = 0$) and the extrinsic 3-curvature $K := -3H + \kappa$ and its perturbation $\kappa := 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi$.

Based on the above gauge transformation properties, we can construct linear-order gauge-invariant quantities. The gauge-invariant variables are

$$\varphi_v := \varphi - aHv \,, \qquad \varphi_\chi := \varphi - H\chi \,, \qquad v_\chi := v - \frac{1}{a}\chi \,, \qquad \delta_v := \delta - a\frac{\dot{\rho}}{\rho}v \,, \tag{4.32}$$
$$\alpha_\chi := \alpha - \frac{1}{a}\chi' \,, \qquad \varphi_\delta := \varphi + \frac{\delta\rho}{3(\rho+p)}, \qquad \Psi^{(v)}_\alpha := B^{(v)}_\alpha + C^{(v)\prime}_\alpha \,, \qquad v^{(v)}_\alpha := U^{(v)}_\alpha - B^{(v)}_\alpha \,.$$

These gauge-invariant variables ($\alpha_\chi, \varphi_\chi, v_\chi, \Psi_\alpha, v^{(v)}_\alpha$) correspond to Φ_A, Φ_H, $v^{(0)}_s$, Ψ, and v_c in Bardeen (1980).

4.1.6 Popular Choices of Gauge Condition

By a suitable choice of coordinates, we can set $T = L = 0$, simplifying the metric. For simplicity, we only consider the scalar perturbations in the following two cases.

• ***The conformal Newtonian Gauge***.— in which we choose the spatial and the temporal gauge conditions:

$$\tilde{\gamma} = \gamma = 0 \;\rightarrow\; L = 0 \,, \qquad \tilde{\beta} = \beta = 0 \;\rightarrow\; T = 0 \,, \qquad \chi = 0 \,. \tag{4.33}$$

All the gauge modes are fixed, and the metric in this gauge condition is

$$ds^2 = -a^2\,(1+2\psi)\,d\eta^2 + a^2\,(1+2\phi)\,\bar{g}_{\alpha\beta}dx^\alpha dx^\beta \,, \qquad \psi := \alpha = \alpha_\chi \,, \qquad \phi := \varphi = \varphi_\chi \,, \tag{4.34}$$

and the velocity vector is then

$$U = v_\chi \,, \qquad \boldsymbol{v} = -\nabla U \,. \tag{4.35}$$

The metric and its equations appear more like the Newtonian equations, and hence the name. We will use this gauge condition to illustrate and simplify the problems.

• ***Synchronous-Comoving Gauge***.— in which we choose the spatial and the temporal gauge conditions:

$$\tilde{\alpha} = \alpha = 0 \;\rightarrow\; (aT)' = 0 \,, \qquad \tilde{\beta} = \beta = 0 \;\rightarrow\; T = L' \,, \tag{4.36}$$

such that the metric becomes

$$ds^2 = -a^2 d\eta^2 + a^2\,(\bar{g}_{\alpha\beta} + 2C_{\alpha\beta})\,dx^\alpha dx^\beta \,. \tag{4.37}$$

All the metric perturbations in this gauge condition are included in the spatial metric tensor. However, as apparent from the above gauge condition, the gauge freedoms are not completely fixed:

$$T = L' = \frac{1}{a}F(\mathbf{x}) \,, \qquad L = \int d\eta\,\frac{F(\mathbf{x})}{a} + G(\mathbf{x}) \,, \tag{4.38}$$

where F and G are two arbitrary functions of spatial coordinates. Typically, this issue is resolved by assuming additional condition at the initial epoch

$$v = 0 \;\rightarrow\; F(\mathbf{x}) = 0 \,, \quad T = 0 \,. \tag{4.39}$$

This condition is indeed the temporal comoving gauge condition, and hence the whole choice is often referred to as the comoving-synchronous gauge (or synchronous-comoving). The comoving gauge is often chosen with a different spatial gauge condition ($\gamma = 0$). Note, however, that the spatial function $G(\mathbf{x})$ is still left unspecified, and hence γ is a gauge mode, whereas U for example is physical. Due to this deficiency, we will not use this gauge condition in the following.

The notation convention in Ma and Bertschinger (1995):

$$h_{ij} := \hat{k}_i\hat{k}_j h + \left(\hat{k}_i\hat{k}_j - \frac{1}{2}\delta_{ij}\right)6\eta \;\rightarrow\; 2C_{ij} \,, \qquad h \rightarrow 6\varphi + 2\Delta\gamma \,, \qquad \eta \rightarrow -\varphi_v \,. \tag{4.40}$$

4.2 Energy-Momentum Tensor

4.2.1 Formal Definition

We will consider a simple action of the matter sector, in addition to the gravity described by the Einstein-Hilbert action:

$$S =: S_g + S_m =: \int \sqrt{-g}\, d^4x \left[\frac{R}{16\pi G} - \frac{\Lambda}{8\pi G} + \mathcal{L}_m\right] , \tag{4.41}$$

where the matter Lagrangian includes the cosmological fluids and other matter fields such as scalars and so on. The variation with respect to the metric,

$$0 = \frac{\delta S}{\delta g^{\mu\nu}} = \frac{M_{\rm pl}^2}{2}\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right] + \frac{\delta S_m}{\delta g^{\mu\nu}} , \tag{4.42}$$

yields the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{1}{M_{\rm pl}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = 8\pi G\, T_{\mu\nu} , \tag{4.43}$$

where the energy-momentum tensor defined by the action

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_m - 2\,\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\, \mathcal{L}_m , \tag{4.44}$$

and it is indeed the conserved current (tensor) of the action under the space-time translation invariance. The Noether theorem says that when there exists a (global) symmetry, there exists a conserved current. The space-time translation invariance is the symmetry of general relativity, and the Noether current associated with this symmetry is the energy-momentum tensor:

$$T_{\mu\nu;\nu} = 0 . \tag{4.45}$$

Note that we can repeat the calculations with upper indicies and obtain

$$T^{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} = g^{\mu\nu}\mathcal{L}_m + 2\,\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\, \mathcal{L}_m , \tag{4.46}$$

but mind the subtlety, for example, for the scalar field action,

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi , \qquad \delta\mathcal{L}_\phi = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi . \tag{4.47}$$

4.2.2 General Decomposition for Cosmological Fluids

For our purposes, we are not interested in the microscopic states of the systems, but interested in their macroscopic states, often described by the density, the pressure, the temperature, and so on. The energy-momentum tensor for a fluid can be expressed in terms of the fluid quantities measured by an observer with four velocity u^μ as (*the most general decomposition*)

$$T_{\mu\nu} := \rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu} , \qquad 0 = \mathcal{H}_{\mu\nu}u^\nu , \tag{4.48}$$

where $\mathcal{H}_{\mu\nu}$ is the projection tensor and

$$\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu , \qquad \mathcal{H}^\mu_\mu = 3 , \qquad u^\mu q_\mu = 0 = u^\mu\pi_{\mu\nu} , \qquad \pi_{\mu\nu} = \pi_{\nu\mu} , \qquad \pi^\mu_\mu = 0 . \tag{4.49}$$

In general, we decompose the fluid quantities into the background and the perturbations:

$$\rho := \bar\rho + \delta\rho , \qquad p := \bar p + \delta p , \qquad \delta p := c_s^2\delta\rho + e , \qquad c_s^2 := \frac{\dot{\bar p}}{\dot{\bar\rho}} , \tag{4.50}$$

$$q_\alpha := aQ_\alpha , \qquad \pi_{\alpha\beta} := a^2\Pi_{\alpha\beta} , \qquad e := \dot p\Gamma , \qquad \Gamma = \frac{\delta p}{\dot p} - \frac{\delta\rho}{\dot\rho} , \tag{4.51}$$

where Q_α and $\Pi_{\alpha\beta}$ are based on $\bar{g}_{\alpha\beta}$. At the background level, all the above fluid quantities vanish, except $\rho = \bar{\rho}$ and $p = \bar{p}$. The sound speed c_s^2 is the response of the pressure perturbation in terms of the density perturbation, and the remaining response is captured in terms of the non-adiabatic perturbation e. For multiple fluids, we can add up the individual energy-momentum tensor to derive the total energy-momentum tensor. In the case of multiple fluids, their fluid velocities are not necessarily identical, and there exist non-vanishing energy flux. Non-vanishing e and Γ parametrize the entropic perturbations of the fluids.

Though these relations are exact, we will be concerned with linear-order perturbations. Raising the index of the energy momentum tensor, we derive

$$T^0_0 = -\rho + \mathcal{O}(2) \,, \qquad T^0_\alpha = (\bar{\rho}+\bar{p})\,(U_\alpha - B_\alpha) + Q_\alpha + \mathcal{O}(2) \,, \tag{4.52}$$

$$T^\alpha_\beta = p\,\delta^\alpha_\beta + \Pi^\alpha_\beta + \mathcal{O}(2) \,, \qquad T^\alpha_0 = -(\bar{\rho}+\bar{p})U^\alpha - Q^\alpha + \mathcal{O}(2) \,, \tag{4.53}$$

where all quantities are those appearing in $T_{\mu\nu}$. Given the conditions $0 = u^\mu q_\mu = u^\mu \pi_{\mu\nu}$, the (spatial) energy flux and the anisotropic pressure should satisfy

$$q_0 = 0 + \mathcal{O}(2) \,, \qquad \pi_{0\mu} = 0 + \mathcal{O}(2) \,. \tag{4.54}$$

• ***Gauge transformation properties of the fluid quantities*.—**

$$\tilde{\delta} = \delta - \frac{\bar{\rho}'}{\bar{\rho}}\,T \,, \qquad \widetilde{\delta p} = \delta p - \bar{p}'\,T \,, \qquad \tilde{Q}_\alpha = Q_\alpha \,, \qquad \tilde{\pi}_{\mu\nu} = \pi_{\mu\nu} \,, \qquad \tilde{e} = e \,. \tag{4.55}$$

Note that the spatial energy flux is gauge-invariant, but dependent upon the observer choice.

4.2.3 Tetrad Approach and Observables

The variables ρ, p, q_μ and $\pi_{\mu\nu}$ are the energy density, the isotropic pressure (including the entropic one), the (spatial) energy flux and the anisotropic pressure, measured by u^μ. If we consider an observer with $u^\mu_{\text{obs}} (\neq u^\mu)$ and the energy tensor $T_{\mu\nu}$ with u^μ above, the observer measures

$$\rho_{\text{obs}} = T_{\mu\nu} u^\mu_{\text{obs}} u^\nu_{\text{obs}} \,, \qquad p_{\text{obs}} = \frac{1}{3} T_{\mu\nu} \hat{\mathcal{H}}^{\mu\nu} \,, \qquad q^{\text{obs}}_\mu = -T_{\rho\sigma} u^\rho_{\text{obs}} \hat{\mathcal{H}}^\sigma_\mu \,, \qquad \pi^{\text{obs}}_{\mu\nu} = T_{\rho\sigma} \hat{\mathcal{H}}^\rho_\mu \hat{\mathcal{H}}^\sigma_\nu - p_{\text{obs}} \hat{\mathcal{H}}_{\mu\nu} \,, \tag{4.56}$$

where $\hat{\mathcal{H}}_{\mu\nu}$ is the projection tensor in terms of u^μ_{obs}. Of course, in case $u^\mu_{\text{obs}} = u^\mu$, the relation yields the same quantities in $T_{\mu\nu}$: $\rho_{\text{obs}} = \rho$ and so on. Remember that these fluid quantities are observer-dependent, hence it would be ideal to provide the energy momentum tensor for each fluid in terms of the fluid quantities that would be measured by a fictitious observer moving together with the fluid $u^\mu_f = u^\mu_{\text{obs}}$ (hence without spatial flux $q_\mu = 0$). This brings us to the tetrad description.

Given the observer with u^μ, one can define a local Lorentz frame (where the metric is Minkowski) by constructing three spacelike orthonormal vectors $[e_i]^\mu$. For example, one can construct three rectangular basis vectors $[e_x]^\mu$, $[e_y]^\mu$, $[e_z]^\mu$ and of course $[e_t]^\mu = u^\mu = -[e^t]^\mu$, where the component index a of $[e_a]^\mu$ represents their coordinates. The orthonormality condition and the spacelike normalization is

$$\eta_{ab} = g_{\mu\nu}[e_a]^\mu[e_b]^\nu \;\; \rightarrow \;\; \delta_{ij} = g_{\mu\nu}[e_i]^\mu[e_j]^\nu \,, \qquad 0 = g_{\mu\nu}[e_i]^\mu[e_t]^\nu \,, \qquad -1 = g_{\mu\nu}[e_t]^\mu[e_t]^\nu \,, \tag{4.57}$$

where $a, b, \cdots = t, x, y, z$ represent the local Lorentz indices.

First, assuming that the four velocity of the observer is indeed the fluid velocity, we can go to the rest-frame of the fluid by using the tetrad expressions as

$$\rho = T_{\mu\nu}[e_t]^\mu[e_t]^\nu \,, \qquad p = \frac{1}{3} T_{\mu\nu} \sum_{i=1}^3 [e_i]^\mu[e_i]^\nu \,, \qquad \mathcal{H}_{\mu\nu} = \sum_{i=1}^3 [e_i]_\mu[e_i]_\nu \,, \qquad q_\mu = \sum_{i=1}^3 q_i [e^i]_\mu \,. \tag{4.58}$$

Noting that the metric is Minkowski in the rest frame, the energy-momentum tensor in the rest-frame of the fluid can be written as

$$T_{ab} = \begin{pmatrix} \rho & -q_x & -q_y & -q_z \\ -q_x & p & \pi_{xy} & \pi_{xz} \\ -q_y & \pi_{yx} & p & \pi_{yz} \\ -q_z & \pi_{zx} & \pi_{zy} & p \end{pmatrix} , \qquad u^\mu \equiv u^\mu_f , \tag{4.59}$$

where in fact $q_i = 0$ because it is in the fluid rest frame. The energy density ρ and the pressure p in $T_{\mu\nu}$ are in fact those measured by the observer rest frame, moving together with the fluid. The orthogonality condition for the energy-flux and the anisotropic stress implies

$$0 = u^\mu q_\mu = q_t , \qquad 0 = u^\mu \pi_{\mu\nu} = \pi_{ta} , \qquad 0 = \pi^\mu_\mu = \pi^t_t + \pi^i_i = \pi_{ii} . \tag{4.60}$$

However, we should pay attention to the difference in the quantities q_μ and $\pi_{\mu\nu}$ expressed in the rest-frame and in the FRW coordinate:

$$q_i := q_\mu [e_i]^\mu = Q^{(1)}_\alpha \delta^\alpha_i + \mathcal{O}(2) , \qquad \pi_{ij} := \pi_{\mu\nu}[e_i]^\mu [e_i]^\nu = \Pi^{(1)}_{\alpha\beta} \delta^\alpha_i \delta^\beta_j + \mathcal{O}(2) , \tag{4.61}$$

where they differ with the scale factor a at the linear order. When the fluid velocity is the same as the observer velocity, the spatial flux $q_i \equiv 0$, or $q_\mu \equiv 0$.

As noted, the fluid quantities are observer-dependent. Given the energy momentum tensor in terms of the fluid rest-frame quantities ($q_\mu = 0$), if the observer is moving with e^μ_t relative to the fluid velocity u^μ_f, the observer measures different fluid quantities from those defined in the rest frame:

$$T^f_{\mu\nu} = \rho_f u^f_\mu u^f_\nu + p_f \mathcal{H}^f_{\mu\nu} + \pi^f_{\mu\nu} , \qquad \mathcal{H}_{\mu\nu} = g_{\mu\nu} + u^f_\mu u^f_\nu , \qquad q^f_\mu \equiv 0 . \tag{4.62}$$

The energy-momentum tensor can be projected into the observer rest-frame $[e_t]^\mu$ as

$$\tilde{T}_{ab} := T^f_{\mu\nu}[e_a]^\mu [e_b]^\nu = \rho_f u_a u_b + p_f(\eta_{ab} + u_a u_b) + \pi_{ab} , \qquad u_a := u^f_\mu [e_a]^\mu , \tag{4.63}$$

where we put tilde to emphasize that the components of $\tilde{T}_{ab}$ are the fluid quantities measured by the observer $[e_t]^\mu$ in the rest frame. The fluid velocity and the anisotropic pressure satisfy

$$-1 = u^\mu_f u^f_\mu = u^a u_a = -u^2_t + u^2_i , \qquad 0 = \pi^\mu_\mu = \pi^a_a \ \rightarrow \ \pi_{tt} = \pi_{ii} . \tag{4.64}$$

Furthermore, the anisotropic pressure is perpendicular to the fluid velocity:

$$0 = u^\mu_f \pi^f_{\mu\nu} = u^a \pi_{ab} = u^t \pi_{ta} + u^i \pi_{ia} , \qquad \pi_{ti} = \pi_{ij}\frac{u^j}{u^t} = \mathcal{O}(2) . \tag{4.65}$$

Therefore, the energy density and the pressure measured by the observer are

$$\tilde{\rho} = T^f_{\mu\nu}[e_t]^\mu [e_t]^\nu = \tilde{T}_{tt} = (\rho + p)_f u^2_t - p_f + \pi_{tt} , \tag{4.66}$$

$$\tilde{p} = \frac{1}{3} T^f_{\mu\nu} \hat{\mathcal{H}}^{\mu\nu} = \frac{1}{3}\tilde{T}_{ii} = \frac{1}{3}(\rho + p)_f u_i u_i + p_f + \frac{1}{3}\pi_{ii} = p_f + \frac{1}{3}\left[(\rho + p)_f (u^2_t - 1) + \pi_{tt}\right] , \tag{4.67}$$

and the anisotropic pressure is

$$\tilde{\pi}_{ij} = (\rho + p)_f u^i u^j + \pi_{ij} - \frac{1}{3}\delta_{ij}\left[(\rho + p)_f (u^2_t - 1) + \pi_{tt}\right] , \qquad 0 = \tilde{\pi}_{ti} = \tilde{\pi}_{tt} . \tag{4.68}$$

Since the velocities of the fluid and the observer are different, the observer measures the non-vanishing spatial energy flux

$$\tilde{q}_i = \tilde{T}^t{}_i = (\rho + p)_f u^t u_i + \pi^t{}_i . \tag{4.69}$$

If the observer velocity is the fluid velocity, we obtain the consistency relation:

$$u_a = \eta_{ta} , \qquad \tilde{\rho} = \rho_f , \qquad \tilde{p} = p_f , \qquad \tilde{q}_i = 0 , \qquad \tilde{\pi}_{ij} = \pi^f_{ij} . \tag{4.70}$$

From T^μ_ν in Eq. (4.52), it is clear that at the linear order in perturbations the energy density ρ, pressure p, and anisotropic pressure $\Pi_{\alpha\beta}$ are the same as those measured by an observer, independent of the fluid or the observer velocity. However, the velocity of the fluid and the spatial flux measured by the observer are

$$u^a = e^a_\mu u^\mu = \left(1, U^i - U^i_{\rm obs}\right) + \mathcal{O}(2) , \qquad \tilde{q}_i = (\rho + p)(U^i - U^i_{\rm obs}) + \mathcal{O}(2) , \tag{4.71}$$

where the presence of the relative velocity between the fluid and the observer is apparent. In addition to the fluid quantities ρ, p, $\pi_{\mu\nu}$ (same for any observer), the energy-momentum tensor $T_{\mu\nu}$ can be written in terms of the fluid velocity without spatial flux $q_\mu \equiv 0$, and the off-diagonal part of the energy-momentum tensor is

$$T^0_\alpha = (\bar\rho + \bar p)\left(U^f_\alpha - B_\alpha\right) + 0 . \tag{4.72}$$

One can also express $T_{\mu\nu}$ in terms of the observer velocity $u^\mu_{\rm obs}$ with non-vanishing spatial flux $\tilde{q}_i$ in Eq. (4.71), which defines

$$\tilde{q}_i =: [e_i]^\mu_{\rm obs} q^{\rm obs}_\mu = Q^{\rm obs}_\alpha \delta^\alpha_i + \mathcal{O}(2) , \tag{4.73}$$

and the off-diagonal part of the energy-momentum tensor is again identical:

$$T^0_\alpha = (\bar\rho + \bar p)\left(U^{\rm obs}_\alpha - B_\alpha\right) + Q^{\rm obs}_\alpha = (\bar\rho + \bar p)\left(U^f_\alpha - B_\alpha\right) . \tag{4.74}$$

In summary, the fluid quantities measured by the observer are

$$0 = \pi_{tt} = \pi_{ti} , \qquad \tilde\rho = \rho , \qquad \tilde p = p , \qquad \tilde\pi_{ij} = \pi_{ij} , \qquad \tilde q_i = (\bar\rho + \bar p)\left(U^i - U^i_{\rm obs}\right) . \tag{4.75}$$

In other words, whoever the observer is, the fluid quantities the observer measures are identical to those at the fluid rest frame, except the spatial energy flux.

4.2.4 Distribution Function

In cosmology, photons and neutrinos are the most important radiation components, and they are not described by the fluid approximation. Their statistical properties are captured by the distribution function F:

$$F := \bar f + f , \tag{4.76}$$

where the background distribution $\bar f$ often follows the equilibrium distribution and the perturbation f describes the deviation from the equilibrium. The equilibrium distribution for massless particles is fully described by the physical momentum and the temperature, and it is independent of position and time.[2] In the rest-frame of an observer, the physical energy E and the momentum P^a can be measured, and the energy-momentum tensor can be re-constructed as

$$T^{ab} = g \int \frac{d^3P}{E} P^a P^b F , \tag{4.77}$$

where the four momentum satisfies the on-shell condition $-m^2 = P_a P^a$ and $E = P^t$ and g is the spin-degeneracy of the particle, equal to two for photons and one for left-handed neutrinos.

The fluid elements can be readily computed as

$$\rho = g \int d^3P \, EF , \qquad q^i = Q_\alpha \delta^\alpha_i = g \int d^3P \, P^i F , \qquad p\delta^{ij} + \pi^{ij} = g \int d^3P \, \frac{P^i P^j}{E} F . \tag{4.78}$$

For later convenience, we introduce an angular decomposition

$$f(P, \hat n) := \sum_{lm} (-i)^l \sqrt{\frac{4\pi}{2l+1}} f_{lm}(P) Y_{lm}(\hat n) , \qquad f_{lm}(P) \equiv i^l \sqrt{\frac{2l+1}{4\pi}} \int d^2\hat n \, Y^*_{lm}(\hat n) f(P, \hat n) , \tag{4.79}$$

[2] In the background, the physical momentum and the temperature redshift in the same way.

where $P^i = Pn^i$ and $\hat{n}$ is the unit directional vector. The normalization convention may differ in literature. The perturbations in the fluid quantities are then related to the distribution function as

$$\delta\rho = 4\pi g \int_0^\infty dP\, P^2 E f_{00}\,, \qquad \delta p = \mathrm{Tr}\,\delta T^{ij} = \frac{4\pi g}{3}\int_0^\infty dP\,\frac{P^4}{E} f_{00}\,, \tag{4.80}$$

where we performed the angular integration. Higher moments of the fluid elements will be related to the higher-moments of the distribution function. At the linear order, the spatial energy flux from the distribution function is related to the relative velocity as

$$q_i = (\bar\rho + \bar p)\left(U^i_f - U^i_{\mathrm{obs}}\right) = Q^{\mathrm{obs}}_\alpha \delta^\alpha_i\,. \tag{4.81}$$

4.2.5 Multiple Cosmological Fluids

The details are summarized in Hwang and Noh (2002). In a universe with multiple fluids $i = 1, \cdots, N$, each fluid component has different velocity $u^\mu_{(i)}$. Given the fluid components in a coordinate system, the energy-momentum tensor $T_{\mu\nu}$ is completely set (i.e., all the components of $T_{\mu\nu}$):

$$T^{\mathrm{tot}}_{\mu\nu} = \sum_i T^{(i)}_{\mu\nu}, \tag{4.82}$$

which defines (ignoring the super-script "total")

$$T^0_0 = -\rho + \mathcal{O}(2)\,, \qquad T^\alpha_\beta = p\,\delta^\alpha_\beta + \Pi^\alpha_\beta + \mathcal{O}(2)\,, \tag{4.83}$$

or defines

$$\rho_{\mathrm{tot}} = (\bar\rho + \delta\rho)_{\mathrm{tot}} = \sum_i \rho_{(i)} = -T^0_0 + \mathcal{O}(2)\,, \qquad p_{\mathrm{tot}} = \sum_i p_{(i)}\,, \qquad \Pi^{\mathrm{tot}}_{\alpha\beta} = \sum_i \Pi^{(i)}_{\alpha\beta}\,, \tag{4.84}$$

such that the energy density, pressure, anisotropic pressure are just the sum of individual fluids, regardless of fluid and observer velocities at the linear order. However, the velocity u^μ_{tot} (and q^{tot}_μ) that would appear in the total energy momentum tensor is yet to be determined. In fact, we can define u^μ_{tot} as one without spatial flux,[3] i.e.,

$$T^0_\alpha = (\bar\rho + \bar p)\,(U_\alpha - B_\alpha)_{\mathrm{tot}} + \mathcal{O}(2) := \sum_i (\bar\rho + \bar p)_{(i)}(U^{(i)}_\alpha - B_\alpha) + \mathcal{O}(2)\,. \tag{4.85}$$

For the case of multiple fluids, it is possible to have interactions between fluids, even in the background, such that the energy conservation law is

$$T^{(i)\mu}{}_{\nu;\mu} =: I^{(i)}_\nu\,, \qquad 0 = \sum_i I^{(i)}_\mu\,, \qquad T^{\mathrm{tot}}_{\mu\nu;\mu} = 0\,. \tag{4.86}$$

In the background the conservation equation becomes

$$\dot{\bar\rho}_{(i)} + 3H\,(\bar\rho + \bar p)_{(i)} = \bar I_{(i)}\,, \qquad \frac{\dot{\bar\rho}_{(i)}}{\rho_{(i)}} = -3H\bar{(}\rho + \bar p)_{(i)}(1 - q_{(i)})\,, \tag{4.87}$$

where we defined (ignoring the vector type)

$$I^{(i)}_0 =: -a\left[\bar I_{(i)}(1+\alpha) + \delta I_{(i)}\right]\,, \qquad I^{(i)}_\alpha =: J^{(i)}_{,\alpha}\,, \qquad \bar I_{(i)} =: 3H(\bar\rho + \bar p)_{(i)} q_{(i)}\,. \tag{4.88}$$

At the background level, the equation of state and the sound speed of the individual components are

$$w_{(i)} := \frac{\bar p_{(i)}}{\bar\rho_{(i)}}\,, \qquad c^2_{s(i)} := \frac{\dot{\bar p}_{(i)}}{\dot{\bar\rho}_{(i)}} = w_{(i)} + \frac{dw_{(i)}}{d\ln\bar\rho_{(i)}}\,, \qquad \frac{1}{1+w} = \sum_i \frac{x_{(i)}}{1+w_{(i)}}\,, \tag{4.89}$$

$$\dot w_{(i)} = -3H(c^2_{s(i)} - w_{(i)})(1 + w_{(i)})(1 - q_{(i)})\,, \tag{4.90}$$

[3] This is possible nonlinearly, as we trade three dof in spatial flux with three dof in velocity.

where we defined

$$x_{(i)} := \frac{\bar\rho_{(i)} + \bar p_{(i)}}{\bar\rho + \bar p} \,, \qquad \sum_i x_{(i)} = 1 \,. \tag{4.91}$$

At the perturbation level, we derive

$$\delta p = \sum_i \delta p_{(i)} = \sum_i c^2_{s(i)} \delta\rho_{(i)} + \sum_i e_{(i)} =: c_s^2 \delta\rho + e \,, \tag{4.92}$$

$$e = \sum_i \left(c^2_{s(i)} - c_s^2\right) \delta\rho_{(i)} + \sum_i e_{(i)} =: e^{\rm rel} + e^{\rm int} \,, \tag{4.93}$$

where we defined the intrinsic and the relative entropy perturbations

$$e^{\rm rel} := \sum_i \left(c^2_{s(i)} - c_s^2\right) \delta\rho_{(i)} \tag{4.94}$$

$$= \frac{1}{2}\sum_{i,j} \frac{(\bar\rho+\bar p)_{(i)}(\bar\rho+\bar p)_{(j)}}{\bar\rho + \bar p}(c^2_{(i)} - c^2_{(j)})S_{ij} + \sum_i \frac{(\bar\rho+\bar p)_{(i)}}{\bar\rho+\bar p} q_{(i)} c^2_{s(i)} \delta\rho \,, \tag{4.95}$$

where the relative fluctuation is

$$S_{ij} := \frac{\delta n_{(i)}}{n_{(i)}} - \frac{\delta n_{(j)}}{n_{(j)}} = \frac{\delta\rho_{(i)}}{\bar\rho_{(i)} + \bar p_{(i)}} - \frac{\delta\rho_{(j)}}{\bar\rho_{(j)} + \bar p_{(j)}} \,. \tag{4.96}$$

• ***Gauge-transformation properties***.—

$$\widetilde{\delta I}_{(i)} = \delta I_{(i)} - \bar I'_{(i)} T \,, \qquad \tilde J_{(i)} = J_{(i)} + a \bar I_{(i)} T \,, \qquad \tilde S_{ij} = S_{ij} - 3\mathcal{H} T (q_{(i)} - q_{(j)}) \,, \tag{4.97}$$

where S_{ij} is gauge-invariant only when there is no energy transfer in the background $q_{(i)} \equiv 0$.

4.3 Einstein Equations

4.3.1 Christoffel Symbols

In the absence of the torsion, the Christoffel symbols are uniquely determined by the metric tensor as

$$\Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial \xi^\sigma}\frac{\partial^2 \xi^\sigma}{\partial x^\nu \partial x^\rho} = \frac{1}{2} g^{\mu\sigma}(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}) \,, \qquad 0 = \frac{d^2\xi^\mu}{d\tau^2} \,, \qquad d\tau^2 = -\eta_{\mu\nu} d\xi^\mu d\xi^\nu \,, \tag{4.98}$$

where ξ^μ is a freely falling coordinate and $d\tau$ is the proper time. To the linear order in perturbations, we derive

$$\Gamma^0_{00} = \frac{a'}{a} + A' \;\rightarrow\; \mathcal{H} + \psi' \,, \qquad \Gamma^0_{0\alpha} = A_{,\alpha} - \frac{a'}{a} B_\alpha \;\rightarrow\; \psi_{,\alpha} \,, \tag{4.99}$$

$$\Gamma^\alpha_{00} = A^{|\alpha} - B^{\alpha\prime} - \frac{a'}{a} B^\alpha \;\rightarrow\; \psi^{,\alpha} \,, \tag{4.100}$$

$$\Gamma^0_{\alpha\beta} = \frac{a'}{a}\bar g_{\alpha\beta} - 2\frac{a'}{a}\bar g_{\alpha\beta} A + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\frac{a'}{a} C_{\alpha\beta} \;\rightarrow\; \mathcal{H}\bar g_{\alpha\beta}\,(1 - 2\psi) + \left(\phi' + 2\mathcal{H}\phi\right)\bar g_{\alpha\beta} \,, \tag{4.101}$$

$$\Gamma^\alpha_{0\beta} = \frac{a'}{a}\delta^\alpha_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right) + C^{\alpha\prime}_\beta \;\rightarrow\; \mathcal{H}\delta^\alpha_\beta + \phi'\delta^\alpha_\beta \,, \tag{4.102}$$

$$\Gamma^\alpha_{\beta\gamma} = \bar\Gamma^\alpha_{\beta\gamma} + \frac{a'}{a}\bar g_{\beta\gamma} B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\alpha} \;\rightarrow\; \bar\Gamma^\alpha_{\beta\gamma} + 2\phi_{,(\gamma}\delta^\alpha_{\beta)} - \phi^{,\alpha}\bar g_{\beta\gamma} \,, \tag{4.103}$$

where the conformal Hubble parameter is $\mathcal{H} = a'/a$ and $\bar\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols based on 3-metric $\bar g_{\alpha\beta}$.

4.3.2 Riemann Tensor

The Riemann tensor can then be constructed in terms of the Christoffel symbols as

$$R^{\mu}_{\nu\rho\sigma} := \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\epsilon}_{\nu\sigma}\Gamma^{\mu}_{\rho\epsilon} - \Gamma^{\epsilon}_{\nu\rho}\Gamma^{\mu}_{\sigma\epsilon} = \Gamma^{\mu}_{\nu\sigma;\rho} - \Gamma^{\mu}_{\nu\rho;\sigma} - \Gamma^{\epsilon}_{\nu\sigma}\Gamma^{\mu}_{\rho\epsilon} + \Gamma^{\epsilon}_{\nu\rho}\Gamma^{\mu}_{\sigma\epsilon} \,, \tag{4.104}$$

and the Riemann tensor has all the information of the geometry, such that how any four vector changes locally is fully determined by the Riemann tensor

$$2u_{\mu;[\nu\rho]} = u_\sigma R^{\sigma}{}_{\mu\nu\rho} \,, \qquad u^{\rho}{}_{;[\nu\mu]} = \frac{1}{2}R^{\rho}{}_{\sigma\mu\nu}u^{\sigma} \,. \tag{4.105}$$

Out of the Riemann tensor, we can construct the Ricci tensor (and Ricci scalar) by contracting the Riemann tensor as

$$R_{\mu\nu} := R^{\rho}_{\mu\rho\nu} \,, \qquad R = R^{\mu}_{\mu} \,, \tag{4.106}$$

and construct the (conformal) Weyl tensor as

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \,. \tag{4.107}$$

The Riemann tensor has the symmetry:

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu} \,, \qquad R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = R_{\mu[\nu\rho\sigma]} = 0 \,, \tag{4.108}$$

such that its 20 independent components can be separated into the Ricci tensor (10 components) and the (traceless) Weyl curvature tensor (also 10 components with all the symmetry of the Riemann tensor)

$$C^{\mu\nu}{}_{\mu\nu} = 0 \,, \qquad C_{\mu\nu\rho\sigma} = C_{[\mu\nu][\rho\sigma]} = C_{\rho\sigma\mu\nu} \,, \qquad C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = C_{\mu[\nu\rho\sigma]} = 0 \,. \tag{4.109}$$

The Ricci tensor is algebraically set by matter distribution through the Einstein equation, but the Weyl tensor is determined by differential equations with suitable boundary conditions.

To the background, we derive

$$R^{0}_{\alpha0\beta} = \mathcal{H}'\bar{g}_{\alpha\beta} \,, \qquad R^{\alpha}_{00\beta} = \mathcal{H}'\delta^{\alpha}_{\beta} \,, \qquad R^{\alpha}_{\beta\gamma\delta} = 2\left(K + \mathcal{H}^2\right)\delta^{\alpha}_{[\gamma}\bar{g}_{\delta]\beta} = \bar{R}^{\alpha}_{\beta\gamma\delta} + 2\mathcal{H}^2\delta^{\alpha}_{[\gamma}\bar{g}_{\delta]\beta} \,, \tag{4.110}$$
$$R_{00} = -3\mathcal{H}' \,, \qquad R_{\alpha\beta} = \left(2K + \mathcal{H}' + 2\mathcal{H}^2\right)\bar{g}_{\alpha\beta} = \bar{R}_{\alpha\beta} + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\bar{g}_{\alpha\beta} \,, \qquad R = \frac{6}{a^2}\left(K + \mathcal{H}' + \mathcal{H}^2\right) \,,$$

where we defined the Riemann tensor for a 3-hypersurface based on 3-metric $\bar{g}_{\alpha\beta}$

$$\bar{R}^{\alpha}_{\beta\gamma\delta} = 2K\delta^{\alpha}_{[\gamma}\bar{g}_{\delta]\beta} \,, \qquad \bar{R}_{\alpha\beta} = 2K\bar{g}_{\alpha\beta} \,, \qquad \bar{R} = 6K \,. \tag{4.111}$$

and we used for any second-rank tensor $F_{\alpha\beta}$

$$F_{\alpha\beta|[\gamma\delta]} = K\left(\bar{g}_{\alpha[\delta}F_{\gamma]\beta} + \bar{g}_{\beta[\delta}F_{\gamma]\alpha}\right) \,. \tag{4.112}$$

To the linear order in perturbations, we derive the Riemann tensor:

$$R^{\mu}{}_{\nu00} = 0 \,, \qquad R^{0}{}_{00\alpha} = -\mathcal{H}'B_\alpha \,, \qquad R^{0}{}_{0\alpha\beta} = 0 \,, \tag{4.113}$$

$$R^{0}{}_{\alpha0\beta} = \mathcal{H}'\bar{g}_{\alpha\beta} - \left[\mathcal{H}A' + 2\mathcal{H}'A\right]\bar{g}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + \mathcal{H}B_{(\alpha|\beta)} + C''_{\alpha\beta} + \mathcal{H}C'_{\alpha\beta} + 2\mathcal{H}'C_{\alpha\beta} \,, \tag{4.114}$$

$$R^{0}{}_{\alpha\beta\gamma} = 2\mathcal{H}\bar{g}_{\alpha[\beta}A_{,\gamma]} - B_{\alpha|[\beta\gamma]} + \frac{1}{2}(B_{\gamma|\alpha\beta} - B_{\beta|\alpha\gamma}) - 2C'_{\alpha[\beta|\gamma]} \,, \tag{4.115}$$

$$R^{\alpha}{}_{00\beta} = \mathcal{H}'\delta^{\alpha}_{\beta} - \mathcal{H}A'\delta^{\alpha}_{\beta} - A^{|\alpha}{}_{\beta} + \frac{1}{2}\left(B_{\beta}{}^{|\alpha} + B^{\alpha}{}_{|\beta}\right)' + \frac{1}{2}\mathcal{H}\left(B_{\beta}{}^{|\alpha} + B^{\alpha}{}_{|\beta}\right) + C^{\alpha\prime\prime}_{\beta} + \mathcal{H}C^{\alpha\prime}_{\beta} \,, \tag{4.116}$$

$$R^{\alpha}{}_{0\beta\gamma} = 2\mathcal{H}\delta^{\alpha}_{[\beta}A_{,\gamma]} - B_{[\beta}{}^{|\alpha}{}_{\gamma]} + B^{\alpha}{}_{|[\beta\gamma]} - 2\mathcal{H}^2\delta^{\alpha}_{[\beta}B_{\gamma]} - 2C^{\alpha\prime}_{[\beta|\gamma]} \,, \tag{4.117}$$

$$R^{\alpha}{}_{\beta0\gamma} = \mathcal{H}\left(\bar{g}_{\beta\gamma}A^{,\alpha} - \delta^{\alpha}_{\gamma}A_{,\beta}\right) + \mathcal{H}'\bar{g}_{\beta\gamma}B^{\alpha} - \mathcal{H}^2\left(\bar{g}_{\beta\gamma}B^{\alpha} - \delta^{\alpha}_{\gamma}B_{\beta}\right) - \frac{1}{2}\left(B_{\beta}{}^{|\alpha} - B^{\alpha}{}_{|\beta}\right)_{|\gamma} + C^{\alpha\prime}_{\gamma|\beta} - C'_{\beta\gamma}{}^{|\alpha} \,, \tag{4.118}$$

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = {} & \bar{R}^{\alpha}_{\beta\gamma\delta} + \mathcal{H}^2\left(\delta^{\alpha}_{\gamma}\bar{g}_{\beta\delta} - \delta^{\alpha}_{\delta}\bar{g}_{\beta\gamma}\right)(1 - 2A) \\ & + \frac{1}{2}\mathcal{H}\left[\bar{g}_{\beta\delta}\left(B_{\gamma}{}^{|\alpha} + B^{\alpha}{}_{|\gamma}\right) - \bar{g}_{\beta\gamma}\left(B_{\delta}{}^{|\alpha} + B^{\alpha}{}_{|\delta}\right) + 2\delta^{\alpha}_{\gamma}B_{(\beta|\delta)} - 2\delta^{\alpha}_{\delta}B_{(\beta|\gamma)}\right] \\ & + \mathcal{H}\left[\bar{g}_{\beta\delta}C^{\alpha\prime}_{\gamma} - \bar{g}_{\beta\gamma}C^{\alpha\prime}_{\delta} + \delta^{\alpha}_{\gamma}C'_{\beta\delta} - \delta^{\alpha}_{\delta}C'_{\beta\gamma} + 2\mathcal{H}\left(\delta^{\alpha}_{\gamma}C_{\beta\delta} - \delta^{\alpha}_{\delta}C_{\beta\gamma}\right)\right] \\ & + 2C^{\alpha}_{(\beta|\delta)\gamma} - 2C^{\alpha}_{(\beta|\gamma)\delta} + C_{\beta\gamma}{}^{|\alpha}{}_{\delta} - C_{\beta\delta}{}^{|\alpha}{}_{\gamma} \,, \end{aligned} \tag{4.119}$$

and by contracting the Riemann tensor we derive the Ricci tensor and the Ricci scalar:

$$R_{00} = -3\mathcal{H}' + 3\mathcal{H}A' + \Delta A - B^{\alpha\prime}{}_{|\alpha} - \mathcal{H}B^{\alpha}{}_{|\alpha} - C^{\alpha\prime\prime}_{\alpha} - \mathcal{H}C^{\alpha\prime}_{\alpha} , \quad (4.120)$$

$$R_{0\alpha} = 2\mathcal{H}A_{,\alpha} - \mathcal{H}'B_\alpha - 2\mathcal{H}^2 B_\alpha + \frac{1}{2}\Delta B_\alpha - \frac{1}{2}B^{\beta}{}_{|\alpha\beta} - C^{\beta\prime}_{\beta|\alpha} + C'_{\alpha\beta}{}^{|\beta} , \quad (4.121)$$

$$R_{\alpha\beta} = 2K\bar{g}_{\alpha\beta} + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\bar{g}_{\alpha\beta}(1-2A) - \mathcal{H}A'\bar{g}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + 2\mathcal{H}B_{(\alpha|\beta)} + \mathcal{H}\bar{g}_{\alpha\beta}B^{\gamma}{}_{|\gamma} \quad (4.122)$$
$$+C''_{\alpha\beta} + 2\frac{a'}{a}C'_{\alpha\beta} + 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)C_{\alpha\beta} + \mathcal{H}\bar{g}_{\alpha\beta}C^{\gamma\prime}_{\gamma} + 2C^{\gamma}_{(\alpha|\beta)\gamma} - C^{\gamma}_{\gamma|\alpha\beta} - \Delta C_{\alpha\beta} ,$$

$$R = \frac{1}{a^2}\Big[6\left(\mathcal{H}' + \mathcal{H}^2 + K\right) - 6\mathcal{H}A' - 12\left(\mathcal{H}' + \mathcal{H}^2\right)A - 2\Delta A \quad (4.123)$$
$$+2B^{\alpha\prime}{}_{|\alpha} + 6\mathcal{H}B^{\alpha}{}_{|\alpha} + 2C^{\alpha\prime\prime}_{\alpha} + 6\mathcal{H}C^{\alpha\prime}_{\alpha} - 4KC^{\alpha}_{\alpha} - 2\Delta C^{\alpha}_{\alpha} + 2C^{\alpha\beta}{}_{|\alpha\beta}\Big] .$$

• HW: derive the Riemann tensor and the Ricci tensor in the conformal Newtonian gauge

4.3.3 Einstein Equation and Background Equation

The Einstein equation is that the Einstein tensor $G_{\mu\nu}$ is proportional to the energy-momentum tensor $T_{\mu\nu}$:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} , \quad (4.124)$$

where G is the Newton's constant and Λ is the cosmological constant. The cosmological constant can be put in the right-hand side as a part of the energy-momentum tensor:

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} . \quad (4.125)$$

The trace of the Einstein equation gives

$$T = -\rho + 3p , \qquad R = 8\pi G(\rho - 3p) + 4\Lambda , \quad (4.126)$$

and the Ricci tensor is completely set by the trace of the energy-momentum tensor. To the background, the Einstein equation yields the Friedmann equation give

$$H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{K}{a^2} + \frac{\Lambda}{3} , \qquad H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\bar{\rho} + 3\bar{p}\right) + \frac{\Lambda}{3} , \quad (4.127)$$

and the energy-momentum conservation yields

$$\dot{\bar{\rho}} + 3H\left(\bar{\rho} + \bar{p}\right) = 0 . \quad (4.128)$$

In terms of the conformal time, the Friedmann equation becomes

$$\mathcal{H}' = a^2(H^2 + \dot{H}) = -\frac{4\pi G}{3}a^2(\bar{\rho} + 3\bar{p}) , \qquad \mathcal{H}^2 + \mathcal{H}' = \frac{a''}{a} = \frac{4\pi G}{3}a^2(\bar{\rho} - 3\bar{p}) - K . \quad (4.129)$$

In a flat Universe ($K = 0$) dominated by an energy component $\bar{\rho} \propto a^{-n}$, we can derive the analytic solutions to the Friedmann equation:

$$a \propto \eta^{\frac{2}{n-2}} \propto t^{2/n} , \qquad t \propto \eta^{\frac{n}{n-2}} , \qquad H = H_o\left(\frac{t_o}{t}\right) = \frac{2}{nt} , \qquad \mathcal{H} = \mathcal{H}_o\left(\frac{\eta_o}{\eta}\right) = \frac{2}{n-2}\,\frac{1}{\eta} , \quad (4.130)$$

or in terms of equation of state $\bar{\rho} \propto a^{-3(1+w)}$,

$$a \propto \eta^{\frac{2}{1+3w}} \propto t^{2/3(1+w)} , \qquad t \propto \eta^{\frac{3(1+w)}{1+3w}} , \qquad H = \frac{2}{3(1+w)t} , \qquad \mathcal{H} = \frac{2}{1+3w}\,\frac{1}{\eta} , \quad (4.131)$$

where we used $n = 3(1+w)$.

• ***Einstein-de Sitter Universe***.— This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations are

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{\eta_0}\right)^2 , \qquad \frac{t}{t_0} = \left(\frac{\eta}{\eta_0}\right)^3 , \qquad \eta_0 = 3t_0 , \tag{4.132}$$

$$H = \frac{2}{3t} , \qquad \mathcal{H} = \frac{2}{\eta} , \qquad \rho_m = \frac{1}{6\pi G t^2} , \qquad r = \eta_0 - \eta = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) , \tag{4.133}$$

where the reference point t_0 satisfies $a(t_0) = 1$, but it can be any time $t_0 \in (0, \infty)$.

4.3.4 Linear-Order Einstein Equation

The Einstein equation can be expanded up to the linear order in perturbations, and decomposed into the evolution equations describing the scalar, the vector, and the tensor perturbations. At the linear order, they do not mix.

• ***Scalar perturbations***.—

$$G^0_0 \quad : \quad H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\rho , \tag{4.134}$$

$$G^0_\alpha \quad : \quad \kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\bar\rho + \bar p)av , \tag{4.135}$$

$$G^\alpha_\alpha - G^0_0 \quad : \quad \dot\kappa + 2H\kappa + \left(3\dot H + \frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\rho + 3\delta p\right) , \tag{4.136}$$

$$G^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta G^\gamma_\gamma \quad : \quad \dot\chi + H\chi - \varphi - \alpha = 8\pi G\Pi , \tag{4.137}$$

where we defined

$$\kappa := 3H\alpha - 3\dot\varphi - \frac{\Delta}{a^2}\chi , \qquad \chi := a\beta + a\gamma' , \qquad \Pi_{\alpha\beta} := \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}\bar g_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta} . \tag{4.138}$$

The energy-momentum conservation yields

$$T^\nu_{0;\nu} \quad : \quad \delta\dot\rho + 3H\left(\delta\rho + \delta p\right) - (\bar\rho + \bar p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right) = 0 , \tag{4.139}$$

$$T^\nu_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v]^\cdot}{a^4(\bar\rho + \bar p)} - \frac{1}{a}\alpha - \frac{1}{a(\bar\rho + \bar p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right) = 0 . \tag{4.140}$$

• ***Vector perturbations***.—

$$G^0_\alpha \quad : \quad \frac{\Delta + 2K}{2a^2}\,\Psi^{(v)}_\alpha + 8\pi G(\bar\rho + \bar p)v^{(v)}_\alpha = 0 , \tag{4.141}$$

$$G^\alpha_\beta \quad : \quad \dot\Psi^{(v)}_\alpha + 2H\Psi^{(v)}_\alpha = 8\pi G\,\Pi^{(v)}_\alpha , \tag{4.142}$$

$$T^\nu_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v^{(v)}_\alpha]^\cdot}{a^4(\bar\rho + \bar p)} + \frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\bar\rho + \bar p} = 0 . \tag{4.143}$$

• ***Tensor perturbations***.—

$$G^\alpha_\beta \quad : \quad \ddot C^{(t)\alpha}{}_\beta + 3H\dot C^{(t)\alpha}{}_\beta - \frac{\Delta - 2K}{a^2}C^{(t)\alpha}{}_\beta = 8\pi G\Pi^{(t)\alpha}{}_\beta . \tag{4.144}$$

• ***Multiple fluids***.— In the presence of interactions in Eq. (4.88), the individual conservation becomes

$$T^{(i)\nu}_{0;\nu} \quad : \quad \delta\dot\rho_{(i)} + 3H\left(\delta\rho + \delta p\right)_{(i)} - \dot{\bar\rho}_{(i)}\alpha - (\bar\rho + \bar p)_{(i)}\left(\kappa + \frac{1}{a}\Delta v_{(i)}\right) = \delta I_{(i)} , \tag{4.145}$$

$$T^{(i)\nu}_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v]^\cdot_{(i)}}{a^4(\bar\rho + \bar p)_{(i)}} - \frac{1}{a}\alpha - \frac{1}{a(\bar\rho + \bar p)_{(i)}}\left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi_{(i)} - J_{(i)}\right) = 0 . \tag{4.146}$$

The first conservation equation can be expressed as

$$\dot{\delta}_{(i)} + 3H(c_s^2 - w)_{(i)}\delta_{(i)} + 3H(1+w_{(i)})q_{(i)}\delta_{(i)} = (1+w_{(i)})\left[\kappa - 3H(1-q_{(i)})\alpha + \frac{\Delta}{a}v_{(i)}\right] + \frac{1}{\bar{\rho}_{(i)}}(-3He + \delta I)_{(i)} \ . \tag{4.147}$$

4.4 Linear-Order Cosmological Solutions

4.4.1 Super-Horizon Solutions

On super-horizon scales $k \ll \mathcal{H}$, many simplifications can be made to derive useful relations. The definition of κ and the first two Einstein equations are

$$\kappa \simeq 3H\alpha - 3\dot{\varphi} \ , \qquad H\kappa \simeq -4\pi G\bar{\rho}\,\delta \ , \qquad \kappa \simeq 12\pi G\bar{\rho}(1+w)av \ , \tag{4.148}$$

where we used $\simeq$ to emphasize the validity only on the super-horizon scales and we ignored the Laplacian term. Also note that the anisotropic pressure Π is *not* necessarily zero on super-horizon scales. For example, photons in RDE are tightly coupled with baryons (hence a fluid), but neutrinos free-stream, generating non-zero anisotropic pressure even on super-horizon scales. Manipulating these equations, we first obtain that the comoving gauge density fluctuation vanishes

$$0 \simeq 4\pi G\bar{\rho}\left[\delta + 3\mathcal{H}\bar{\rho}(1+w)v\right] \ , \qquad \therefore \ \ 0 \simeq \delta_v = \delta - \frac{\bar{\rho}'}{\bar{\rho}}v = \delta + 3\mathcal{H}\bar{\rho}(1+w)v \ , \tag{4.149}$$

which can be understood in terms of $k^2\varphi_\chi = 4\pi G\bar{\rho}a^2\delta_v$. This also implies the equivalence

$$\varphi_\delta := \varphi + \frac{\delta}{3(1+w)} \ , \qquad \varphi_\delta - \varphi_v = \frac{\delta}{3(1+w)} + \mathcal{H}v \ \simeq 0, \qquad \therefore \ \ \varphi_v \simeq \varphi_\delta \ , \tag{4.150}$$

for the total comoving and total uniform-matter curvature fluctuations, independent of adiabatic conditions.

The other important conservation law deals with the comoving-gauge curvature (often denoted as $\mathcal{R}$):

$$\varphi_v := \varphi - \mathcal{H}v \ . \tag{4.151}$$

As we derive in Section 5, the comoving-gauge curvature perturbation φ_v in a flat universe $K = 0$ is conserved on large scales throughout the evolution:

$$\dot{\varphi}_v \simeq 0 \ , \tag{4.152}$$

if the total matter content of the Universe is *adiabatic*, which is the case in the standard model. However, without assuming the adiabatic condition, we can also manipulate

$$\mathcal{H}v_\chi \simeq \frac{2}{9H}\frac{\kappa_\chi}{1+w} \simeq \frac{2(\alpha_\chi - \dot{\varphi}_\chi/H)}{3(1+w)} \ , \tag{4.153}$$

to arrive at

$$\varphi_v = \varphi_\chi - \mathcal{H}v_\chi \simeq \varphi_\chi - \frac{2(\alpha_\chi - \dot{\varphi}_\chi/H)}{3(1+w)} \simeq \frac{5+3w}{3(1+w)}\,\varphi_\chi + \frac{2\dot{\varphi}_\chi}{3H(1+w)} + \frac{16\pi G\Pi}{3(1+w)} \ , \tag{4.154}$$

where we used the second Einstein equation and the definition of κ.

In the standard model, the anisotropic pressure is $\Pi = 0$ in MDE, and the quarupole moments from photons and neutrinos in RDE give rise to

$$8\pi G\Pi = \frac{2}{5}f_\nu\alpha_\chi \simeq 0.175\ \alpha_\chi \ , \qquad f_\nu := \frac{\bar{\rho}_\nu}{\bar{\rho}_{\rm rad}} \approx 0.438 \ . \tag{4.155}$$

Ignoring this correction and assuming the conservation of φ_v with the adiabatic conditions, we derive

$$\alpha_\chi(t) \simeq -\varphi_\chi(t) \ , \qquad \varphi_v \simeq \frac{5+3w}{3(1+w)}\,\varphi_\chi(t) \ . \tag{4.156}$$

While φ_v is conserved on super-horizon scales throughout the whole evolution, the Newtonian gauge potential evolves, as the Universe transitions from the radiation dominated to the matter dominated eras:

$$\varphi_\chi^{\rm RDE} = \frac{2}{3}\varphi_v \, , \qquad \varphi_\chi^{\rm MDE} = \frac{3}{5}\varphi_v \, . \tag{4.157}$$

Without the adiabatic conditions, the comovoing-gauge curvature φ_v evolves as well, but the relation in Eq. (4.154) is still valid on super-horizon scales.

Finally, the energy-momentum conservation in Eq. (4.139) can be re-arranged as

$$3(1+w)\dot\varphi_\delta + \frac{3H}{\bar\rho}\left(\delta p - \bar p\,\delta\right) + (1+w)\frac{\Delta}{a^2}(\chi - av) = 0 \, , \tag{4.158}$$

implies that in the super-horizon limit we have a conservation law for a medium with adiabatic condition $e = \delta p - c_s^2\delta\rho \equiv 0$:

$$\dot\varphi_\delta \simeq 0 \, , \tag{4.159}$$

where we assumed the equation of state is constant. The uniform-density gauge curvature is often denoted as $\zeta = \varphi_\delta$. Note that in the conservation equation we assumed no energy transfers between any fluids, and this conservation law holds for individual adiabatic fluids.

4.4.2 Einstein Equation in the conformal Newtonian Gauge

In the conformal Newtonian gauge we have

$$\kappa = 3H\psi - 3\dot\phi \, , \qquad \chi = 0 \, , \qquad U = v_\chi \, , \qquad \boldsymbol{v} = -\nabla U \, . \tag{4.160}$$

By substituting into the Einstein equation, we derive

$$H\kappa + \frac{\Delta + 3K}{a^2}\phi = -4\pi G\delta\rho \, , \qquad \phi + \psi = -8\pi G\Pi \, , \tag{4.161}$$

$$\kappa = 12\pi G(\bar\rho + \bar p)av \, , \qquad \dot\kappa + 2H\kappa + \left(3\dot H + \frac{\Delta}{a^2}\right)\psi = 4\pi G\left(\delta\rho + 3\,\delta p\right) \, . \tag{4.162}$$

To remove κ in favor of the other variables, we use Eq. (4.162) to arrive at

$$\dot\phi + H\phi = -4\pi G(\bar\rho + \bar p)av - 8\pi G H\Pi \, , \tag{4.163}$$

and Eq. (4.161) can be further manipulated as

$$-(\Delta + 3K)\phi = 4\pi G a^2\bar\rho\,\delta + a\mathcal{H}\kappa = 4\pi G a^2\bar\rho\left[\delta + 3\mathcal{H}v(1+w)\right] \equiv 4\pi G a^2\bar\rho\,\delta_v \, , \tag{4.164}$$

where δ_v is the density fluctuation in the comoving gauge:

$$\delta_v := \delta - \frac{\dot{\bar\rho}}{\bar\rho}av = \delta + 3\mathcal{H}v \, . \tag{4.165}$$

Finally, the equation for the velocity can be obtained from Eq. (4.162) as

$$v' + \mathcal{H}v = -\phi + \frac{\delta p_v}{\bar\rho + \bar p} - 8\pi G\Pi + \frac{2}{3}\frac{\Delta + 3K}{a^2}\frac{\Pi}{\bar\rho + \bar p} \, , \tag{4.166}$$

where the pressure fluctuation in the comoving gauge is

$$\delta p_v := \delta p - \dot{\bar p}\,av \, . \tag{4.167}$$

Assuming a flat Universe ($K = 0$) and a pressureless medium ($p = \delta p = 0$), we can further simplify the equation as

$$\phi + \psi = 0 \, , \qquad \kappa = \frac{3}{a}\left(a\psi\right)^{\cdot} = 12\pi G\bar\rho av \, , \qquad \Delta\phi = -4\pi G a^2\bar\rho\,\delta_v \, , \qquad v' + \mathcal{H}v = \psi \, . \tag{4.168}$$

4.4.3 Newtonian Correspondence

As apparent, the relativistic equations appear quite similar or identical to those in the Newtonian dynamics. Here we identify such correspondence made available in a particular choice of gauge. However, keep in mind that the relativistic dynamics is intrinsically different from the Newtonian, and such correspondence is only identified in a limited case (e.g., linear order for pressureless media).

With $\beta \equiv 0$ in the conformal Newtonian gauge, we find the velocity potential in the standard Newtonian perturbation theory

$$U = v \,, \qquad \mathbf{v} = -\nabla v \,, \qquad \theta := -\frac{1}{a}\nabla\cdot\mathbf{v} = \frac{1}{a}\Delta v \,, \tag{4.169}$$

and by taking the divergence of v in Eq. (4.168), we obtain the governing equation

$$\Delta\psi = \frac{1}{a}\Delta(av)' = (a^2\theta)^{\cdot} \,, \qquad \therefore\quad \dot\theta + 2H\theta = \frac{1}{a^2}\Delta\psi = 4\pi G\bar\rho\,\delta_v \,. \tag{4.170}$$

The last remaining equation in the SPT can be obtained by taking the time derivative of δ_v in Eq. (4.168):

$$\dot\delta_v = -\frac{\Delta(a\phi)^{\cdot}}{4\pi G a^3\bar\rho} = \theta \,. \tag{4.171}$$

where we assumed in this case $\bar\rho \propto 1/a^3$. With a proper identification of gauge-invariant variables to the standard Newtonian perturbation theory

$$v_\chi \to U \,, \qquad \alpha_\chi = -\varphi_\chi \to \delta\Phi \,, \qquad \delta_v \to \delta_m \,, \tag{4.172}$$

we find the governing equation in SPT is fully relativistic at the linear order.

Manipulating the Newtonian gauge equations, we find that the density fluctuation then follows the same evolution equation as in the standard Newtonian perturbation theory

$$\ddot\delta_v + 2H\dot\delta_v - 4\pi G\bar\rho_m\delta_v = 0 \,. \tag{4.173}$$

With a mathematical identity

$$\frac{1}{a^2H}\left[a^2H^2\left(\frac{\delta}{H}\right)^{\cdot}\right]^{\cdot} = \ddot\delta + 2H\dot\delta - \delta\left(\frac{\ddot H}{H} + 2\dot H\right) \,, \tag{4.174}$$

and by using the Friedmann equation with $\bar p = w\bar\rho$ (valid for any K)

$$\left(\frac{\ddot H}{H} + 2\dot H\right) = 4\pi G\bar\rho(1+w)(1+3w) \,, \tag{4.175}$$

we can derive a formal solution for the differential equation for δ_v in case $w = 0$:

$$\delta_v(\mathbf{k}, t) = c_1(\mathbf{k})H(t)\int\frac{dt}{a^2H^2} + c_2(\mathbf{k})H(t) \,, \tag{4.176}$$

where the first term is the growing solution and the second term is the decaying solution.

4.4.4 General Solutions

In fact, the most general evolution equation for the density fluctuation was derived already in Bardeen (1980); Hwang and Noh (1999) by solving the Einstein equations with full generality:

$$\begin{aligned}
&\ddot\delta_v + (2+3c_s^2-6w)H\dot\delta_v + \left[c_s^2\frac{k^2}{a^2} - 4\pi G\bar\rho(1-6c_s^2+8w-3w^2) + 12(w-c_s^2)\frac{K}{a^2} + (3c_s^2-5w)\Lambda\right]\delta_v \\
&\equiv \frac{1+w}{a^2H}\left[\frac{H^2}{a(\bar\rho+\bar p)}\left(\frac{a^3\bar\rho}{H}\delta_v\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\delta_v = -\frac{k^2-3K}{a^2}\frac{1}{\bar\rho}\left\{e + 2H\dot\Pi + 2\left[-\frac{1}{3}\frac{k^2}{a^2} + 2\dot H + 3(1+c_s^2)H^2\right]\Pi\right\} \,,
\end{aligned} \tag{4.177}$$

where the equation of state, the sound speed c_s^2, and the entropy perturbation are defined as

$$w := \frac{\bar{p}}{\bar{\rho}} , \qquad c_s^2 := \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} , \qquad \delta p := c_s^2 \delta\rho + e . \tag{4.178}$$

This equation is fully general for any Λ and K. In the same way, the most general evolution equation for the potential fluctuation was also derived in Hwang and Noh (1999)

$$\ddot{\varphi}_\chi + (4+3c_s^2)H\dot{\varphi}_\chi + \left[c_s^2\frac{k^2}{a^2} + 8\pi G\bar{\rho}(c_s^2 - w) - 2(1+3c_s^2)\frac{K}{a^2} + (1+c_s^2)\Lambda\right]\varphi_\chi$$
$$\equiv \frac{\bar{\rho}+\bar{p}}{H}\left[\frac{H^2}{a(\bar{\rho}+\bar{p})}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\varphi_\chi = \mathcal{F}(e,\Pi) , \tag{4.179}$$

where the source term $\mathcal{F}$ is some function of e and Π. These equations are greatly simplified in the absence of anisotropic pressure Π and entropy perturbation e. The equations in terms of conformal time can be obtained by using

$$\dot{f} = \frac{1}{a}f' , \qquad \ddot{f} = \frac{1}{a^2}f'' - \frac{H}{a}f' . \tag{4.180}$$

In a universe with constant equation of state and $K = \Lambda = e = \Pi = 0$, the differential equation becomes

$$\varphi_\chi'' + 3(1+w)\mathcal{H}\varphi_\chi' + wk^2\varphi_\chi = 0 , \qquad \mathcal{H} = \frac{2}{1+3w}\frac{1}{\eta} , \tag{4.181}$$

and the solutions are the Bessel functions of order α:

$$\varphi_\chi = y^{-\alpha}\left[c_1(k)J_\alpha(y) + c_2(k)Y_\alpha(y)\right] , \qquad y := \sqrt{w}k\eta , \qquad \alpha := \frac{1}{2}\left(\frac{5+3w}{1+3w}\right) . \tag{4.182}$$

In RDE ($w = 1/3$) and MDE ($w = 0$), the solutions are

$$\varphi_\chi = \frac{1}{y^2}\left[c_1(k)\left(\frac{\sin y}{y} - \cos y\right) + c_2(k)\left(\frac{\cos y}{y} + \sin y\right)\right] , \qquad w = \frac{1}{3} , \qquad \alpha = \frac{3}{2} , \tag{4.183}$$

$$\varphi_\chi = c_1(k) + \frac{c_2(k)}{y^5} , \qquad w = 0 , \qquad \alpha = \frac{5}{2} , \tag{4.184}$$

where c_2 is the decaying mode. The growing mode of the gravitational potential is constant at all k in MDE and also outside the horizon in RDE, while it decays inside the horizon in RDE. Ignoring the oscillatory part inside the horizon in RDE, the gravitational potential can be well approximated as

$$\varphi_\chi = \frac{2}{3}\frac{1}{1+(k\eta)^2} \qquad \text{for } \eta \leq \eta_{\rm eq} , \quad \forall k , \tag{4.185}$$

where the normalization is set at the super-horizon scale, compared to φ_v. The solution for the density fluctuation is from the Einstein equation

$$\delta_v = \frac{k^2 - 3K}{4\pi G\bar{\rho}a^2}\varphi_\chi . \tag{4.186}$$

4.4.5 ΛCDM Universe

The Universe today is best described by a flat universe ($K = 0$) with a cosmological constant and cold dark matter. Here we study analytical solutions in ΛCDM universe.

• ***Background equations***.— A flat homogeneous universe with pressureless matter and a cosmological constant is governed by

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} , \qquad H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3} , \qquad \dot{H} = -4\pi G\rho_m = -\frac{3}{2}H^2\Omega_m(z) ,$$
$$\mathcal{H}' = -\frac{4\pi G}{3}a^2\rho_m - \frac{\Lambda}{3}a^2 , \qquad \Omega_m(z) := \frac{\rho_m}{\rho_c} = \frac{8\pi G\rho_m}{3H^2} , \qquad H^2\Omega_m(z) = \frac{8\pi G}{3}\rho_m . \tag{4.187}$$

The Einstein-de Sitter Universe is obtained by setting $\Lambda = 0$ and $\Omega_m(z) = 1$. In this section, we will use $\Omega_m \equiv \Omega_m(z)$.

• ***Perturbations***.— We will derive the solutions first in the comoving gauge and derive the relation to the conformal Newtonian gauge. The energy conservation and the energy constraint equations give

$$0 = \dot{\delta}_v - \kappa_v \,, \qquad H\kappa_v + 4\pi G\rho_m\delta_v = -\frac{1}{a^2}\Delta\varphi_v \equiv \delta R^{(h)} \quad \rightarrow \quad H\dot{\delta}_v + \frac{3}{2}H^2\Omega_m\delta_v = -\frac{1}{a^2}\Delta\varphi_v \,. \tag{4.188}$$

Using the background solution for H, the homogeneous solution for δ_v (where the RHS is set zero) can be readily derived as $\delta_v^{(h)} \propto H$. The homogeneous solution is the decaying mode, and the particular solution (or the growing mode) can be derived as

$$\delta_v^{(p)} = \delta_v^{(h)} \int \frac{dt}{\delta_v^{(h)}} \left(-\frac{\Delta\varphi_v}{a^2 H}\right) := -D\Delta\varphi_v \,, \qquad D := H \int \frac{dt}{\mathcal{H}^2} \,, \qquad \dot{\varphi}_v = 0 \,, \tag{4.189}$$

where the momentum constraint equation gives the conservation of the comoving gauge curvature and we defined the growth function D for the density.[4] This solution is identical to that in the Newtonian dynamics despite the presence of Λ, and it is indeed consistent with the general equation (4.177) with $w = c_s^2 = 0$.

If we define the logarithmic growth rate f,

$$f := \frac{d\ln D}{d\ln a} = \frac{1}{\mathcal{H}}\frac{d}{d\eta}\ln D = \frac{1}{H}\frac{d}{dt}\ln D \quad \rightarrow \quad \dot{D} = HfD \,, \tag{4.190}$$

the energy constraint equation can be re-arranged as

$$H^2 fD + \frac{3}{2}H^2\Omega_m D = \frac{1}{a^2} \quad \rightarrow \quad \therefore \; D = \frac{1}{\mathcal{H}^2 f\Sigma} \,, \qquad \Sigma := 1 + \frac{3}{2}\frac{\Omega_m}{f} \xrightarrow[\Omega_m=1]{} \frac{5}{2} \,, \qquad \delta_v = -\frac{\Delta\varphi_v}{\mathcal{H}^2 f\Sigma} \,. \tag{4.191}$$

The remaining perturbations are

$$\chi_v := a\beta_v \,, \qquad \kappa_v \equiv -\frac{\Delta}{a^2}\chi_v \equiv \dot{\delta}_v \equiv -\frac{\Delta\varphi_v}{a^2 H\Sigma} \,, \qquad \chi_v = \frac{\varphi_v}{H\Sigma} \,, \tag{4.192}$$

where we used $\dot{D} = HfD$.

• ***Newtonian correspondence***.— The velocity and the gravitational potential in the Newtonian dynamics correspond to the conformal Newtonian gauge quantities: $U^i = -v_\chi^{,i}$ and $\alpha_\chi = -\varphi_\chi$. The simplest way to derive the relations is the gauge transformation from the comoving gauge to the conformal Newtonian gauge:

$$\gamma_v = \gamma_\chi \equiv 0 \quad \rightarrow \quad L = 0 \,, \qquad v_\chi = 0 - T \,, \qquad \varphi_\chi = \varphi_v - \mathcal{H}T \,, \tag{4.193}$$

$$0 = \beta_\chi = \beta_v - T \qquad \rightarrow \qquad T = \beta_v = \frac{1}{a}\chi_v = \frac{\varphi_v}{\mathcal{H}\Sigma} \,, \tag{4.194}$$

such that we derive

$$v_\chi = -\beta_v = -\frac{1}{a}\chi_v = -\frac{\varphi_v}{\mathcal{H}\Sigma} \,, \qquad \varphi_\chi = \varphi_v + \mathcal{H}v_\chi = \left(1 - \frac{1}{\Sigma}\right)\varphi_v = \dot{\chi}_v \,, \tag{4.195}$$

where we used a useful relation in ΛCDM

$$1 \equiv \frac{1}{a}\left(\frac{a}{H\Sigma}\right)^{\cdot} = \frac{1}{\Sigma} + \left(\frac{1}{H\Sigma}\right)^{\cdot} \quad \rightarrow \quad 1 - \frac{1}{\Sigma} = \left(\frac{1}{H\Sigma}\right)^{\cdot} \,. \tag{4.196}$$

The remaining relations are then

$$\delta_v = \delta_\chi + 3\mathcal{H}v_\chi \,, \qquad \Delta\varphi_\chi = -\frac{3}{2}\mathcal{H}^2\Omega_m\delta_v \,, \qquad \Delta v_\chi = \delta_v' \,, \qquad \kappa_v = \dot{\delta}_v = \theta \,. \tag{4.197}$$

[4] Note that the solution D is unique with the boundary condition $D = 0$ at $t = 0$. So, the usual growth function $\hat{D}$ that is normalized today is then $\hat{D} := D(t)/D(t_0)$ and the density is $\delta_v(t) = \hat{D}(t)\delta_v(t_0)$.

4.4.6 Cosmological Gravitational Waves

In the absence of anisotropic pressure $\Pi^{(t)}_{\alpha\beta}$ in the tensor component, the cosmological gravitational waves $C^{(t)}_{\alpha\beta}$ propagate freely in an expanding universe. By decomposing the transverse traceless tensor $C^{(t)}_{\alpha\beta}$ in terms of two polarization basis $e^s_{\alpha\beta}(\mathbf{k})$, with $s=+,\times$, the propagation equation (4.144) becomes

$$\ddot{h}^s_{\mathbf{k}}+3H\dot{h}^s_{\mathbf{k}}-\frac{1}{a^2}\Delta h^s_{\mathbf{k}}=0\,,\qquad h^{s\prime\prime}_{\mathbf{k}}+2\mathcal{H}h^{s\prime}_{\mathbf{k}}-\Delta h^s_{\mathbf{k}}=0\,,\tag{4.198}$$

where we assumed $K=0$ and used

$$h^{(t)}_{\alpha\beta}=2C^{(t)}_{\alpha\beta}(\eta,\mathbf{k})\equiv e^+_{\alpha\beta}(\mathbf{k})h^+(\eta,\mathbf{k})+e^\times_{\alpha\beta}(\mathbf{k})h^\times(\eta,\mathbf{k})\,,\qquad e^s_{\alpha\beta}(\mathbf{k})e^{s'\alpha\beta}(\mathbf{k})=2\delta^{ss'}\,.\tag{4.199}$$

Mind the normalization convention for the polarization basis with $s=+,\times$. By change of variable $v^s_{\mathbf{k}}:=ah^s_{\mathbf{k}}$, the propagation equation becomes

$$(v^s_{\mathbf{k}})''+\left(k^2-\frac{a''}{a}\right)v^s_{\mathbf{k}}=0\,,\qquad \mathcal{H}^2+\mathcal{H}'=\frac{a''}{a}\,.\tag{4.200}$$

On large scales ($k^2\ll a''/a$), we can readily find the solution

$$\frac{1}{a}v^s_{\mathbf{k}}=h^s_{\mathbf{k}}=c^s_1(\mathbf{k})+c^s_2(\mathbf{k})\int\frac{dt}{a^3}\,,\tag{4.201}$$

where the first one is the growing mode and is constant on large scales. Furthermore, by assuming $\Lambda=0$ and a constant equation of state, the exact solution can be obtained in terms of Bessel functions as

$$h^s_{\mathbf{k}}=c_1\frac{J_\nu(k\eta)}{(k\eta)^\nu}+c_2\frac{Y_\nu(k\eta)}{(k\eta)^\nu}\,,\qquad \nu:=\frac{3(1-w)}{2(1+3w)}\,.\tag{4.202}$$

In the absence of parity violating process, two different polarization states behave statistically in the same way, and we can omit the superscript s. In RDE ($a\propto\eta$, $\nu=1/2$), two solutions are

$$(v^s_{\mathbf{k}})''+k^2v^s_{\mathbf{k}}=0\,,\qquad v^s_{\mathbf{k}}\propto\sin(k\eta)\,,\quad\cos(k\eta)\,,\qquad h^s_{\mathbf{k}}\propto\frac{1}{\eta}\sin(k\eta)\,,\quad\frac{1}{\eta}\cos(k\eta)\,.\tag{4.203}$$

In MDE ($a\propto\eta^2$; $\nu=3/2$), two solutions are the spherical Bessel functions

$$(v^s_{\mathbf{k}})''+\left(k^2-\frac{2}{\eta^2}\right)v^s_{\mathbf{k}}=0\,,\qquad v^s_{\mathbf{k}}\propto\eta j_1(k\eta)\,,\quad\eta y_1(k\eta)\,,\qquad h^s_{\mathbf{k}}\propto\frac{1}{\eta}j_1(k\eta)\,,\quad\frac{1}{\eta}y_1(k\eta)\,.\tag{4.204}$$

The second solution blows up at $k\to0$. Given the normalization convention, the total power spectrum of the cosmological gravitational waves is

$$P_T=2(P_{h^+}+P_{h^\times})=P_{h^{+2}}+P_{h^{-2}}\,,\tag{4.205}$$

where $P_{h^{\pm2}}$ is the power in the helicity basis.

5 Standard Inflationary Models

Standard single field inflationary models provide a mechanism for the inflationary expansion (horizon problem) and the perturbation generation (initial condition) by a single scalar field, called inflaton. The scalar field Lagrangian has the canonical kinetic term, but various single field models differ in the scalar field potential, according to which the inflaton rolls over. In most cases, the slow-roll condition is adopted, such that the scalar field dynamics is insensitive to the details of the scalar field potential.

The outcome of the standard model predictions is as follows: The curvature fluctuations are scale-invariant ($n_s \simeq 1$) and highly Gaussian. The tensor fluctuations are also scale-invariant, but its amplitude is very small compared to the scalar fluctuations. The running of the indices is very small. Recent observations confirm these predictions and constrain the parameters with high precision. However, beyond these basic features/constraints, we do not have a solid model for inflation. Note that the energy scale of inflation is beyond the validity of the standard model physics, and most inflationary models have many theoretical issues, when quantum corrections are considered.

5.1 Single Scalar Field

5.1.1 Scalar Field Action

In addition to the Einstein-Hilbert action for gravity, we consider the action for a scalar field with canonical kinetic term and the potential V:

$$S = \int \sqrt{-g}\, d^4x \left[\frac{c^4}{16\pi G}\, R - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi)\right] , \tag{5.1}$$

where the kinetic term in the Minkowski spacetime reduces to the standard form

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi = \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2\right] . \tag{5.2}$$

The Euler-Lagrange equation yields the equation of motion for the scalar field

$$\Box\phi - V_{,\phi} = 0 , \qquad \Box := g^{\mu\nu}\nabla_\mu\nabla_\nu , \tag{5.3}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\,\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\, \phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu} . \tag{5.4}$$

The equation of motion in the Minkowski spacetime reduces to the usual:

$$\ddot\phi - \nabla^2\phi + V_{,\phi} = 0 . \tag{5.5}$$

It is often in literature that the Planck unit is adopted, and there exist two different conventions:

$$M_{\rm pl}^2 := \frac{1}{8\pi G} , \qquad m_{\rm pl}^2 := \frac{1}{G} . \tag{5.6}$$

The recent analysis of the Planck CMB mission yields that the scalar fluctuation amplitude $A_s \simeq 2.1\times10^{-9}$ and hence the energy scale of the inflation is

$$A_s = \frac{H^2}{8\pi^2\varepsilon M_{\rm pl}^2} = 2.1\times10^{-9} , \qquad \therefore\ \ H = 4.1\times10^{-4}\sqrt{\varepsilon}M_{\rm pl} . \tag{5.7}$$

5.1.2 Background Relation and Evolution Equations

In the background, the non-vanishing fluid quantities for a scalar field are the energy density and the pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \,, \tag{5.8}$$

and accounting for the covariant derivative the equation of motion can be obtained as

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \,, \qquad \phi'' + 2\mathcal{H}\phi' + a^2 V_{,\phi} = 0 \,. \tag{5.9}$$

The Friedmann equation for a scalar field is

$$H^2 = \frac{\rho_\phi}{3M_{\rm pl}^2} \,, \qquad \dot{H} = -\frac{\rho_\phi + p_\phi}{2M_{\rm pl}^2} = -\frac{\dot{\phi}^2}{2M_{\rm pl}^2} \,, \qquad H^2 + \dot{H} = \frac{\ddot{a}}{a} = \frac{1}{3M_{\rm pl}^2}\left(V - \dot{\phi}^2\right) \,, \tag{5.10}$$

where we assumed a flat universe and no cosmological constant. If the potential energy of the scalar field is the dominant energy component of the Universe or the kinetic energy is smaller than the potential energy (slow-roll), the expansion of the Universe is accelerating $\ddot{a} > 0$. Various inflationary models with slow-roll condition state that the potential is sufficiently flat, such that $V(\phi)$ is nearly constant during the inflationary period and ϕ slowly evolves (rolls over V).

5.1.3 de-Sitter Spacetime

The de-Sitter universe is a highly symmetric spacetime, defined as a background FRW universe with no matter and constant Hubble parameter. A constant Hubble parameter leads to an exponential expansion, and we parametrize the de-Sitter solution as

$$H^2 := \frac{\Lambda}{3} \,, \qquad a(t) = e^{Ht} = -\frac{1}{H\eta} \,, \qquad a = (0, \infty) \,, \qquad t = (-\infty, \infty) \,, \qquad \eta = (-\infty, 0) \,, \tag{5.11}$$

where the scale factor is normalized at $t = 0$. The slow-roll parameter for the de-Sitter spacetime is

$$\varepsilon := -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right) = 0 \,. \tag{5.12}$$

5.1.4 Slow-Roll Parameters

In general, inflationary models slightly deviate from the de-Sitter phase ($\varepsilon \neq 0$), and its deviation is captured by the slow-roll parameter:

$$\varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot{H}}{H^2} \,, \qquad \dot{H} = -H^2\varepsilon \,, \qquad \frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1-\varepsilon) \,, \qquad (3-\varepsilon)H^2 = \frac{V}{M_{\rm pl}^2} \,. \tag{5.13}$$

To solve the horizon problem, we know that the comoving horizon has to decrease in time

$$0 > \frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) = -\frac{\ddot{a}}{a^2H^2} = -\frac{1-\varepsilon}{a} \,. \tag{5.14}$$

The background evolution of a scalar field can be re-phrased in terms of the slow-roll parameters as

$$\varepsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2 M_{\rm pl}^2} = \frac{3}{2}(1+w) \,, \qquad \dot{\phi}^2 = \rho_\phi + p_\phi \,. \tag{5.15}$$

If we ignore the second derivative of the field ($\ddot{\phi} \simeq 0$) in the equation of motion,

$$3H\dot{\phi} \simeq -V_{,\phi} \,, \qquad \rho_\phi + p_\phi \simeq \left(\frac{V_{,\phi}}{3H}\right)^2 \,, \tag{5.16}$$

the slow-roll parameters are then further related to the slow-roll parameters defined in terms of the derivatives of the potential also used below)

$$\varepsilon_V := \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \simeq \varepsilon \,, \qquad \eta_V := M_{\rm pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) \simeq \varepsilon+\eta \,, \qquad \xi_V := \frac{M_{\rm pl}^4 V_{,\phi}V_{,\phi\phi\phi}}{V^2} \,, \tag{5.17}$$

where we used the second slow-roll parameter

$$\eta := -\frac{\ddot\phi}{H\dot\phi} \,. \tag{5.18}$$

In fact, one can show the exact relation

$$\varepsilon = \varepsilon_V\left(1-\frac{4}{3}\varepsilon_V+\frac{2}{3}\eta_V\right) \,. \tag{5.19}$$

In literature, different convention for slow-roll parameters are often used, in particular, in terms of Hubble flow:

$$\varepsilon_1 := \varepsilon \,, \qquad \varepsilon_2 := \frac{1}{H}\frac{d\ln\varepsilon}{dt} = 2(\varepsilon-\eta) \,, \qquad \varepsilon_{i+1} := \frac{1}{H}\frac{d\ln\varepsilon_i}{dt} \,. \tag{5.20}$$

Furthermore, the inflation has to last for some time, such that the modes we measure in CMB have to expand at least by $40-60$ e-folds. So it is convenient to define the number of e-folding for a given mode as the number of e-folds the mode k expanded from the horizon crossing until the end of inflation,[1]

$$N(\phi_k) := \ln\frac{a_{\rm end}}{a(\phi_k)} = \int_{t_k}^{t_{\rm end}} H\,dt \,, \qquad k = aH \,, \tag{5.21}$$

where t_k is the time the k-mode crosses the horizon. Using the e-folding number, we can express the slow-roll parameters as

$$dN = Hdt = d\ln a \,, \qquad \varepsilon = -\frac{d\ln H}{dN} \,, \qquad \varepsilon_{i+1} = \frac{d\ln\varepsilon_i}{dN} \,. \tag{5.22}$$

5.1.5 Linear-Order Evolution

Given the energy momentum tensor, we can derive the fluid quantities for a scalar field:

$$\delta\rho_\phi = \dot\phi\dot{\delta\phi}-\dot\phi^2\alpha+V_{,\phi}\delta\phi = \delta\rho_v-3H\dot\phi\,\delta\phi \,, \qquad \delta\rho_v := \delta\rho-\rho' v \,, \tag{5.23}$$

$$\delta p_\phi = \dot\phi\dot{\delta\phi}-\dot\phi^2\alpha-V_{,\phi}\delta\phi = \delta\rho_v-3c_s^2H\dot\phi\,\delta\phi \,, \qquad v_\phi = \frac{\delta\phi}{\phi'} \,, \tag{5.24}$$

$$e := \delta p - c_s^2\,\delta\rho = (1-c_s^2)\delta\rho_v \,, \qquad \pi^\phi_{\alpha\beta} = q^\phi_\alpha = 0 \,, \tag{5.25}$$

where we used the following relation and the sound speed is defined as

$$\dot\rho_\phi = \dot\phi(\ddot\phi+V_{,\phi}) = -3H\dot\phi^2 \,, \qquad \dot p_\phi = \dot\phi(\ddot\phi-V_{,\phi}) = \dot\phi(2\ddot\phi+3H\dot\phi) \,, \qquad c_s^2 := \frac{\dot p_\phi}{\dot\rho_\phi} = -1-\frac{2\ddot\phi}{3H\dot\phi} \,. \tag{5.26}$$

Note tht the sound speed defined above is negative, and in particular $c_s^2 \simeq -1$ for slow-roll inflation. The entropy perturbation is gauge-invariant, and it is non-negligible for scalar field ($e \simeq 2\delta\rho_v$ for slow roll models). Therefore, the comoving gauge corresponds to the uniform field gauge for the single-field models:

$$\varphi_v = \varphi - \mathcal{H}v = \varphi - H\frac{\delta\phi}{\dot\phi} = \varphi_{\delta\phi} \,. \tag{5.27}$$

Other useful gauge-invariant variables are

$$\widetilde{\delta\phi} = \delta\phi-\phi' T \,, \qquad \delta\phi_\varphi := \delta\phi-\frac{\dot\phi}{H}\varphi \,, \qquad \delta\phi_\chi := \delta\phi-\dot\phi\chi \,. \tag{5.28}$$

[1]The end of inflation is a bit ill-defined, as we do not have a concrete model. However, in terms of N we can safely use the condition that the slow-roll parameter becomes order unity $\varepsilon \simeq 1$.

The equation of motion for a scalar field is then

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(V_{,\phi\phi} + \frac{k^2}{a^2}\right)\delta\phi = \dot{\phi}(\dot{\alpha} + \kappa) + (2\ddot{\phi} + 3H\dot{\phi})\alpha\ . \tag{5.29}$$

There exists a very important conservation law on super horizon scales. First, we define a gauge-invariant variable Φ, which is essentially the comoving-gauge curvature:

$$\Phi := \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)}\varphi_\chi = \frac{H^2}{4\pi G(\rho+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{2H^2\Pi}{\rho+p}\ , \tag{5.30}$$

where the last equality can be readily verified by using Eq. (4.163). Taking the time derivative of the definition of Φ and using the Einstein equations (4.166) and (4.163) to remove $\dot{\varphi}_\chi$ and $\dot{v}_\chi$, we derive the governing equation for

$$\dot{\Phi} = \frac{H}{4\pi G(\rho+p)}\frac{c_s^2}{a^2}\Delta\varphi_\chi - \frac{H}{\rho+p}\left(e + \frac{2}{3a^2}\Delta\Pi\right)\ , \tag{5.31}$$

where the sound speed is $c_s^2 := \dot{p}/\dot{\rho}$ and the entropy perturbation is $\delta p =: c_s^2\delta\rho + e$. The derivation is fully based on the Einstein equation (no conservation equation at the perturbation level), so that the fluid components are for the total energy-momentum tensor. For the inflaton field with $\Pi = 0$ and $e = (1-c_s^2)\delta\rho_v$, we define the physical sound speed c_A for inflaton

$$\dot{\Phi} = \frac{H}{4\pi G(\rho+p)}\frac{c_A^2}{a^2}\Delta\varphi_\chi\ , \qquad c_A^2\Delta\varphi_\chi := c_s^2\Delta\varphi_\chi - 4\pi Ga^2(1-c_s^2)\delta\rho_v = \Delta\left[1 + (1-c_s^2)\frac{3K}{\Delta^{-1}}\right]\varphi_\chi\ , \tag{5.32}$$

so that the physical sound speed for the inflaton is $c_A \equiv 1$ in a flat universe. In other words, when the scalar field is treated as a fluid, c_A appears in the fluid equations as the proper sound speed, instead of c_s^2, and it is a relativistic object. It is clear that the comoving-gauge curvature is conserved on super horizon scales.

5.2 Quantum Fluctuations in Quadratic Action

The background relation describes the inflationary expansion, and the equation of motion we derived describes the evolution of the perturbations at the linear order. Here we will derive their statistical properties. However, before we proceed, we need to better understand the structure of the theory. Even for the standard inflationary models of a single field, the theory is not a free-field, but an interacting field theory.

This can be illustrated as follows. To simplify the calculations, we choose the comoving gauge

$$0 = v_\phi = \frac{\delta\phi}{\phi'}\ , \qquad \phi(x) = \bar{\phi}(t)\ , \qquad \zeta := \varphi_v = \varphi_{\delta\phi}\ , \tag{5.33}$$

and it coincides with the uniform field gauge. Our main variable for scalar fluctuation is then the comoving gauge curvature ζ, as the scalar field is uniform. We can expand the action perturbatively to give

$$S = S_0[\bar{\phi}, \bar{g}_{ab}] + S_2[\zeta^2] + S_3[\zeta^3] + \cdots\ , \qquad H = H_0 + H_{\rm int}\ , \qquad H_{\rm int} = \sum_i F_i(\varepsilon, \eta, \cdots)\zeta^3(\tau) + \cdots\ , \tag{5.34}$$

where the background action S_0 defines the background evolution and its slow-roll parameters. Here we will study the quadratic action S_2 in great detail to derive the power spectrum of the scalar and tensor fluctuations, and the quadratic action is indeed a free-field action in the de-Sitter background (or with small deviations around it). However, remember that the full theory is interacting, and we cannot use the free-field theory to quantize the fluctuations, if we go beyond the quadratic action or compute the high-order correlation functions.

5.2.1 Quadratic Action for Scalars

To derive the linear-order equation of motion, we need to expand the action to the quadratic in perturbations. To simplify the calculations, we choose the comoving gauge. After some integrations by part of the quadratic action, the quadratic action for scalars in the comoving gauge becomes[2]

$$S_{(2)} = \frac{1}{2}\int dt\, d^3\mathbf{x}\;\, a^3\frac{\dot\phi^2}{H^2}\left[\dot\zeta^2 - \frac{1}{a^2}(\nabla\zeta)^2\right] = \frac{1}{2}\int d\eta\, d^3\mathbf{x}\;\left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]\,, \tag{5.35}$$

where we integrate by part, $M_{\rm pl} = 1$, and we defined the canonically-normalized (Mukhanov-Sasaki) variable

$$v := z\zeta\,, \qquad \zeta := \varphi_v\,, \qquad z^2 := a^2\frac{\dot\phi^2}{H^2} = 2a^2\varepsilon\,. \tag{5.36}$$

At the quadratic action, scalar, vector, and tensor do not mix, while they do mix in general quadratic terms. In the action, their indices need to be contracted, e.g., $\phi_{i|j}h^{ij}$, and an integration by parts yields vanishing contribution due to the divergence free condition for vector and tensor contributions. The Lagrangian now takes the form of the simple harmonic oscillator, but with time-dependent mass term

$$m^2(\eta) := -\frac{z''}{z} \quad \underset{\rm dS}{\longrightarrow} \quad -\frac{a''}{a} = -\frac{2}{\eta^2}\,. \tag{5.37}$$

where we took the de-Sitter limit ($\varepsilon = z = 0$). The canonical momentum and the Hamiltonian are then

$$\pi = \frac{\delta\mathcal{L}}{\delta v'} = v'\,, \qquad \mathcal{H} = \pi v' - \mathcal{L} = \frac{1}{2}\left[(v')^2 + (\nabla v)^2 + m^2v^2\right]\,. \tag{5.38}$$

The equation of motion for the Mukhanov-Sasaki variable is the Klein-Gordon equation:

$$\left(\Box - m^2\right)v = 0\,, \qquad v''_{\mathbf{k}} + \omega_k^2 v_{\mathbf{k}} = 0\,, \qquad \omega_k^2 := k^2 + m^2\,, \qquad v_{\mathbf{k}} = v^*_{-\mathbf{k}}\,. \tag{5.39}$$

The mode functions take the simple solution for the time-dependence under the assumption that $\omega_k \simeq k$ is time-independent in the limit $\eta \to -\infty$:

$$v_{\mathbf{k}}(\eta) \equiv v^+_{\mathbf{k}}e^{i\omega_k\eta} + v^-_{\mathbf{k}}e^{-i\omega_k\eta} := v^+_{\mathbf{k}}(\eta) + v^-_{\mathbf{k}}(\eta)\,, \qquad v^+_{\mathbf{k}} = (v^-_{-\mathbf{k}})^\dagger\,, \tag{5.40}$$

where the amplitude of the mode functions are undetermined. Therefore, the general solution can be written as

$$v(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_{\mathbf{k}}e^{i\omega_k\eta} + v^-_{\mathbf{k}}e^{-i\omega_k\eta}\right)e^{i\mathbf{k}\cdot\mathbf{x}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_{-\mathbf{k}}\,e^{-ikx} + v^-_{\mathbf{k}}e^{ikx}\right)\,, \qquad k := (\omega_k, \mathbf{k})\,. \tag{5.41}$$

5.2.2 Canonical Quantization

So far, we have derived a classical solution of the quadratic action for scalars. By promoting the Mukhanov-Sasaki field v and its canonical momentum field π to quantum fields, we need to impose the canonical quantization relation ($\hbar = 1$)

$$[\hat v(\eta,\mathbf{x}), \hat\pi(\eta,\mathbf{y})] = i\delta^{\rm 3D}(\mathbf{x}-\mathbf{y})\,, \qquad [\hat v(\eta,\mathbf{x}), \hat v(\eta,\mathbf{y})] = [\hat\pi(\eta,\mathbf{x}), \hat\pi(\eta,\mathbf{y})] = 0\,, \tag{5.42}$$

where we work in the Heisenberg picture for the time-dependent operators. Apparent from the notation, we want to define the creation and annihilation operators as

$$v^-_{\mathbf{k}} := \hat a_{\mathbf{k}}v^-_k\,, \qquad \left(v^-_{\mathbf{k}}\right)^\dagger = v^+_{-\mathbf{k}} = \hat a^\dagger_{\mathbf{k}}v^+_k\,, \qquad (v^-_k)^* = v^+_k\,, \tag{5.43}$$

such that we derive

$$\hat v(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\hat a^\dagger_{\mathbf{k}}v^+_k\;e^{-ikx} + \hat a_{\mathbf{k}}v^-_k\;e^{ikx}\right) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left[\hat a^\dagger_{\mathbf{k}}v^+_k(\eta)\;e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat a_{\mathbf{k}}v^-_k(\eta)\;e^{i\mathbf{k}\cdot\mathbf{x}}\right]\,, \tag{5.44}$$

where we defined

$$v^\pm_k(\eta) := v^\pm_k e^{\pm i\omega_k\eta}\,. \tag{5.45}$$

[2]Here, "scalars" are used to refer to the scalar fluctuations, not to be confused with the scalar field.

By substituting into the canonical quantization relation, we can derive that the ladder operators indeed satisfy the standard quantization relation at the equal time

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{3D}(\mathbf{k}-\mathbf{k}') , \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0 , \tag{5.46}$$

if the normalization for the mode functions is properly chosen

$$W[v_k^-, v_k^+] := v_k^- v_k^{+\prime} - v_k^{-\prime} v_k^+ := i . \tag{5.47}$$

With the properly normalized operators, we obtain the usual relations

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 , \qquad \langle 0|0\rangle = 1 , \qquad |n_{\mathbf{k}}\rangle = \sqrt{\frac{2E_k}{n!}}\,[(\hat{a}^\dagger_{\mathbf{k}})^n]|0\rangle , \tag{5.48}$$

where $\sqrt{2E}$ is put to make it Lorentz invariant. One can quantize the field, starting with the time-independent Harmonic oscillators, then applying the Heisenberg picture with the free-field Hamiltonian, as in Peskin & Schröder.

5.2.3 Vacuum Expectation Value

While we imposed the normalization condition for the mode functions $v_k^\pm(\eta)$ in terms of their Wronskian, the physical vacuum is yet to be fully determined, due to the arbitrariness in the mode functions. Note that we can change $v_k^\pm$ and $\hat{a}_k$ together, while $\hat{v}(x)$ remains unchanged. Consider a different set of mode functions $u_k^\pm$ that are related to the original mode functions as

$$u_k^-(\eta) = \alpha_k v_k^-(\eta) + \beta_k v_k^+(\eta) , \tag{5.49}$$

and construct the creation and annihilation operators $\hat{b}^\pm_{\mathbf{k}}$ with $u_k^\pm$

$$u^-_{\mathbf{k}} := \hat{b}_{\mathbf{k}} u_k^- . \tag{5.50}$$

Using this relation, we can write the operator $\hat{v}$ and its canonical momentum $\hat{\pi}$ in terms of $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$. These two sets of quantum operators are then related as by, so called, the Bogolyubov transformation:

$$\hat{a}_{\mathbf{k}} = \alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}^\dagger_{-\mathbf{k}} , \qquad \hat{a}^\dagger_{\mathbf{k}} = \alpha_k \hat{b}^\dagger_{\mathbf{k}} + \beta_k^* \hat{b}_{-\mathbf{k}} , \qquad |\alpha_k|^2 - |\beta_k|^2 = 1 , \tag{5.51}$$

where the normalization for the transformation coefficients is due to the Wronskian normalization. Note that the vacuum defined by one set of operators $\hat{a}_{\mathbf{k}}$ is not the vacuum with respect to the other set of operators $\hat{b}_{\mathbf{k}}$. To properly determine the physical vacuum, we need to fix the mode function completely.

In terms of the mode functions, the Hamiltonian in Minkowski spacetime is

$$\hat{H} = \int d^3\mathbf{x}\, \hat{\mathcal{H}} , \qquad \hat{\mathcal{H}} = \frac{1}{2}\left[\hat{\pi}^2 + (\nabla\hat{v})^2\right] , \qquad m \to 0 . \tag{5.52}$$

Using the expression for the mode function in Eq. (5.44), we derive the Hamiltonian

$$\hat{H} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\left((v_k^{+\prime})^2 + k^2(v_k^+)^2\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \left((v_k^{-\prime})^2 + k^2(v_k^-)^2\right)\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right)\left(|v_k^{-\prime}|^2 + k^2|v_k^-|^2\right)\right] , \tag{5.53}$$

acting on the vacuum $|0\rangle$ as

$$\hat{H}|0\rangle = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\left((v_k^{+\prime})^2 + k^2(v_k^+)^2\right)\hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}} + \left(|v_k^{-\prime}|^2 + k^2|v_k^-|^2\right)(2\pi)^3\delta^{3D}(0)\right]|0\rangle . \tag{5.54}$$

The vacuum $|0\rangle$ is an eigenstate of the Hamiltonian, and indeed the first round bracket vanishes. The remaining term in the Hamiltonian should be minimized by a proper choice of the mode function. Given the normalization of the Wronskian and the time dependence of the mode function, the physical mode function is then found to be[3]

$$W[v_k^-, v_k^+] = 2ik|v_k^-|^2 = i , \qquad v_k^-(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} . \tag{5.55}$$

Therefore, we derive the vacuum expectation values

$$\left\langle 0|\hat{v}^\dagger_{\mathbf{k}}\hat{v}_{\mathbf{k}'}|0\right\rangle = (2\pi)^3\delta^{3D}(\mathbf{k}-\mathbf{k}')P_v(k) , \qquad P_v(k) = |v_k^-|^2 = \frac{1}{2k} . \tag{5.56}$$

[3]In fact, given the Bogolyubov transformation, one cannot use the time-dependence $\pm i\omega_k\eta$ to find the physical mode function, rather one has to find a general solution with $v_k^- = A_k e^{iB_k\eta}$.

5.2.4 Scalar Fluctuations

Now we consider the time-dependent mass term in the equation of motion, and following the same procedure we pick the vacuum that corresponds to the solution in the Minkowski spacetime as the modes were deep inside the horizon in the far past

$$v_k(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}}\, e^{-i\omega_k(\eta)\eta} \,, \qquad \lim_{\eta\to-\infty} v_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \,, \tag{5.57}$$

and this choice of the mode function is called the Bunch-Davis vacuum. Note that with time-dependent mass term (or spacetime) the vacuum defined as the minimum of the Hamiltonian is also evolving in time, i.e., the vacuum state a moment ago is not a vacuum, but a state of particles.

To the zero-th order in the slow-roll approximation ($\varepsilon = 0$), the inflationary period is the de-Sitter spacetime, in which

$$m^2(\eta) = -\frac{a''}{a} = -\frac{2}{\eta^2} \,, \qquad \omega_k^2 = k^2 - \frac{2}{\eta^2} \,, \tag{5.58}$$

and we can derive the exact solution for the mode functions:

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) . \tag{5.59}$$

When the k-mode is stretched beyond the horizon, the amplitude of the mode function is

$$\lim_{k\eta\to0} v_k(\eta) = \frac{1}{i\sqrt{2}}\frac{1}{k^{3/2}\eta} \,, \qquad \lim_{k\eta\to0} k^3|v_k|^2 = \frac{1}{2\eta^2} = \frac{a^2H^2}{2} \,, \tag{5.60}$$

and the power spectra of the mode function and the comoving-gauge curvature are

$$P_v \equiv |v_k|^2 = \frac{a^2H^2}{2k^3} \,, \qquad \Delta_\zeta^2 := \frac{k^3}{2\pi^2}P_\zeta = \frac{1}{2a^2\varepsilon}\Delta_v^2 = \frac{H^2}{8\pi^2\varepsilon} \,. \tag{5.61}$$

5.2.5 Tensor Fluctuations: Gravity Waves

We can repeat the exercise for the scalar fluctuations to derive the tensor fluctuations. The tensor perturbations are decomposed in terms of two helicity eigenstates as

$$h_{ij} := 2C_{ij}^{(t)} = 2h^{(\pm2)}Q_{ij}^{(\pm2)} \,, \tag{5.62}$$

and the quadratic action for tensor is

$$S_{(2)} = \frac{M_{\rm pl}^2}{8}\int d\eta\, d^3\mathbf{x}\, a^2\left[(h'_{ij})^2 - (\nabla h_{ij})^2\right] = \sum_{s=\pm2}\int d\eta\, d^3\mathbf{k}\,\frac{a^2}{4}\,M_{\rm pl}^2\Big[(h_{\mathbf{k}}^{s\prime})^2 - k^2(h_{\mathbf{k}}^s)^2\Big] \,. \tag{5.63}$$

Using the Mukhanov-Sasaki variable for tensor fluctuations, the quadratic action is

$$v_{\mathbf{k}}^s := \frac{a}{2}M_{\rm pl}h_{\mathbf{k}}^s \,, \qquad S_{(2)} = \sum_{s=\pm2}\frac{1}{2}\int d\eta\, d^3\mathbf{k}\left[(v_{\mathbf{k}}^{s\prime})^2 - k^2(v_{\mathbf{k}}^s)^2 + \frac{a''}{a}(v_{\mathbf{k}}^s)^2\right] , \qquad m^2 = -\frac{a''}{a} \,, \tag{5.64}$$

where we integrate by part and used $\mathcal{H}^2 + \mathcal{H}' = a''/a$. The quadratic action for tensor is identical, and we can readily derive the tensor power spectrum

$$P_v = \frac{(aH)^2}{2k^3} \,, \qquad P_T := 2P_{h_k^s} = 2\left(\frac{2}{aM_{\rm pl}}\right)^2 P_v = \frac{4}{k^3}\frac{H^2}{M_{\rm pl}^2} \,. \tag{5.65}$$

The amplitude of the tensor power spectrum is the energy scale of the inflation in the early Universe, and its ratio to the scalar power spectrum is

$$r := \frac{\Delta_t^2}{\Delta_s^2} = \frac{8}{M_{\rm pl}^2}\frac{\dot\phi^2}{H^2} = 16\varepsilon \,, \tag{5.66}$$

slow-roll suppressed. Note that GR is a proper low-energy EFT of quantum gravity and hence there are no issues in quantizing gravity in the standard QFT as above. The problems arise only when the energy scale approaches the Planck scale.

5.3 Predictions of the Standard Inflationary Models

5.3.1 Consistency Relations

For the standard single field inflationary models with the slow-roll approximation, we summarize the predictions for scalar fluctuations

$$P_\zeta = \left(\frac{2\pi^2}{k^3}\right) A_s \,, \qquad A_s := \frac{H^2}{8\pi^2\varepsilon M_{\rm pl}^2} = \frac{1}{24\pi^2\varepsilon}\frac{V}{M_{\rm pl}^4} \,, \tag{5.67}$$

$$n_s - 1 := \frac{d\ln k^3 P_\zeta}{d\ln k} = (-2\varepsilon - \varepsilon_2)(1-\varepsilon)^{-1} \simeq 2\eta_V - 6\varepsilon_V \,, \tag{5.68}$$

the predictions for tensor fluctuations

$$P_T = \frac{4}{k^3}\frac{H_*^2}{M_{\rm pl}^2} = \left(\frac{2\pi^2}{k^3}\right) A_T \,, \qquad A_T := \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2} = \frac{2V}{3\pi^2 M_{\rm pl}^4} \,, \qquad n_t := \frac{d\ln k^3 P_T}{d\ln k} \simeq -2\varepsilon \,, \tag{5.69}$$

and the consistency relations

$$r := \frac{A_T}{A_s} = \frac{8\dot\phi_*^2}{H_*^2} = 16\varepsilon = -8n_t \,. \tag{5.70}$$

By measuring the power spectrum amplitude and its slope for both scalar and tensor fluctuations, we can ensure that the fluctuations are indeed generated by a single field inflaton or rule out the standard inflationary models. There exist other predictions in the standard inflationary models (and of course, for the beyond the standard models) that can be used to test models, such as the primordial non-Gaussianity and so on.

5.3.2 Lyth Bound

Given the definition of the e-folds, we can further manipulate it by using the inflaton as a time clock:

$$N(\phi_k) = \int_{\phi_k}^{\phi_{\rm end}} d\phi\, \frac{H}{\dot\phi} = \int_{\phi_k}^{\phi_{\rm end}} \frac{d\phi}{M_{\rm pl}\sqrt{2\varepsilon}} \,, \qquad r = 16\varepsilon = \frac{8}{M_{\rm pl}^2}\left(\frac{d\phi}{dN}\right)^2 \,, \tag{5.71}$$

and this relation further implies that the excursion of the inflaton field is related to the tensor-to-scalar ratio as

$$\frac{\Delta\phi_k}{M_{\rm pl}} \simeq \int_{N_{\rm end}}^{N_{\rm cmb}} dN\, \sqrt{\frac{r}{8}} \,, \tag{5.72}$$

where $\varepsilon(\phi_{\rm end}) \equiv 1$. To solve the horizon problem, the mode k should have expanded at least $40-60$ in e-folds. So, this consistency relation (Lyth, 1997) implies that an inflationary field variation of the order of the Planck mass is needed to produce $r > 0.01$. From the theoretical point of view, this sets the upper bound on the amplitude of gravitational waves. Indeed, the standard inflationary model predictions are very small.

Note that the uncertainty in e-folds N is due to our ignorance in the reheating era: After the inflationary period ends, the inflaton field decays into other particles and reheats the Universe. This period is expected to be described by a matter-dominated era, as the inflaton oscillates around the minimum of the potential, effectively acting as a matter. However, we know very little about this period.

The current observational constraint is

$$A_s \simeq 2.2\times 10^{-9} \,, \qquad n_s \simeq 0.96 \,, \qquad \varepsilon \simeq 0.01 \,. \tag{5.73}$$

indicating the energy scale of the inflation is

$$A_T = \frac{2V}{3\pi^2 M_{\rm pl}^4} = 16\varepsilon A_s \,, \qquad H^2 = \frac{V}{3M_{\rm pl}^2} = \varepsilon\left(2\times 10^{14}\ \text{GeV}\right)^2 \,. \tag{5.74}$$

5.3.3 A Worked Example

Here we consider a very simple inflationary model with a power-law potential:

$$V = \frac{1}{2} m^{4-\alpha} \phi^\alpha \,, \tag{5.75}$$

where the mass m and the slope α are the free parameters of the model. It chaotically starts everywhere at any time in field configurations, and its predictions are then

$$\varepsilon_V = \frac{\alpha^2}{2} \left(\frac{M_{\rm pl}}{\phi}\right)^2 \,, \qquad \eta_V = \alpha(\alpha-1)\left(\frac{M_{\rm pl}}{\phi}\right)^2 \,, \tag{5.76}$$

$$N \simeq \int \frac{d\phi}{M_{\rm pl}^2}\frac{V}{V'} = \frac{\phi^2 - \phi^2_{\rm end}}{2M^2_{\rm pl}\alpha} \,, \qquad r \simeq 16\varepsilon_V \,, \qquad n_s - 1 \simeq 2\eta_V - 6\varepsilon_V \,. \tag{5.77}$$

Approximating $\phi_{\rm end} \simeq 0$, we further derive

$$N \simeq \frac{1}{2\alpha}\left(\frac{\phi}{M_{\rm pl}}\right)^2 \,, \qquad \varepsilon_V \simeq \frac{\alpha}{4N} \,, \qquad \eta_V = \frac{\alpha - 1}{2N} \,, \qquad 1 - n_s \simeq \frac{\alpha+2}{2N} \,, \qquad r \simeq \frac{4\alpha}{N} \,. \tag{5.78}$$

5.4 Adiabatic Modes and Isocurvature Modes

• ***Adiabatic modes***.— Assuming a flat Universe, we can arrange Eq. (5.31) to show

$$\dot\varphi_v = \Xi - \frac{H}{\rho+p}\frac{k^2}{a^2}\left(\frac{c_s^2}{4\pi G}\,\varphi_\chi - \frac{2}{3}\Pi\right) \,, \qquad \Xi := \frac{\dot{\bar\rho}\,\delta p - \dot{\bar p}\,\delta\rho}{3(\bar\rho + \bar p)^2} \equiv -\frac{He}{\rho+p} \,, \tag{5.79}$$

where the entropy perturbation is gauge invariant at the linear order

$$e := \delta p - c_s^2 \delta\rho \,. \tag{5.80}$$

If the pressure of a fluid is just a function of the density, it satisfies the adiabatic condition ($\Xi \equiv e \equiv 0$):

$$p = p(\rho) = p(\bar\rho) + \left.\frac{dp}{d\rho}\right|_0 \delta\rho + \cdots = \bar p + c_s^2 \delta\rho + \cdots \,, \qquad c_s^2 := \frac{\dot{\bar p}}{\dot{\bar\rho}} \,, \qquad e := 0 \,. \tag{5.81}$$

Therefore, in the limit $k \to 0$, if $\Xi = 0$ vanishes, the comoving-gauge curvature perturbation is conserved, regardless of contents in the Universe.

$$\lim_{k\to 0} \varphi_v = \text{constant in time} \quad \text{if} \;\; \Xi = 0 \,. \tag{5.82}$$

Indeed, the adiabatic condition $\Xi = 0$ is satisfied for the matter-dominated era, the radiation-dominated era, and for the single field inflation,[4] which are essentially cases with a single fluid.

For multi-fluid cases, the adiabatic condition can be imposed for individual components, *fluctuating at the same rate at a given point*:

$$\frac{\delta\rho_i}{\dot{\bar\rho}_i} = \frac{\delta\rho_{\rm tot}}{\dot{\bar\rho}_{\rm tot}} = \frac{\delta p_i}{\dot{\bar p}_i} = \frac{\delta p_{\rm tot}}{\dot{\bar p}_{\rm tot}} =: -\varphi_v\,\mathcal{I} \,, \qquad \frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \quad \text{for} \,\forall\, a, b \,, \tag{5.83}$$

and in the limit $k \to 0$ we can indeed derive the adiabatic condition

$$\mathcal{I} := \frac{1}{a}\int_{t_i}^t dt\, a(t) \,, \qquad v_\chi \equiv -\frac{1}{a}\mathcal{I}\,\varphi_v \,. \tag{5.84}$$

This is a non-trivial condition, as opposed to the single-fluid case. For example, consider radiation and matter components:

$$\dot{\bar\rho}_\gamma = -4H\bar\rho_\gamma \,, \qquad \dot{\bar\rho}_m = -3H\bar\rho_m \,, \qquad \Xi = \frac{H\bar\rho_m\bar\rho_\gamma}{(3\bar\rho_m + 4\bar\rho_\gamma)^2}(4\delta_m - 3\delta_\gamma) \,, \tag{5.85}$$

[4] It vanishes only in the limit $k = 0$ for single field models.

such that φ_v is conserved, only when the adiabatic condition

$$\delta_m = \frac{\delta_\gamma}{1+1/3} \tag{5.86}$$

is satisfied in the limit $k \to 0$. Even for single-field inflationary scenarios, there should have existed many other matter fields, and some energy transfer to these fields are inevitable. However, these non-adiabatic perturbations decay fast as the inflation proceeds, and they become exponentially suppressed when these matter fields dominate the energy budget during the reheating era. For adiabatic case, the curvature fluctuations for each fluid are identical:

$$\varphi_{\delta_a} := \varphi + \frac{\delta_a}{3(1+w_a)} = \varphi_{\delta_b} \,, \quad \text{for} \,\forall\, a,\ b \,. \tag{5.87}$$

• ***Isocurvature/entropy mode***.— Isocurvature perturbations represent non-adiabatic fluctuations that arise from a decay of a single source (or the inflaton). With the specific definition of $\mathcal{S}_{XY}$ below, one can set up the initial conditions for N-component fluids in terms of one adiabatic fluctuation and $N-1$ isocurvature fluctuations (no isocurvature fluctuations for a single component). Note, however, that this independent setup is valid only at the initial time. The evolution of isocurvature perturbations depends not only on inflationary dynamics, but also on post-inflationary evolution. For example, if all particles thermalize after inflation, all isocurvature perturbations become adiabatic perturbations eventually. The details are in CPT.pdf.

The isocurvature perturbations and the entropy perturbations are interchangeably used, because they do represent the perturbations between species and it does conserve the curvature. In practice, the entropy perturbations are parametrized by two free parameters at some pivot scale k_0 (0.002/Mpc in WMAP), i.e., ratio α of the isocurvature to the adiabatic perturbations in their amplitudes and their correlation β

$$\frac{P_\mathcal{S}}{P_\zeta} := \frac{\alpha}{1-\alpha} \,, \qquad\qquad \beta := \frac{P_{\mathcal{S}\zeta}}{\sqrt{P_\mathcal{S} P_\zeta}} \,, \tag{5.88}$$

where the relative entropy perturbation (or specific entropy) is defined as

$$\mathcal{S}_{XY} \equiv \delta\left(\frac{n_X}{n_Y}\right) \Big/ \left(\frac{n_X}{n_Y}\right) = \frac{\delta n_X}{n_X} - \frac{\delta n_Y}{n_Y} = \frac{\delta_X}{1+w_X} - \frac{\delta_Y}{1+w_Y} \,. \tag{5.89}$$

By defining the gauge-invariant curvature perturbation in the uniform-density gauge

$$\varphi_\delta = \varphi - H\frac{\delta\rho}{\dot\rho} = \varphi + \frac{\delta}{3(1+w)} \,, \tag{5.90}$$

we can readily show that the entropy perturbation is gauge invariant and conserved on large scales in the absence of mutual interactions.

$$\mathcal{S}_{XY} = 3\left(\varphi_\delta^X - \varphi_\delta^Y\right) \,. \tag{5.91}$$

In literature, it is often the case that the reference species is set for photons, and three extra components are considered such as baryon, cdm, neutrino (sometimes neutrino velocity) for isocurvature perturbations. In the most general case, we need to consider $\mathcal{S}_{X\gamma}$ with $X = b,\ \mathrm{cdm},\ \nu$ in addition to P_ζ, such that the initial power spectra are characterized by 4-4 matrix of auto and cross power spectra, each of which is described by the initial amplitude and the slope. A pure isocurvature model is ruled out, because the Sachs-Wolfe plateau is six times larger than in the adiabatic case and the contributions on small scales are further suppressed $\mathcal{T}_l \propto (k/k_\mathrm{eq})^{-2}$ in the isocurvature case.

6 Weak Gravitational Lensing

6.1 Gravitational Lensing by a Point Mass

In classical mechanics, the gravitational interaction due to a point mass M provides a perturbation along the transverse direction to a test particle moving with the relative speed $v_{\rm rel}$:

$$\Delta v_\perp = \frac{2GM}{b\, v_{\rm rel}} \,, \tag{6.1}$$

where G is the Newton's constant and b is the transverse separation (or the impact parameter). The prediction for the light deflection angle $\hat\alpha$ in Einstein's general relativity is well-known to follow the same result in classical mechanics, but with additional factor two:

$$\hat\alpha = \frac{4GM}{b\,c^2} = 8.155\times10^{-3}\text{ arcsec}\left(\frac{M}{M_\odot}\right)\left(\frac{b}{\rm AU}\right)^{-1} \,. \tag{6.2}$$

Given the deflection angle $\hat\alpha$, we can readily write down *the lens equation* in terms of the angular diameter distances

$$\mathcal{D}_s\hat s = \mathcal{D}_s\hat n - \mathcal{D}_{ls}\hat\alpha \,, \qquad\qquad \hat s = \hat n - \theta_E^2/\hat n \,, \tag{6.3}$$

where the Einstein radius is

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{\mathcal{D}_{ls}}{\mathcal{D}_l\mathcal{D}_s}} = 2.853\times10^{-3}\text{ arcsec}\left(\frac{M}{M_\odot}\right)\left(\frac{\mathcal{D}_l}{\rm kpc}\right)^{-1/2}\left(1-\frac{\mathcal{D}_l}{\mathcal{D}_s}\right)^{1/2} \,. \tag{6.4}$$

For a point mass lens and a point source, two lensed image positions are readily obtained as

$$\hat n_1 = \frac12\left(\hat s + \sqrt{\hat s^2 + 4\theta_E^2}\right) \,, \qquad \hat n_2 = \frac12\left(\hat s - \sqrt{\hat s^2 + 4\theta_E^2}\right) < 0 \,, \qquad \hat n_1 + \hat n_2 = \hat s \,, \tag{6.5}$$

and when the source and the lens are aligned, the lensed images form a ring with radius θ_E.

• microlensing, probe of MACHOs or exoplanets

6.2 Standard Weak Lensing Formalism

6.2.1 Lens Equation and Distortion Matrix

This light deflection due to a point mass can be generalized to derive the standard weak lensing formalism by considering the gravitational potential fluctuation $\psi = -GM/r$ of the general matter distribution ρ (but still a single lens plane), instead of a point mass (ψ indeed corresponds to the metric fluctuation α_χ). The lensing potential Φ is the line-of-sight integration of the metric fluctuation,

$$\Phi := \frac{1}{c^2}\frac{\mathcal{D}_{ls}}{\mathcal{D}_l\mathcal{D}_s}\int dz\; 2\psi \,, \tag{6.6}$$

and using the Poisson equation, we can relate the lensing potential with the surface density Σ as

$$\nabla^2\psi = 4\pi G\bar\rho a^2\delta \,, \qquad \hat\nabla^2\Phi = 2\frac{\Sigma}{\Sigma_c} \,, \qquad \Phi(\hat n) = \int d^2\hat n'\,\frac{\Sigma}{\pi\Sigma_c}\ln|\hat n - \hat n'| \,, \tag{6.7}$$

where we ignored the boundary term when the Poisson equation is integrated and the critical surface density is defined as

$$\Sigma_c^{-1} := \frac{4\pi G}{c^2}\frac{\mathcal{D}_{ls}\mathcal{D}_l}{\mathcal{D}_s} \,, \qquad \Sigma_c = 1.663\times10^6\; hM_\odot\text{ pc}^{-2}\left(\frac{\mathcal{D}_s}{\mathcal{D}_{ls}}\right)\left(\frac{\mathcal{D}_l}{h^{-1}\rm Mpc}\right)^{-1} \,. \tag{6.8}$$

Though the lensing potential is formally divergent for a point mass, its angular derivative is well defined:

$$\hat\alpha = \left(\frac{\mathcal{D}_l\mathcal{D}_s}{\mathcal{D}_{ls}}\right)\nabla_\perp\Phi = \frac{\mathcal{D}_s}{\mathcal{D}_{ls}}\hat\nabla\Phi \,, \tag{6.9}$$

such that the lens equation becomes

$$\hat{\boldsymbol{s}} = \hat{\boldsymbol{n}} - \hat{\nabla}\Phi \,, \tag{6.10}$$

where $\hat{\nabla}$ is the angular gradient. When the lensing material is distributed over the redshift, the lensing potential is then obtained by integrating the potential fluctuation over the line-of-sight distance as

$$\Phi = \int_0^{\bar{r}_s} d\bar{r} \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) 2\psi \equiv \int_0^{\bar{r}_s} d\bar{r}\, \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2}\, 2\psi \,, \tag{6.11}$$

where we switched to a comoving angular diameter distance $\bar{r}$ and we defined the weight function g (or lensing kernel) for later convenience

$$g := \bar{r}^2 \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) \,, \tag{6.12}$$

which peaks at the half the distance to $\bar{r}_s$ and vanishes at both ends $\bar{r} = 0$ and $\bar{r} = \bar{r}_s$. When the background source galaxies are also spread over some redshift with the distribution $n_g(r_s)$, the lensing potential can be readily generalized by replacing the weight function with

$$g := \bar{r}^2 \int_{\bar{r}}^{\infty} d\bar{r}_s \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) n_g(\bar{r}_s) \,, \qquad \Phi = \int_0^{\infty} d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, 2\psi \,, \qquad 1 = \int_0^{\infty} d\bar{r}_s\, n_g(\bar{r}_s) \,, \tag{6.13}$$

where the upper limit for the integration is indeed $\bar{r}(z = \infty)$ and the source distribution is normalized.[1]

Using the lens equation, the distortion matrix $\mathbb{D}$ (or sometimes called the amplification matrix) is defined as

$$\mathbb{D}_{ij} \equiv \frac{\partial s_i}{\partial n_j} = \mathbb{I}_{ij} - \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} , \qquad \Phi_{ij} := \hat{\nabla}_j \hat{\nabla}_i \Phi \,, \tag{6.14}$$

where $\mathbb{I}$ is the two-dimensional identity matrix and we defined a short hand notation for the angular derivatives of the lensing potential. The distortion matrix is conventionally decomposed into the trace, the traceless symmetric and the anti-symmetric matrices:

$$\mathbb{D} := \mathbb{I} - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} , \qquad \det \mathbb{D} = (1-\kappa)^2 - \gamma^2 + \omega^2 \,, \tag{6.15}$$

where the trace is the gravitational lensing convergence κ and the symmetric traceless part is the lensing shear $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$:

$$\kappa \equiv 1 - \frac{1}{2}\mathrm{Tr}\, \mathbb{D} = \frac{1}{2}\left(\Phi_{11} + \Phi_{22}\right) \,, \qquad \omega \equiv \frac{\mathbb{D}_{21} - \mathbb{D}_{12}}{2} = 0 \,, \tag{6.16}$$

$$\gamma_1 \equiv \frac{\mathbb{D}_{22} - \mathbb{D}_{11}}{2} = \frac{1}{2}\left(\Phi_{11} - \Phi_{22}\right) \,, \qquad \gamma_2 \equiv -\frac{\mathbb{D}_{12} + \mathbb{D}_{21}}{2} = \Phi_{12} = \Phi_{21} \,. \tag{6.17}$$

Since the distortion matrix in Eq. (6.14) is symmetric, the rotation ω vanishes in the standard formalism at all orders.

The standard lensing formalism is based on the lens equation and the lensing potential in Eq. (6.10). However, the source angular position $\hat{\boldsymbol{s}} := (\theta + \delta\theta, \phi + \delta\phi)$ is gauge-dependent, and the lensing potential that is responsible for the angular distortion $(\delta\theta, \delta\phi)$ is also gauge-dependent. Indeed, we already know that 2ψ in Eq. (6.10) should be $(\alpha_\chi - \varphi_\chi)$ to match the leading terms for $\delta\theta$ and the Poisson equation in Eq. (6.23) is indeed an Einstein equation with ψ there replaced by $-\varphi_\chi$.[2] Furthermore, there exist no contributions from the vector and the tensor perturbations in the standard lensing formalism. Finally, while the derivations in this subsection assume no linearity, all formulas of the standard lensing formalism turn out to be valid only at the linear order in perturbations.

[1] Sometimes it is normalized when integrated over redshift.

[2] Additional condition of a vanishing anisotropic pressure is needed to guarantee $\alpha_\chi = -\varphi_\chi$ and hence the consistency in the lensing equation.

6.2.2 Convergence and Shear

By using

$$\frac{1}{2}\Phi_{ij} \equiv \frac{1}{2}\hat{\nabla}_i\hat{\nabla}_j\Phi = \int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\hat{\nabla}_i\hat{\nabla}_j\psi\,, \tag{6.18}$$

we derive the individual components of the distortion matrix

$$\kappa = \int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\hat{\nabla}^2\psi\,, \qquad \gamma_1 = \int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\left(\hat{\nabla}_1^2-\hat{\nabla}_2^2\right)\psi\,, \qquad \gamma_2 = 2\int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\hat{\nabla}_1\hat{\nabla}_2\psi\,, \tag{6.19}$$

where the indices $i, j = 1, 2$ represent the angular components.

While the distortion matrix is defined in terms of angles, it is often assumed in literature that the line-of-sight direction is along z-axis ($\hat{\boldsymbol{n}}/\!/\hat{\boldsymbol{z}}$, i.e., $\theta = 0$), and two angles are aligned with x-y plane. In such a setting, consider two small angular vectors at the source position subtended respectively by $d\theta$ and $d\phi$ at the observer position

$$\Delta s_i^{d\theta} = \mathbb{D}_{i1}d\theta\,, \qquad \Delta s_i^{d\phi} = \mathbb{D}_{i2}d\phi\,. \tag{6.20}$$

The solid angle at the source subtended by these two angular vectors is then related to the solid angle at the observer as

$$d\Omega_s = \left|\Delta\boldsymbol{s}^{d\theta}\times\Delta\boldsymbol{s}^{d\phi}\right| = \det\mathbb{D}\;d\theta d\phi = \det\mathbb{D}\;d\Omega_o\,, \tag{6.21}$$

and hence the gravitational lensing magnification μ is then

$$\mu^{-1} \equiv \frac{d\Omega_s}{d\Omega_o} = \det\mathbb{D} = (1-\kappa)^2-\gamma^2+\omega^2 \simeq 1-2\kappa\,. \tag{6.22}$$

For this reason, the distortion matrix is often called the inverse magnification matrix.

Using the Poisson equation in cosmology,

$$\nabla^2\psi = 4\pi G\bar{\rho}a^2\delta_m = \frac{3H_0^2}{2}\Omega_m\frac{\delta_m}{a}\,, \tag{6.23}$$

the gravitational lensing convergence can be computed in terms of the matter density fluctuation δ_m in the comoving gauge as

$$\kappa = \int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\hat{\nabla}^2\psi = \frac{3H_0^2}{2}\Omega_m\int_0^\infty d\bar{r}\;g(\bar{r})\;\frac{\delta_m}{a}\,, \tag{6.24}$$

where we used the Laplacian

$$\nabla^2 = \frac{1}{\bar{r}^2}\frac{\partial}{\partial\bar{r}}\left(\bar{r}^2\frac{\partial}{\partial\bar{r}}\right)+\frac{1}{\bar{r}^2}\hat{\nabla}^2\,, \tag{6.25}$$

and ignored the boundary terms. The dominant contribution to the convergence arises at half the distance to the source due to the lensing kernel $g(\bar{r},\bar{r}_s)$.

6.2.3 Angular Decomposition and Power Spectrum

The convergence field is

$$\kappa(\hat{\mathbf{n}}) := \int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\;\hat{\nabla}^2\psi = \frac{3H_0^2}{2}\;\Omega_m\int_0^\infty d\bar{r}\;g(\bar{r})\;\frac{\delta_m}{a}\,, \tag{6.26}$$

and the convergence can be angular decomposed as

$$\begin{aligned}\kappa_{lm} &= \int d^2\hat{\mathbf{n}}\;Y_{lm}^*(\hat{\mathbf{n}})\int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\hat{\nabla}^2\left[\int\frac{d^3k}{(2\pi)^3}\,4\pi\sum_{LM}i^Lj_L(kr_l)Y_{LM}(\hat{\mathbf{n}})Y_{LM}^*(\hat{\boldsymbol{k}})T_\psi(k,r_l)\mathcal{R}(\mathbf{k})\right] \\ &= 4\pi i^l\int\frac{dk\,k^2}{2\pi^2}\left[-l(l+1)\int_0^\infty d\bar{r}\,\frac{g(\bar{r})}{\bar{r}^2}\;T_\psi(k,\bar{r})j_l(k\bar{r})\right]\int\frac{d\Omega_k}{4\pi}\;Y_{lm}^*(\hat{\boldsymbol{k}})\mathcal{R}(\mathbf{k})\,,\end{aligned} \tag{6.27}$$

or in terms of the matter fluctuation as

$$\kappa_{lm} = 4\pi i^l \int \frac{dk\, k^2}{2\pi^2} \left[\frac{3H_0^2}{2}\Omega_m \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{a}\, T_\delta(k,\bar{r}) j_l(k\bar{r})\right] \int \frac{d\Omega_k}{4\pi}\, Y^*_{lm}(\hat{\boldsymbol{k}})\mathcal{R}(\mathbf{k}) \,, \tag{6.28}$$

where the square bracket in the first line is just a Fourier transformation and we expanded the exponential in terms of plane waves. By defining the convergence transfer function

$$\mathcal{T}_l^\kappa(k) := -l(l+1) \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, T_\psi(k,\bar{r}) j_l(k\bar{r}) = \frac{3H_0^2}{2}\Omega_m \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{a}\, T_\delta(k,\bar{r}) j_l(k\bar{r}) \,, \tag{6.29}$$

the angular power spectrum is then obtained as

$$C_l = \frac{1}{2l+1}\sum_m |\kappa_{lm}|^2 = 4\pi \int d\ln k\, \Delta^2_{\mathcal{R}}(k) \left|\mathcal{T}_l^\kappa(k)\right|^2 . \tag{6.30}$$

Using the flat sky approximation (or the Limber approximation; and see LoVerde and Afshordi (2008) for the discussion)

$$j_l(x) \to \sqrt{\frac{\pi}{2l+1}}\, \delta^D\left(l + \frac{1}{2} - x\right) , \tag{6.31}$$

we can simplify the transfer function as

$$\mathcal{T}_l^\kappa(k) = -l(l+1) \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, T_\psi(k,\bar{r}) \times \frac{1}{k}\sqrt{\frac{\pi}{l+1/2}}\, \delta^D\left(\frac{l+1/2}{k} - \bar{r}\right) = -l(l+1)\sqrt{\frac{\pi}{l+1/2}}\, \frac{g(\bar{r})}{k\bar{r}^2}\, T_\psi(k,\bar{r}) \,, \tag{6.32}$$

where $k\bar{r} = l + 1/2$ should be imposed. With the other transfer function manipulated in terms of $\delta^D(k)$ as

$$\mathcal{T}_l^\kappa(k) = -l(l+1) \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, T_\psi(k,\bar{r}) \times \frac{1}{\bar{r}}\sqrt{\frac{\pi}{l+1/2}}\, \delta^D\left(\frac{l+1/2}{\bar{r}} - k\right) , \tag{6.33}$$

the angular power spectrum can be simplified as

$$C_l = l^2(l+1)^2 \int d\bar{r} \left(\frac{g^2(\bar{r})}{\bar{r}^6}\right) P_\psi\left(k = \frac{l+1/2}{\bar{r}}, \bar{r}\right) . \tag{6.34}$$

Using the Poisson equation

$$P_\psi = \left(\frac{3H_0^2}{2}\Omega_m\right)^2 \frac{P_m}{a^2k^4} = \left(\frac{3H_0^2}{2}\Omega_m\right)^2 \frac{P_m}{a^2}\left(\frac{\bar{r}}{l+1/2}\right)^4 , \qquad kr = l + \frac{1}{2} \,, \tag{6.35}$$

the angular power spectrum in terms of the matter density fluctuation is

$$C_l = \left(\frac{2^4 l^2(l+1)^2}{(2l+1)^4}\right)\left(\frac{3H_0^2}{2}\Omega_m\right)^2 \int d\bar{r} \left(\frac{g^2(\bar{r})}{a^2\bar{r}^2}\right) P_m\left(k = \frac{l+1/2}{r}, \bar{r}\right) , \tag{6.36}$$

where the whole round bracket in front with l is approximately one.

6.2.4 Flat-Sky Computation

Assuming that the survey area is small, we will utilize the angular Fourier transformation in Eq. (3.17) by again computing

$$\begin{aligned}
\Phi(\mathbf{l}) &= \int d^2\theta\, e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, 2\psi = \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2} \int d^2\theta \int \frac{dk_\parallel}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{ik_\parallel\bar{r}}\, e^{i(\boldsymbol{k}_\perp - \mathbf{l}/\bar{r})\cdot\bar{r}\hat{\mathbf{n}}}\, 2\psi(k,\bar{r}) \\
&= \int_0^\infty d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^4} \int \frac{dk_\parallel}{2\pi}\, 2\psi\left(k_\perp = \frac{l}{\bar{r}}\right) e^{ik_\parallel\bar{r}} \,, \qquad \bar{r}_\perp = \bar{r}\hat{\mathbf{n}} \,,
\end{aligned} \tag{6.37}$$

such that the lensing observables are

$$\kappa(\mathbf{l}) = -\frac{l^2}{2}\Phi(l) \,, \qquad \gamma_1(\mathbf{l}) = -\frac{l_1^2 - l_2^2}{2}\,\Phi(l) = \cos 2\phi_l\,\kappa(\mathbf{l}) \,, \qquad \gamma_2(\mathbf{l}) = -l_1 l_2\,\Phi(l) = \sin 2\phi_l\,\kappa(\mathbf{l}) \,, \tag{6.38}$$

and the angular power spectra can be readily derived as

$$P_\kappa(l) = \frac{l^4}{4}P_\Phi(l) \,, \qquad P_{\gamma_1} = \cos^2 2\phi_l\, P_\kappa(l) \,, \qquad P_{\gamma_2} = \sin^2 2\phi_l\, P_\kappa(l) \,, \tag{6.39}$$

where we used $\mathbf{l} = (l_1, l_2) = l(\cos\phi_l, \sin\phi_l)$. By computing

$$\langle \Phi(l_1)\Phi(l_2)\rangle = \int_0^\infty d\bar r_1\,\frac{g(\bar r_1)}{\bar r_1^4}\int_0^\infty d\bar r_2\,\frac{g(\bar r_2)}{\bar r_2^4}\int\frac{dk_\parallel}{2\pi}\,4P_\psi\left(k_\perp = \frac{l_1}{\bar r_1}\right)e^{ik_\parallel(\bar r_1 - \bar r_2)}(2\pi)^2\delta^D\left(\frac{\mathbf{l}_1}{\bar r_1} + \frac{\mathbf{l}_2}{\bar r_2}\right) \,, \tag{6.40}$$

we obtain the angular power spectrum of the lensing potential as

$$P_\Phi(l) = \int_0^\infty d\bar r\,\frac{g^2(\bar r)}{\bar r^6}\,4P_\psi\left(k_\perp = \frac{l}{\bar r}\right) \,, \tag{6.41}$$

where the line-of-sight integration over $k_\parallel$ gives rise to another Dirac delta function.

From the relation of the lensing observables, we find it useful to construct E and B-modes as

$$E(\mathbf{l}) := \cos 2\phi_l\,\gamma_1(\mathbf{l}) + \sin 2\phi_l\,\gamma_2(\mathbf{l}) \,, \qquad B(\mathbf{l}) := -\sin 2\phi_l\,\gamma_1(\mathbf{l}) + \cos 2\phi_l\,\gamma_2(\mathbf{l}) \,. \tag{6.42}$$

We can readily derive

$$E(\mathbf{l}) = \kappa(\mathbf{l}) \,, \qquad B(\mathbf{l}) = 0 \,, \qquad P_E(l) = P_\kappa(l) \,, \qquad P_B(l) = P_{EB}(l) = 0 \,, \tag{6.43}$$

in the absence of any systematics and/or physics other than the gravitational lensing, such that it provides a consistency check of the measurements, where the convergence power spectrum is again related to the matter power spectrum as

$$P_\kappa(l) = \left(\frac{3H_0^2}{2}\Omega_m\right)^2\int_0^\infty d\bar r\,\frac{g^2(\bar r)}{\bar r^2 a^2}\,P_m\left(k_\perp = \frac{l}{\bar r}\right) \,. \tag{6.44}$$

Now we compute the angular correlation function by Fourier transforming the angular power spectrum. Out of two shear components, we construct three angular correlation functions as

$$w_{ij}(\theta) := \langle\gamma_i(0)\gamma_j(\theta)\rangle = \int\frac{d^2l}{(2\pi)^2}\,e^{i\mathbf{l}\cdot\boldsymbol{\theta}}\begin{pmatrix}\cos^2 2\phi_l & \cos 2\phi_l\sin 2\phi_l\\ \cos 2\phi_l\sin 2\phi_l & \sin^2 2\phi_l\end{pmatrix}P_\kappa(l) \tag{6.45}$$

$$= \frac{1}{2}\int_0^\infty\frac{dl\,l}{2\pi}\,P_\kappa(l)\begin{pmatrix}J_0(l\theta) + J_4(l\theta) & 0\\ 0 & J_0(l\theta) - J_4(l\theta)\end{pmatrix} \,, \tag{6.46}$$

where J_n is the Bessel function and used its integral representation

$$J_0(x) = \int\frac{d\phi}{2\pi}\,e^{ix\cos\phi} \,, \qquad J_4(x) = \int\frac{d\phi}{2\pi}\,e^{ix\cos\phi}\cos 4\phi \,. \tag{6.47}$$

6.2.5 Worked Examples

For the simplest case, where the lens and the source are at two definite redshift slices, the lensing observables can be written in a polar coordinate as

$$2\kappa = \Phi_{rr} + \frac{\Phi_r}{r} + \frac{\Phi_{\theta\theta}}{r^2} = 2\frac{\Sigma}{\Sigma_c} \,, \tag{6.48}$$

$$2\gamma_1 = \cos 2\theta\Phi_{rr} - \frac{2\sin 2\theta}{r}\Phi_{r\theta} - \frac{\cos 2\theta}{r}\Phi_r - \frac{\cos 2\theta}{r^2}\Phi_{\theta\theta} + \frac{2\sin 2\theta}{r^2}\Phi_\theta \,, \tag{6.49}$$

$$2\gamma_2 = \sin 2\theta\Phi_{rr} + \frac{2\cos 2\theta}{r}\Phi_{r\theta} - \frac{\sin 2\theta}{r}\Phi_r - \frac{\sin 2\theta}{r^2}\Phi_{\theta\theta} - \frac{2\cos 2\theta}{r^2}\Phi_\theta \,, \tag{6.50}$$

$$\gamma^2 := \gamma_1^2 + \gamma_2^2 = \frac{1}{4}\left(\Phi_{rr} - \frac{\Phi_r}{r} - \frac{\Phi_{\theta\theta}}{r^2}\right)^2 + \left(\frac{\Phi_{r\theta}}{r} - \frac{\Phi_\theta}{r^2}\right)^2 \,. \tag{6.51}$$

For an axisymmetric lens, the lensing observables are further simplified, and the convergence and shear are

$$2\kappa = \hat{\nabla}^2\Phi = \Phi_{rr} + \frac{\Phi_r}{r} = 2\,\frac{\Sigma}{\Sigma_c}\,, \qquad \gamma = \frac{1}{2}\left(\frac{\Phi_r}{r} - \Phi_{rr}\right) = \frac{\Phi_r}{r} - \frac{\Sigma}{\Sigma_c} = \frac{\bar{\Sigma}(<r) - \Sigma}{\Sigma_c}\,, \tag{6.52}$$

where $\bar{\Sigma}(<r)$ is the average surface density enclosed in radius r and $\bar{\Sigma}(<r) = \Phi_r/r$ from the first relation. The magnification is determined by the surface density of the lensing material, and the gravitational shear is set by the excess surface density of the enclosed mass $\Delta\Sigma := \bar{\Sigma}(<r) - \Sigma(r)$.

For a point mass, the convergence and the shear are

$$\psi = -\frac{GM}{r}\,, \qquad \nabla^2\psi = 4\pi GM\delta^D(x)\,, \qquad \kappa = \frac{\Sigma}{\Sigma_c} = \frac{M\delta^D(R)}{\Sigma_c}\,, \tag{6.53}$$

$$\gamma = \frac{\bar{\Sigma}(<R) - \Sigma}{\Sigma_c} = \frac{\bar{\Sigma}}{\Sigma_c} = \frac{\theta_E^2}{\theta^2}\,, \qquad \bar{\Sigma} := \frac{M}{\pi R^2}\,. \tag{6.54}$$

The lensing magnification is then

$$\mu^{-1} = \det\,\mathbb{D} = 1 - \Phi_{rr} - \frac{\Phi_r}{r} + \frac{\Phi_{rr}\Phi_r}{r} = \frac{s}{r}\frac{\partial s}{\partial r}\,, \tag{6.55}$$

where we used the lens equation

$$s = r - \Phi_r\,, \qquad \partial_r s = 1 - \Phi_{rr}\,. \tag{6.56}$$

For a point mass, there exist two lensed images. When two images are not spatially resolved, the magnification of the lensed images is the sum of two, and we derive the master equation for microlening

$$\mu = \left(\frac{s}{r}\frac{\partial s}{\partial r}\right)_1^{-1} + \left(\frac{s}{r}\frac{\partial s}{\partial r}\right)_2^{-1} = \frac{u^2+2}{u\sqrt{u^2+4}}\,, \qquad u := \frac{s}{\theta_E}\,. \tag{6.57}$$

6.2.6 Galaxy-Galaxy Lensing

Galaxy-galaxy lensing is used to refer to the two-point correlation of the galaxies at one point and the lensing signal measured by background galaxies at the other point. In short, it measures the galaxy-matter cross-correlation. Compared to the cosmic shear measurements, the advantage here is that we have well-defined lenses (lens galaxies) in the foreground, such that the shear measurements in galaxy-galaxy lensing are less susceptible to other systematics.

Assuming spherical symmetry, we can readily derive the lensing convergence and the shear as

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c}\,, \qquad \gamma(\theta) = \bar{\kappa}(\theta) - \kappa(\theta)\,, \tag{6.58}$$

where the "comoving" critical surface density is[3]

$$\Sigma_c(z_1, z_2) = \frac{c^2}{4\pi G}\frac{r_s}{r_l r_{ls}}\frac{1}{1+z_l} = 1.663\times 10^{18}\ hM_\odot\mathrm{Mpc}^{-2}\ \frac{r_s}{r_l}\left(\frac{r_{ls}}{h^{-1}\mathrm{Mpc}}\right)^{-1}\frac{1}{1+z_l}\,. \tag{6.59}$$

Since we are measuring the excess matter around galaxies, the lensing observables are related to the projected galaxy-matter correlation function:

$$w(R) = \int_{-\infty}^{\infty} dz\ \xi_{\rm gm}\left(r = \sqrt{R^2+z^2}\right) = \int_0^\infty \frac{dk_\perp k_\perp}{2\pi}\ P_{\rm gm}(k_\perp)\ J_0(k_\perp R)\,, \tag{6.60}$$

where the integration along the line-of-sight is performed. However, note that this is valid only on small angle, as the observed angular separation θ, not the physical separation R is kept fixed. Under the small-angle approximation, the convergence at a given separation can be derived as

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c} = \int dz\ \frac{\bar{\rho}_m}{\Sigma_c}\ (1+\xi_{gm}) = \frac{\bar{\rho}_m}{\Sigma_c}\ w(R)\,, \tag{6.61}$$

[3]Multiply by $(1+z_l)^2$ for physical critical surface density. Note that sometimes people use the physical angular diameter distances, while using comoving coordinates for other quantities, in which $(1+z_l)^2$ appears in the equation, instead of $(1+z_l)$.

and the remaining lensing observables are

$$\bar{\kappa}(\theta) = \frac{2\pi}{\pi\theta^2}\int_0^\theta d\theta\,\theta\,\kappa(\theta) = \frac{3H_0^2\Omega_m}{2}\frac{(\bar{r}_s-\bar{r}_l)\bar{r}_l}{a_l\bar{r}_s}\int\frac{dk_\perp\,k_\perp}{2\pi}\,P_{gm}(k_\perp;\bar{r}_l)\frac{2J_1(k_\perp\bar{r}\theta)}{k_\perp\bar{r}\theta}\,, \tag{6.62}$$

$$\gamma_T(\theta) = \bar{\kappa}(\theta)-\kappa(\theta) = \frac{3H_0^2\Omega_m}{2}\frac{(\bar{r}_s-\bar{r}_l)\bar{r}_l}{a_l\bar{r}_s}\int\frac{dk_\perp\,k_\perp}{2\pi}\,P_{gm}(k_\perp;\bar{r}_l)\;J_2(k_\perp\bar{r}\theta)\,, \tag{6.63}$$

$$\Delta\Sigma(R) = \Sigma_c\gamma_T = \bar{\rho}_m\int_{-\infty}^{\infty}dz\left[\frac{2}{R^2}\int_0^R dR'\,R'\,\xi_{\rm gm}(R',z)-\xi_{\rm gm}(R,z)\right]\,. \tag{6.64}$$

6.3 Weak Lensing Observables

6.3.1 Ellipticity of Galaxies

The ellipticity ϵ of galaxies is measured in terms of its semi-major axis a and the semi-minor axis b or in terms of the axis ratio q as

$$\epsilon := \frac{a^2-b^2}{a^2+b^2} \equiv \frac{1-q^2}{1+q^2} \equiv \frac{\delta-\frac{1}{2}\delta^2}{1-\delta+\frac{1}{2}\delta^2} \simeq \delta\,, \qquad q := \frac{b}{a} := 1-\delta\,. \tag{6.65}$$

In an idealized case of round galaxies, the ellipticity ϵ, the axis ratio q, and the distortion δ are a measure of gravitational lensing effects of intervening matter, and they are equivalent in the weak lensing regime. In observations, the center of the galaxy and its ellipticity moment are measured by using some weight function $W[I_\nu(\hat{n})]$ of the observed intensity as

$$\hat{n}_o := \frac{\int d^2\hat{n}\;\hat{n}\;W[\hat{n}]}{\int d^2\hat{n}\;W[\hat{n}]}\,, \qquad \mathcal{M}_{ij} := \frac{\int d^2\hat{n}\;(\hat{n}-\hat{n}_o)_i(\hat{n}-\hat{n}_o)_j\;W[\hat{n}]}{\int d^2\hat{n}\;W[\hat{n}]}\,, \tag{6.66}$$

where the simplest weight function is just the observed intensity $W = I[\hat{n}]$. Given the ellipticity moment, we can define the ellipticity vector and the position angle as

$$\boldsymbol{\epsilon} := \left(\frac{\mathcal{M}_{xx}-\mathcal{M}_{yy}}{\mathcal{M}_{xx}+\mathcal{M}_{yy}}\,,\;\frac{2\mathcal{M}_{xy}}{\mathcal{M}_{xx}+\mathcal{M}_{yy}}\right) := (\epsilon_+,\epsilon_\times) = \epsilon(\cos 2\Theta,\;\sin 2\Theta)\,, \qquad \tan 2\Theta \equiv \frac{2\mathcal{M}_{xy}}{\mathcal{M}_{xx}-\mathcal{M}_{yy}}\,. \tag{6.67}$$

Note that the ellipticity vector is headless, such that it is identical under 180 degree rotation, or spin 2. Since only the ellipticity vector matters, the ellipticity moments $\mathcal{M}$ are often defined without the denominator.

6.3.2 Lensing Polarization

The ellipticity moments of the source galaxies would be what we measure in the absence of gravitational lensing. However, the gravitational lensing changes the observed ellipticity moments. Now, for simplicity, we will ignore rotation ($\omega = 0$) and express the distortion matrix in our coordinate:

$$\mathbb{D} = \mathbb{I} - \begin{pmatrix}\kappa & 0\\ 0 & \kappa\end{pmatrix} - \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_2 & -\gamma_1\end{pmatrix}\,, \tag{6.68}$$

and the magnification matrix is then the inverse of the distortion matrix:

$$\mathbf{M}_{ij} := \mathbb{D}^{-1}_{ij} = \frac{1}{|\mathbb{D}|}\begin{pmatrix}1-\kappa+\gamma_1 & \gamma_2\\ \gamma_2 & 1-\kappa-\gamma_1\end{pmatrix}\,, \qquad \mu := |\mathbf{M}| = |\mathbb{D}|^{-1} = \frac{1}{(1-\kappa)^2-\gamma^2}\,. \tag{6.69}$$

Further, assuming the surface brightness conservation due to gravitational lensing (i.e., no frequency change), the observed ellipticity moments are related to those in the source rest-frame as

$$\mathcal{M}^I_{ij} := \int d^2\hat{n}\;\hat{n}_i\hat{n}_j W[\hat{n}] = \int |\mathbf{M}|d^2\hat{s}\;\;\mathbf{M}_{ik}\hat{s}_k\;\mathbf{M}_{jl}\hat{s}_l\;W[\hat{s}] \simeq \mu\,\mathbf{M}_{ik}\mathbf{M}_{jl}\mathcal{M}^s_{kl}\,, \tag{6.70}$$

where we assumed $\hat{n}_o = 0$ and the source size is small that the magnification matrix is constant over the area. Using the definition of the source ellipticity moments

$$\mathcal{M}^s_{11} = \frac{1+\epsilon^s_+}{2}\mathcal{M}\,, \qquad \mathcal{M}^s_{22} = \frac{1-\epsilon^s_+}{2}\mathcal{M}\,, \qquad \mathcal{M}^s_{12} = \frac{\epsilon^s_\times}{2}\mathcal{M}\,, \qquad \mathcal{M} := \mathcal{M}^s_{11}+\mathcal{M}^s_{22}\,, \tag{6.71}$$

the observed ellipticity can be derived in terms of the magnification matrix as

$$\epsilon^I_+ = \frac{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times \mathbf{M}_{12}(\mathbf{M}_{11}-\mathbf{M}_{22}) - 2\epsilon^s_+\mathbf{M}^2_{12} - (1-\epsilon^s_+)\mathbf{M}^2_{22}}{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times \mathbf{M}_{12}(\mathbf{M}_{11}+\mathbf{M}_{22}) + 2\epsilon^s_+\mathbf{M}^2_{12} + (1-\epsilon^s_+)\mathbf{M}^2_{22}} , \tag{6.72}$$

$$\epsilon^I_\times = \frac{2\mathbf{M}_{12}\left[\epsilon^s_\times \mathbf{M}_{12} + (1-\epsilon^s_+)\mathbf{M}_{22}\right] + 2\mathbf{M}_{11}\left[(1+\epsilon^s_+)\mathbf{M}_{12} + \epsilon^s_\times \mathbf{M}_{22}\right]}{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times \mathbf{M}_{12}(\mathbf{M}_{11}+\mathbf{M}_{22}) + 2\mathbf{M}^2_{12} + (1-\epsilon^s_+)\mathbf{M}^2_{22}} , \tag{6.73}$$

where the relation is exact. In terms of the lensing convergence and shear,[4] we derive

$$\epsilon^I_+ = \frac{\epsilon^s_+\left[(1-\kappa)^2 + \gamma_1^2 - \gamma_2^2\right] + 2\epsilon^s_\times\gamma_1\gamma_2 + 2\gamma_1(1-\kappa)}{2\epsilon^s_+\gamma_1(1-\kappa) + 2\epsilon^s_\times\gamma_2(1-\kappa) + (1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} , \tag{6.74}$$

$$\epsilon^I_\times = \frac{\epsilon^s_\times\left[(1-\kappa)^2 - \gamma_1^2 + \gamma_2^2\right] + 2\epsilon^s_+\gamma_1\gamma_2 + 2\gamma_2(1-\kappa)}{2\epsilon^s_+\gamma_1(1-\kappa) + 2\epsilon^s_\times\gamma_2(1-\kappa) + (1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} . \tag{6.75}$$

For circular sources, where $\mathcal{M}^s_{11} = \mathcal{M}^s_{22} \neq 0$, and $\mathcal{M}_{12} = 0$ (or $\epsilon^s_+ = \epsilon^s_\times = 0$), the observed ellipticity becomes

$$\epsilon^I_+ = \frac{2\gamma_1(1-\kappa)}{(1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_1 , \qquad \epsilon^I_\times = \frac{2\gamma_2(1-\kappa)}{(1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_2 , \tag{6.76}$$

where we expanded to the linear order in the last step. Assuming there is no ellipticity correlation of the source galaxies

$$\langle\epsilon_+\rangle = \langle\epsilon_\times\rangle = \langle\epsilon_+\epsilon_\times\rangle = \frac{1}{2}\left\langle\epsilon^2\right\rangle\langle\sin 4\phi\rangle = 0 , \tag{6.77}$$

$$\left\langle\epsilon^2_+\right\rangle = \left\langle\epsilon^2_\times\right\rangle = \left\langle\epsilon^2\right\rangle\left\langle\cos^2 2\phi\right\rangle = \frac{1}{2}\left\langle\epsilon^2\right\rangle , \tag{6.78}$$

the observed ellipticity correlation becomes

$$\xi_{\epsilon^I_+}(\theta) = \left\langle\epsilon^I_+(0)\epsilon^I_+(\theta)\right\rangle = \xi_\delta(\theta)\left(1 - \sigma^2_\epsilon + \frac{1}{4}\sigma^4_\epsilon\right) = \xi_{\epsilon^I_\times} , \tag{6.79}$$

to be compared to the typical ellipticity

$$\sigma_\epsilon = \langle\epsilon^s\rangle^{1/2} \simeq 0.3 . \tag{6.80}$$

• modified gravity, no galaxy bias

[4] With rotation, the magnification matrix is not symmetric, $\mathbf{M}_{12} \neq \mathbf{M}_{21}$.

7 CMB Temperature Anisotropies

In this chapter, we will perform very simple calculations to gain intuitive understanding of the observed CMB anisotropies. We adopt the conformal Newtonian gauge and ignore the vector and tensor perturbation:

$$\alpha \to \alpha_\chi \,, \qquad \varphi \to \varphi_\chi \,, \qquad \beta = \gamma = \chi \equiv 0 \,, \qquad U \to v_\chi \,, \qquad V_\alpha = -v_{\chi,\alpha} + v_\alpha^{(v)} \to -v_{\chi,\alpha} \,. \tag{7.1}$$

7.1 Basics

In the early Universe, the radiation dominates the overall energy density, and due to high pressure the fluctuations cannot grow within the horizon. In particular, the tight-coupling between the baryons and the photons leads to a single fluid, or the baryon-photon fluid, oscillating with a unique sound speed. As a fluid, the density (monopole) and the velocity (dipole) characterize the fluid, and the higher multipoles are negligible. The monopole and the dipole are oscillating in units of the sound speed of the fluid, and the first harmonics (or the first acoustic peak) is the fundamental mode that matches the distance the fluid can travel at the sound speed for the age of the universe at the time of the recombination. Once the baryons recombine at later time, the photons are released and free-stream to the observer today. This free-streaming of the monopole and the dipole generates the temperature anisotropies we measure today, and they show the acoustic oscillations of the baryon-photon fluid at the recombination epoch.

7.2 Collisionless Boltzmann Equation for Photons

To describe the angular multipoles of the observed anisotropies, we need a phase space information beyond the fluid approximation. Its evolution is described by the Boltzmann equation.

7.2.1 Geodesic Equation

For simplicity, we consider fictitious observers, who are at rest in a given coordinate:

$$[e_t]^\mu = \frac{1}{a}\left(1-\alpha_\chi,\, 0\right) \,, \qquad [e_i]^\mu = \frac{1}{a}\left[0\,,\ \delta_i^\alpha\left(1-\varphi_\chi\right)\right] \,, \tag{7.2}$$

where we ignored the rotation of tetrad vectors against FRW coordinates. These observers will measure the energy and momentum of the CMB photons in their rest frame. The physical momentum is written in capital letters in their internal coordinates as

$$P^a = (E, P^i) \,, \qquad E = -p^\mu [e_t]_\mu \,, \qquad P^i = p^\mu e^i_\mu \,, \qquad E^2 = m^2 + P^2 \,, \tag{7.3}$$

where p^μ is the photon four-momentum in a FRW coordinate. Using the tetrad expression, we derive the physical momentum in FRW coordinates

$$p^\eta = \frac{(1-\alpha_\chi)E}{a} \,, \qquad p^\alpha = \frac{1}{a}\left[P^\alpha - \varphi_\chi\, P^\alpha\right] \,, \tag{7.4}$$

and the covariant momentum is

$$p_\eta = -a(1+\alpha_\chi)E \,, \qquad p_\alpha = aP_\alpha(1+\varphi_\chi) \,. \tag{7.5}$$

In the background universe, the geodesic equation yields

$$0 = p^\nu p^\mu{}_{;\nu} = p^\eta p^{\mu\prime} + \bar\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \quad \mapsto \quad 0 = p^\eta p^{\eta\prime} + \mathcal{H} p^\eta p^\eta + \mathcal{H} p^\alpha p_\alpha \,, \qquad 0 = p^\eta p^{\alpha\prime} + 2\mathcal{H} p^\eta p^\alpha \,, \tag{7.6}$$

and the last equation says

$$p^\alpha \propto \frac{1}{a^2} \,, \qquad P^\alpha \propto \frac{1}{a} \,, \tag{7.7}$$

the physical momentum for both massless and massive particles in the background universe redshift as $1/a$. In the presence of perturbations, these relations change, so it is convenient to define the "comoving momentum q" and "comoving energy ε" that remain unchanged in the background universe

$$q := aP \,, \qquad \varepsilon := aE = \sqrt{q^2 + a^2 m^2} \,, \qquad q^i := q n^i \,. \tag{7.8}$$

In terms of the comoving quantities, the momentum in FRW coordinates is now

$$p^\eta = \frac{(1-\alpha_\chi)\varepsilon}{a^2} \,, \qquad p^\alpha = \frac{1}{a^2}\left(q^\alpha - \varphi_\chi q^\alpha\right) \,. \tag{7.9}$$

To compute the change in the comoving momentum as the particle propagates, we need to solve the geodesic equation in an inhomogeneous universe:

$$\frac{dq_\alpha}{d\eta} = -\varepsilon\, \alpha_{\chi,\alpha} - \varphi'_\chi q_\alpha - \frac{q^\beta q^\gamma}{\varepsilon}\left(\varphi_{\chi,\gamma}\delta_{\alpha\beta} - \varphi_{\chi,\alpha}\delta_{\beta\gamma}\right) \,, \tag{7.10}$$

where we use the background geodesic equation (valid for massive & massless)

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \frac{q^\beta}{\varepsilon}\frac{\partial}{\partial x^\beta} \quad \text{for } m \neq 0 \,, \qquad \bar{\varepsilon}' = \frac{a^2 \mathcal{H} m^2}{\varepsilon} \,, \tag{7.11}$$

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + n^\alpha \frac{\partial}{\partial x^\alpha} =: \frac{d}{d\lambda} \quad \text{for } m = 0 \,, \qquad \frac{d}{d\Lambda} = \bar{p}^\eta \frac{\partial}{\partial\eta} + \bar{p}^\alpha \frac{\partial}{\partial x^\alpha} = \frac{q}{a^2}\frac{d}{d\eta} \,. \tag{7.12}$$

For the linear-order evolution, the propagation direction is simply the straight path in the background universe, and only the comoving momentum changes as

$$\frac{d\ln q}{d\eta} = -\frac{\varepsilon}{q}\alpha_{\chi,\parallel} - \varphi'_\chi \,. \tag{7.13}$$

Indeed, the comoving momentum is constant in the background. For massless particles ($m = 0$), the comoving momentum is the comoving energy ($q = \varepsilon$), which further simplifies the propagation equation as

$$\frac{d}{d\eta}\left(\ln q + \alpha_\chi\right) = \left(\alpha_\chi - \varphi_\chi\right)' \,, \tag{7.14}$$

and the whole quantity in the bracket is affected by the structure growth along the path. Note that the gravitational potential α_χ becomes more negative as the structure grows in time.

7.2.2 Collisionless Boltzmann Equation

The Liouville theorem in GR states that the phase-space volume dV_p is conserved along the path parametrized by Λ with momentum p^μ. The total number particles in terms of the phase-space density F is

$$0 = \Delta(dN) = \left(\frac{\partial F}{\partial x^\mu}\Delta x^\mu + \frac{\partial F}{\partial p^\mu}\Delta p^\mu\right) dV_p = \left(p^\mu \frac{\partial F}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial F}{\partial p^\mu}\right) \Delta\Lambda\, dV_p \,, \tag{7.15}$$

it translates into the relativistic collisionless Boltzmann equation:

$$0 = p^\mu \frac{\partial F}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial F}{\partial p^\mu} \,, \qquad \text{or} \qquad 0 = p^\mu \frac{\partial F}{\partial x^\mu} + \Gamma^\rho_{\mu\sigma} p_\rho p^\sigma \frac{\partial F}{\partial p_\mu} \,, \tag{7.16}$$

where we used the geodesic equation

$$0 = \frac{d}{d\Lambda} p^\mu + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = p^\nu p^\mu{}_{,\nu} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \,, \qquad 0 = \frac{d}{d\Lambda} p_\mu - \Gamma^\rho_{\mu\sigma} p_\rho p^\sigma \,. \tag{7.17}$$

Despite the presence of the Christoffel symbol, the equation is indeed invariant under diffeomorphisms. We further need to impose the on-shell condition in the collisionless Boltzmann equation.

The Boltzmann equation is further simplified, when we switch the variables (η, x^α, p^μ) to (η, x^α, q^i), where the on-shell condition removes one component of the physical momentum:

$$0 = p^\mu \frac{\partial F}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial F}{\partial p^\mu} = p^\mu \frac{\partial F}{\partial x^\mu} + \frac{dq^i}{d\Lambda}\frac{\partial F}{\partial q^i} , \tag{7.18}$$

where the partial derivatives fix (x^μ, q^i), instead of (x^μ, p^μ). Splitting the distribution function F into the background $\bar f$ and the perturbation f, the Boltzmann equation in the background is obtained as

$$F := \bar f + f , \qquad 0 = \bar f' , \qquad \therefore \ \bar f = \bar f(q) , \tag{7.19}$$

i.e., the phase-space distribution is constant in time and space, but only a function of the comoving momentum. The linear-order perturbation equation can be derived as

$$0 = \bar p^\eta f' + \bar p^\alpha f_{,\alpha} + \frac{dq^i}{d\Lambda}\frac{n_i}{q}\frac{d\bar f}{d\ln q} = \bar p^\eta \left(f' + n^\alpha \frac{\partial f}{\partial x^\alpha} + \frac{d\bar f}{d\ln q}\frac{a^2 n_i}{q^2}\frac{dq^i}{d\Lambda}\right) , \qquad \bar p^\eta = \frac{q}{a^2} . \tag{7.20}$$

Noting that the last term is

$$\frac{a^2 n_i}{q^2}\frac{dq^i}{d\Lambda} = \frac{a^2}{q}\frac{d\ln q}{d\Lambda} = \frac{d\ln q}{d\eta} , \tag{7.21}$$

the collisionless Boltzmann equation can be simplified at the linear order in perturbations as

$$0 = \frac{df}{d\eta} + \frac{d\bar f}{d\ln q}\frac{d\ln q}{d\eta} , \qquad \frac{d}{d\eta}\left[f - \frac{d\bar f}{d\ln q}\alpha_\chi \right] = -\frac{d\bar f}{d\ln q}\left(\alpha_\chi - \varphi_\chi\right)' . \tag{7.22}$$

7.2.3 Massless Particles

Here we consider photons and neutrinos, though neutrinos are massive, massless neutrinos are in most cases a good approximation, with which equations are greatly simplified. We define the temperature anisotropies Θ

$$\rho = aT^4 = a\bar T^4 \left(1 + 4\frac{\delta T}{\bar T}\right) + \mathcal{O}(2) , \qquad \Theta(\hat n) := \frac{\delta T}{\bar T} = \frac{1}{4}\frac{\delta\rho}{\bar\rho} . \tag{7.23}$$

Assuming that the massless particles are in thermal equilibrium, the phase-space distribution function is

$$F(x) = \left[\exp\left(\frac{E(x)}{T(x)}\right) - 1\right]^{-1} = \left[\exp\left(\frac{q}{a\bar T(\eta)[1+\Theta(x)]}\right) - 1\right]^{-1} =: \bar f + f(x) , \tag{7.24}$$

where the background distribution function depends only on q:

$$\bar f(q) = \left[\exp\left(\frac{q}{a\bar T(\eta)}\right) - 1\right]^{-1} , \qquad \bar T(\eta) \propto \frac{1}{a} . \tag{7.25}$$

Expanding in perturbations,

$$F \simeq \bar f \left[1 + \bar f e^{q/a\bar T}\frac{q}{a\bar T}\,\Theta\right] = \bar f\left(1 - \frac{d\ln\bar f}{d\ln q}\,\Theta\right) , \qquad \frac{d\ln\bar f}{d\ln q} = -\bar f e^{q/a\bar T}\frac{q}{a\bar T} , \tag{7.26}$$

we derive

$$\therefore \ f(x) = -\frac{d\bar f}{d\ln q}\,\Theta(x) , \qquad \frac{d}{d\eta} f = -\frac{d\bar f}{d\ln q}\frac{d\Theta}{d\eta} , \tag{7.27}$$

and the Boltzmann equation for photons is finally

$$\frac{d}{d\eta}\left(\Theta + \alpha_\chi\right) = \left(\alpha_\chi - \varphi_\chi\right)' . \tag{7.28}$$

7.3 Observed CMB Power Spectrum

7.3.1 Convention for Multipole Decomposition

Now we introduce the angular decomposition of the perturbation variables in the Boltzmann equation. Eventually, we are interested in the angular power spectrum of the observed CMB anisotropies, but to this goal, we first decompose the Boltzmann equation and evolve each multipole component. Schematically, consider a perturbation variable $P(x^\mu, q^i)$ along the photon (background) path $\mathbf{x} = -\bar{r}n^i$, which is related to the photon momentum $q^i = qn^i$ (n^i is the photon propagation direction, not the observed direction). It can be Fourier transformed,

$$P(x^\mu, q, \hat{n}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, P(\mathbf{k}, \eta, q, \hat{n}) \,, \qquad\qquad x^i = -\bar{r}n^i \,, \tag{7.29}$$

and its Fourier component can be angular decomposed as

$$P(\mathbf{k}, \eta, q, \hat{n}) =: \sum_{lm}(-i)^l\sqrt{\frac{4\pi}{2l+1}}P_{lm}(\mathbf{k}, \eta, q)Y_{lm}(\hat{n}) \,, \qquad\qquad P_{lm}(\mathbf{k}, \eta, q) \equiv i^l\sqrt{\frac{2l+1}{4\pi}}\int d^2n\, Y^*_{lm}(\hat{n})P(\mathbf{k}, \eta, q, \hat{n}) \,. \tag{7.30}$$

Naturally, P_{lm} are helicity eigenstates, such that under a rotation $\phi \to \phi - \Phi$ in a coordinate ($\mathbf{k}/\!\!/\boldsymbol{z}$) they transform as

$$\tilde{P}_{lm} = P_{lm}e^{im\Phi} \,. \tag{7.31}$$

Since we deal with only scalar, vector, tensor types of perturbations, the helicity eigenstates are limited to $|m| \leq 2$ in such coordinate. At the end, we will set $\mathbf{x} = 0$ to derive the observed CMB anisotropies.

In literature, there exists a different convention (up to $2l + 1$ factor) for decomposition in terms of the Legendre polynomial $L_l(x)$,

$$P(\mathbf{k}, \eta, q, \hat{n}) =: \sum_l(-i)^lP_l(\mathbf{k}, \eta, q)\; L_l(\hat{n}\cdot\hat{k}) = \sum_l(-i)^lP_l(\mathbf{k}, \eta, q)\sum_m\frac{4\pi}{2l+1}Y_{lm}(\hat{n})Y^*_{lm}(\hat{k}) \,. \tag{7.32}$$

Consider that Fourier modes are assumed to be aligned as $\mathbf{k}/\!\!/\boldsymbol{z}$, where the spherical harmonics is

$$Y_{lm}(\boldsymbol{z}) = \sqrt{\frac{2l+1}{4\pi}}\;\delta_{m0} \,. \tag{7.33}$$

So we derive the correspondence to our decomposition convention:

$$P(\mathbf{k}, \eta, q, \hat{n}) = \sum_l(-i)^lP_l(\mathbf{k}, \eta, q)\;\delta_{m0}\sqrt{\frac{4\pi}{2l+1}}\;Y_{lm}(\hat{n}) \,, \qquad\qquad \therefore\;\; P_l \to P_{l0} \,, \tag{7.34}$$

and the decomposition with the Legendre polynomial in Eq. (7.32) is *valid only for the scalar modes*.

• ***Notation convention in literature***.— A common notation convention: Seljak and Zaldarriaga (1996); Zaldarriaga and Seljak (1997); Dodelson (2003)

$$P(\hat{k}\cdot\hat{n}) := \sum_l(-i)^l(2l+1)\hat{P}_l\;L_l(\hat{n}\cdot\hat{k}) \,, \qquad\qquad \therefore\;\; (2l+1)\hat{P}_l \equiv P_l \equiv P_{l0} \,. \tag{7.35}$$

7.3.2 Free Streaming: Line-of-Sight Integration

Here we derive a formal integral solution by performing the line-of-sight integration by using the Boltzmann equation (7.28) and accounting for collisions. Before the recombination, the CMB photons interact with free electrons, and free electrons are tightly coupled with protons, such that they form a baryon-photon fluid. The scattering process at this low energy is described by the Thompson scattering with cross-section σ_T, which has quadrapolar dependence on the angular distribution and generates polarization. Here we ignore this subtlety and use the simplified collisional process:

$$\frac{d}{d\eta}\left(\Theta + \alpha_\chi\right) = \left(\alpha_\chi - \varphi_\chi\right)' + \tau'\left[\Theta - \Theta_0\right] \,, \tag{7.36}$$

where $\Theta_0 = \delta_\gamma/4$ is the monopole and the optical depth for photons is

$$\tau(\eta) := \int_\eta^{\eta_0} d\eta \, a n_e \sigma_T \,, \qquad \tau' = -a n_e \sigma_T \,. \tag{7.37}$$

When the optical depth is large, the photon distribution Θ converges to the monopole (and the dipole) without any other higher moments, i.e., the baryon-photon fluid. Once the collisional process becomes inefficient, the photon distribution develops higher-order moments.

We want to derive an analytical solution for the temperature anisotropy Θ. Collecting Θ-terms and noting that the derivative along the path in Fourier space becomes

$$\frac{d}{d\eta}\Theta(x^\mu) - \tau'\Theta(x^\mu) = \Theta'(x^\mu) + (\nabla_r - \tau')\Theta(x^\mu) \tag{7.38}$$
$$\to \quad \Theta'(k,\eta) + (ik\mu_k - \tau')\Theta(k,\eta) = e^{-ik\mu_k\eta+\tau(\eta)}\frac{d}{d\eta}\left[\Theta \, e^{ik\mu_k\eta-\tau(\eta)}\right] ,$$

the Boltzmann equation can be re-arranged and integrated to yield the line-of-sight integral solution in Fourier space

$$\Theta(k,\eta) = -\int_0^{\eta_0} d\eta \, e^{-ik\mu_k(\eta_0-\eta)}e^{-\tau(\eta)}\left[ik\mu_k\alpha_\chi + \left(\varphi_\chi' + \tau'\Theta_0\right)\right](k,\eta) \,, \tag{7.39}$$

where we used $\tau(\eta_i) = \infty$, $\tau(\eta_0) = 0$, and $\Theta(k,\eta_i)$ at the initial time $\eta_i = 0$ was neglected due to $\tau(\eta_i)$. While $d\eta$ is the line-of-sight integration, the position dependence in Fourier space is taken out with e^{ikx} and the integrand is independent of position.

By replacing the angular dependence μ_k with the derivative, the solution can be further simplified as

$$\Theta = -\int_0^{\eta_0} d\eta \, e^{-\tau}\left[\left(\varphi_\chi' + \tau'\Theta_0\right) + \alpha_\chi\frac{d}{d\eta}\right]e^{-ik\mu_k(\eta_0-\eta)} \,, \qquad ik\mu_k \to \frac{d}{d\eta} \,. \tag{7.40}$$

Expanding the *exponential term* and performing the multipole decomposition in Eq. (7.30) on both sides, we derive

$$\frac{\Theta_l}{2l+1} = -\int_0^{\eta_0} d\eta \, e^{-\tau}\left[\left(\varphi_\chi' + \tau'\Theta_0\right) + \alpha_\chi\frac{d}{d\eta}\right]j_l(x) = -\int_0^{\eta_0} d\eta \left[e^{-\tau}\left(\varphi_\chi' + \tau'\Theta_0\right) - \frac{d}{d\eta}\left(e^{-\tau}\alpha_\chi\right)\right]j_l(x) \,, \tag{7.41}$$

where $x := k(\eta_0 - \eta)$ and we integrated by part for the second term in the square bracket. By defining the visibility function

$$g(\eta) := -\tau' e^{-\tau} \,, \tag{7.42}$$

the integral solution can be rearranged as

$$\frac{\Theta_l}{2l+1} = \int_0^{\eta_0} d\eta \left[g\left(\Theta_0 + \alpha_\chi\right) + e^{-\tau}\left(\alpha_\chi - \varphi_\chi\right)'\right]j_l(x) \,, \tag{7.43}$$

Since the visibility is close to a sharp Dirac delta function at the recombination time

$$g(\eta) \simeq \delta^D(\eta - \eta_\star) \,, \tag{7.44}$$

the temperature anisotropies are

$$\frac{\Theta_l}{2l+1} \approx \left(\Theta_0 + \alpha_\chi\right)_\star j_l[k(\eta_0 - \eta_\star)] \,, \tag{7.45}$$

where we ignored the time evolution of the potential term with the exponential damping. Evident from the equation, we in fact consider only $m = 0$ scalar fluctuations. The observed temperature anisotropies are essentially the "monopole" (and the "dipole" we ignored here) of the baryon-photon fluid at the recombination epoch $\eta_\star$, free-streaming to the observer after the recombination and generating all angular multipoles.

7.3.3 CMB Angular Power Spectrum on Large Scales

Finally, we need to connect our theoretical predictions to the observation. The observed CMB temperature can be harmonically decomposed as

$$\Theta(\hat{n}) =: \sum_{lm} a_{lm} Y_{lm}(\hat{n}) \,, \qquad a_{lm} \equiv \int d^2\hat{n}\; Y^*_{lm}(\hat{n})\Theta(\hat{n}) \,, \tag{7.46}$$

and the observed CMB power spectrum can be obtained as

$$C_l = \frac{1}{2l+1}\sum_m |a_{lm}|^2 \,. \tag{7.47}$$

We can derive a simple approximation to the observed power spectrum on large scales. At $k\eta \ll 1$, the Boltzmann equation yields

$$0 = \Theta_0' + \varphi_\chi' \,, \qquad \therefore\; \Theta_0(k,\eta) = -\varphi_\chi(k,\eta) + C(k) \approx -\varphi_\chi(k,\eta) + \frac{3}{2}\varphi_\chi(k,\eta_i) \,, \tag{7.48}$$

where the integral constant is fixed by the initial condition (we did not discuss here). The comoving-gauge curvature perturbation is conserved on large scales all the time, while the conformal Newtonian gauge curvature transitions its value from RDE to MDE. Using Eq. (4.156), we derive

$$\varphi_v = \frac{5+3w}{3(1+w)}\varphi_\chi = \frac{3}{2}\varphi_\chi(\eta_i) = \frac{5}{3}\varphi_\chi(\eta_\star) \,, \qquad \therefore\; \varphi_\chi(\eta_i) = \frac{10}{9}\varphi_\chi(\eta_\star) \,, \qquad \Theta_0(\eta_\star) = \frac{2}{3}\varphi_\chi(\eta_\star) \,. \tag{7.49}$$

The same calculation can be done for the matter density on large scales:

$$0 = \delta' + 3\varphi_\chi' \,, \qquad \therefore\; \delta(\eta_\star) = -3\varphi_\chi(\eta_\star) + \frac{9}{2}\varphi_\chi(\eta_i) = 2\varphi_\chi(\eta_\star) \,, \qquad (\Theta_0 + \alpha_\chi)_\star = -\frac{1}{3}\varphi_\chi(\eta_\star) = -\frac{1}{6}\delta(\eta_\star) \,. \tag{7.50}$$

The CMBtemperature anisotropies are then

$$\frac{\Theta_l}{2l+1} \approx (\Theta_0 + \alpha_\chi)_\star\, j_l[k(\eta_0 - \eta_\star)] \approx -\frac{1}{3}\varphi_\chi(\eta_\star) j_l(k\eta_0) \,, \qquad \eta_* \ll \eta_0 \,, \tag{7.51}$$

and the angular power spectrum is

$$C_l^{\rm SW} \approx \left[\frac{\varphi_\chi(k,\eta_*)}{3}\right]^2 4\pi \int d\ln k\; j_l^2(k\eta_0) \propto \frac{1}{2l(l+1)} \,, \qquad l(l+1)C_l^{\rm SW} = \text{constant} \,, \tag{7.52}$$

where the potential is constant on large scales and we used a mathematical identity

$$\int_0^\infty \frac{dk}{k}\, j_l^2(k) = \frac{1}{2l(l+1)} \,. \tag{7.53}$$

From Θ_l, there exists a factor $(2l+1)^2$, one of which is removed from the average in C_l and the other of which is removed by the rotation of wave vectors. Implicitly our calculations were done in a coordinate ($\mathbf{k}/\!\!/\boldsymbol{z}$) given $\hat{n}$. This coordinate system needs to be rotated back to the original, and this rotation matrix gives $(2l+1)$ factor in the power spectrum.

This constant angular power spectrum on large scales is called the Sachs-Wolfe plateau. At the recombination, the overdense region with $\delta > 0$ corresponds to the hotter spot $\Theta_0 > 0$, but the observed temperature today is colder due to the energy loss by the gravitational redshift from the overdense region. Furthermore, given the level that the temperature anisotropies are $\sim 10^{-5}$, the density growth from the recombination epoch $z \sim 1100$ will lead only to $\delta \sim 10^{-2}$, unless it is further boosted by the nonlinear growth of dark matter prior to the recombination epoch.

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