ETH
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Theoretical Astrophysics and Cosmology

Exercise 0

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This optional exercise sheet brings together a collection of reading assignments and exercises about General Relativity (GR). It is intended for students who would like a review or an introduction to GR.

Reading material

A suggested reading to learn the basics of General Relativity is Section 2.1 of Dodelson, S., *Modern Cosmology*, 2003, Academic Press.

Contraction of tensors

Let $T^{\mu\nu}{}_{\rho\lambda}$ be a (2,2) tensor. The contraction of the tensor is given by

$$S^{\mu}{}_{\nu} = \sum_{\nu} T^{\mu\nu}{}_{\nu\lambda} = T^{\mu\nu}{}_{\nu\lambda}\,,$$

where we have used Einstein's summation convention in the last equality. Show that the contraction of a (2,2) tensor is again a tensor by showing that $S^{\mu}{}_{\nu}$ transforms like a tensor under coordinate transformations.

Metric on spheres

1. A 2-dimensional sphere of constant radius r in 3-dimensional space can be described by the equation $x^2 + y^2 + z^2 = r^2$.

   Show that each point on the sphere can be expressed in terms of spherical coordinates (r, θ, ϕ) as:

   $$x = r\sin\theta\cos\phi$$
   $$y = r\sin\theta\sin\phi$$
   $$z = r\cos\theta.$$

   Derive the metric tensor $g_{\mu\nu}$ of spacetime describing the geometry on the surface of the 2-dimensional sphere in spherical coordinates.

2. Now imagine a 3-dimensional sphere of constant radius r in 4-dimensional space, described by the equation $x^2 + y^2 + z^2 + w^2 = r^2$. In analogy to the first question, show

that each point on the surface of the 3-dimensional sphere can be expressed in terms of 4-dimensional spherical coordinates (r, χ, θ, ϕ) as

$$\begin{aligned} x &= r \sin\chi \sin\theta \cos\phi \\ y &= r \sin\chi \sin\theta \sin\phi \\ z &= r \sin\chi \cos\theta \\ w &= r \cos\chi . \end{aligned}$$

Derive the metric tensor $g_{\mu\nu}$ of spacetime describing the geometry on the surface of the 3-dimensional sphere in spherical coordinates.

Geodesics on a sphere

1. Consider the 2-dimensional sphere introduced in the previous exercise. The metric on the surface of the sphere of a given radius r is given by

$$ds^2 = -\mathrm{d}t^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 + r^2 \mathrm{d}\theta^2 ,$$

   where we have used units in which the speed of light c satisfies $c = 1$.

   Calculate the Christoffel symbols for this metric and express them in terms of θ. Show that the only non-zero terms are $\Gamma^{\theta}{}_{\phi\phi}$, $\Gamma^{\phi}{}_{\phi\theta}$ and $\Gamma^{\phi}{}_{\theta\phi}$.

2. Use the results derived above and the geodesic equation to find the equation of motion for a test (unaccelerated) particle trapped on the surface of the sphere.

3. A great circle on the sphere can be described by the following two equations

$$\theta(s) = s + a; \qquad \phi(s) = b.$$

   where s parametrises the path length along the curve and $a, b =$ const. Show that this family of great circles satisfy the geodesic equation derived in the previous question. Explain why this can be generalised to all great circles. Describe what happens for two particles travelling along the geodesics when they start parallel to each other at the equator.

4. Find the Ricci tensor for this metric. Show that contraction of the tensor leads to

$$R \equiv g^{\mu\nu} R_{\mu\nu} = \frac{2}{r^2} .$$