ETH
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Theoretical Astrophysics and Cosmology

Exercise Sheet 10

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Exercise 1: Linear evolution of the inflaton fluid quantities

a) Starting from the energy-momentum tensor of the inflaton field,

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi - g_{\mu\nu}V(\phi)\,, \tag{1}$$

and consider scalar perturbations in conformal Newtonian gauge,

$$\mathrm{d}s^2 = -a^2(1+2\alpha)\mathrm{d}\eta^2 + a^2\delta_{\alpha\beta}(1+2\varphi)\mathrm{d}x^\alpha\mathrm{d}x^\beta\,, \qquad u^\mu = \frac{1}{a}(1-\alpha, -U^{,\alpha})\,, \tag{2}$$

derive the linear-order fluid quantities:

$$\begin{aligned}\delta\rho &= \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 + \partial_\phi V\delta\phi\,,\\ \delta p &= \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 - \partial_\phi V\delta\phi\,.\end{aligned} \tag{3}$$

Hint: Recall that ρ and p are obtained from the stress-energy tensor as $\rho = T_{\mu\nu}u^\mu u^\nu$, and $p = \frac{1}{3}T_{\mu\nu}(g^{\mu\nu} + u^\mu u^\nu)$. Be careful to not forget any linear order contributions! E.g., we have $\dot\phi = \partial_0\phi$, but $\partial^0\phi = g^{0\mu}\partial_\mu\phi \neq \dot\phi$.

b) Derive the equation of motion for the linear perturbation in the inflaton field:

$$\delta\ddot\phi + 3H\delta\dot\phi + \left(\partial_\phi^2 V + \frac{k^2}{a^2}\right)\delta\phi = \dot\phi(\dot\alpha + \kappa) + (2\ddot\phi + 3H\dot\phi)\alpha\,. \tag{4}$$

Note that this equation is in Fourier space and k is the wavenumber. Also, recall that $\kappa = 3H\alpha - 3\dot\varphi$, and that $\alpha = -\varphi$ in a universe without anistropic stress.

Hint: You can start from the equation

$$0 = -(\partial_\mu\partial^\mu\phi + \Gamma^\alpha_{\alpha\mu}\partial^\mu\phi) + \frac{\partial V}{\partial\phi}\,, \tag{5}$$

from Solution Sheet 7. Use the metric $\mathrm{d}s^2 = -(1+2\alpha)\mathrm{d}t^2 + a^2\delta_{\alpha\beta}(1+2\varphi)\mathrm{d}x^\alpha\mathrm{d}x^\beta$, with the cosmic time coordinate t.

Exercise 2: Linear evolution of the curvature perturbation

We define the gauge-invariant Mukhanov variable Φ, corresponding to the comoving gauge curvature perturbation, as

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)}\varphi_\chi \,, \tag{6}$$

where K is the extrinsic curvature and the gauge-invariant variables φ_v and φ_χ are defined in the lecture notes. Using the Einstein equations, derive the governing equations for the Mukhanov variable (eqs. (4.27)-(4.28) in the notes)

$$\begin{aligned} \Phi &= \frac{H^2}{4\pi G(\rho+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{2H^2\,\Pi}{\rho+p}\,, \\ \dot{\Phi} &= -\frac{H}{4\pi G(\rho+p)}\frac{k^2c_s^2}{a^2}\varphi_\chi - \frac{H}{\rho+p}\left(e - \frac{2k^2}{3a^2}\Pi\right), \end{aligned} \tag{7}$$

where $e = \delta p - c_s^2\delta\rho$, $c_s^2 = \dot{p}/\dot{\rho}$ is the speed of sound, and Π is the scalar part of the anisotropic stress tensor.

Furthermore, assuming a flat universe with $K = 0$, show that the second equation can be rearranged as (eq. (4.76) in the notes)

$$\dot{\Phi} = \Xi - \frac{H}{\rho+p}\frac{k^2}{a^2}\left(\frac{c_s^2}{4\pi G}\varphi_\chi - \frac{2}{3}\Pi\right), \qquad \Xi \equiv \frac{\dot{\rho}\,\delta p - \dot{p}\,\delta\rho}{3(\rho+p)^2}\,. \tag{8}$$

This means that the comoving gauge curvature perturbation is conserved in the super horizon limit ($k \to 0$) if $\Xi = 0$ (i.e. $\delta p = \delta\rho\,\mathrm{d}P/\mathrm{d}\rho = \delta\rho\,\dot{P}/\dot{\rho}$, which sets the adiabatic initial condition).

Exercise 3: Inflation parameters for a given model

Work out the equations in Section 4.3.3 of the lecture notes for $\alpha = 2$.