ETH

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Theoretical Astrophysics and Cosmology

Exercise Sheet 7

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Exercise 1

During the epoch of inflation, the Einstein–Hilbert action can be written as

$$\mathcal{S}\left[g^{\mu\nu},\phi\right] = \int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{R}{2\kappa} + \mathcal{L}_\phi\right], \tag{1}$$

where

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \tag{2}$$

is the Lagrangian density for the inflation field ϕ with potential $V(\phi)$.

Vary the action $\mathcal{S}$ with respect to ϕ and show that the evolution of the inflation field is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{3}$$

Hint: Neglect spatial derivatives of ϕ. You may also need to remember that

$$\partial_\mu\left(\sqrt{-g}\,A^\mu\right) = \sqrt{-g}\,\nabla_\mu A^\mu = \sqrt{-g}\,\left(\partial_\mu A^\mu + \Gamma^\mu_{\mu\alpha}A^\alpha\right). \tag{4}$$

Note that here you may assume a flat FLRW metric – this is a reasonable assumption because inflation drives the curvature towards zero.

Exercise 2

Consider a scalar field $\phi(\mathbf{x},t)$ with a potential energy density $V(\phi)=0$. The field is initially rolling ($\mathrm{d}\phi/\mathrm{d}t \neq 0$) and the kinetic energy associated with the rolling dominates the energy density of the Universe.

a) Recalling the condition $\rho + 3P \ll 0$, show that such a scalar field does **not** lead to inflation.

b) Show that the energy density of a Universe filled with only the scalar field $\phi(\mathbf{x},t)$ drops as $\propto a^{-6}$.

Hint: Read Chapter 3.6.3 of Mo/Bosch/White. Then, to solve Exercise b), combine what you found out about the pressure and energy density of the scalar field with the first law of thermodynamics,

$$\frac{\mathrm{d}\rho}{\mathrm{d}a} + 3\left(\frac{\rho + P/c^2}{a}\right) = 0\,. \tag{5}$$

Exercise 3

Assume that the potential energy density of the inflation field obeys

$$V(\phi) = \lambda\,\phi^4, \tag{6}$$

with $V(\phi)$ being the potential. Also recall the slow roll condition

$$\epsilon(\phi) = \frac{m_{\mathrm{Pl}}^2}{2}\left(\frac{1}{V}\frac{\partial V}{\partial \phi}\right)^2 \ll 1\,, \qquad m_{\mathrm{Pl}} = \frac{1}{\sqrt{8\pi G}}\,, \tag{7}$$

where the field is rolling towards $\phi = 0$ from the positive side.

a) At what value of ϕ is this condition broken? Assuming that inflation ends at $\epsilon = 1$, show that the number of e-foldings,

$$\ell(t_i) = \ln\left(\frac{a[t(\epsilon = 1)]}{a(t_i)}\right)\,, \tag{8}$$

occuring during inflation is given by

$$\ell(t_i) = \pi G\left(\phi_i^2 - 8m_{\mathrm{Pl}}^2\right) \tag{9}$$

for some initial value $\phi_i = \phi(t_i)$.

Hint: What is $\dot{\ell}$? Recall the evolution equation (3) for the scalar field, and that we can neglect $\ddot{\phi}$ in the slow-roll approximation. Furthermore, recall that H is related to the energy density of the universe via the first Friedmann equation. What dominates the energy density during inflation?

b) Starting from equation (9) and the first Friedmann equation, show that

$$\phi = \phi_i\,\exp\left[-\sqrt{\frac{16\lambda m_{\mathrm{Pl}}^2}{3}}\,(t - t_i)\right]\,. \tag{10}$$

Then, again starting from the first Friedmann equation and using the solution for ϕ, show that

$$a = a_i\,\exp\left\{\frac{\phi_i^2}{8m_{\mathrm{Pl}}^2}\left[1 - \exp\left(-\sqrt{\frac{64\lambda m_{\mathrm{Pl}}^2}{3}}\,(t - t_i)\right)\right]\right\}\,. \tag{11}$$

c) Expand the solution for a at small time differences $t - t_i$ and show that inflation is approximately exponential in the initial stage.

Calculate the time constant ω in $a \propto \exp \omega t$ and show that it is equal to the slow-roll Hubble parameter during inflation.