ETH
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Theoretical Astrophysics and Cosmology

Exercise Sheet 8

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Exercise 1

Consider a fluid of n components with mass densitites ρ_i, pressures p_i and velocities $\mathbf{v}_i$. The Euler equations for conservation of mass and momentum and Poisson's equation are given by:

$$\dot{\rho}_i + \nabla \cdot (\rho_i \mathbf{v}_i) = 0\,, \tag{1}$$

$$\dot{\mathbf{v}}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = -\frac{1}{\rho_i}\nabla p_i - \nabla\Phi\,, \tag{2}$$

$$\nabla^2\Phi = 4\pi G \sum_{j=1}^{n} \rho_j\,. \tag{3}$$

Starting from these equations, derive the equations

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot (\delta_i \mathbf{u}_i)\;, \tag{4}$$

$$\dot{\mathbf{u}}_i + H\mathbf{u}_i + \frac{1}{a}\mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{1}{a\bar{\rho}_i}\frac{\nabla \delta p_i}{1+\delta_i} - \frac{1}{a}\nabla\delta\Phi\,, \tag{5}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G \sum_{i=1}^{n} \bar{\rho}_i \delta_i\,, \tag{6}$$

by changing to the comoving coordinate $\mathbf{x}$, where $\mathbf{r} := a(t)\mathbf{x}$, and introducing the perturbation variables δ_i, δp_i, $\mathbf{u}_i$ an $\delta\Phi$ as:

$$\rho_i = \bar{\rho}_i + \delta\rho_i = \bar{\rho}_i\,(1+\delta_i)\;, \qquad p_i = \bar{p}_i + \delta p_i\,, \qquad \mathbf{v}_i = H\mathbf{r} + \mathbf{u}_i\,, \qquad \Phi = \bar{\Phi} + \delta\Phi\,. \tag{7}$$

Hint: First, you need to find the relation between the operators $\nabla_\mathbf{x}$ and $\nabla_\mathbf{r}$, and between the time derivatives $\frac{\partial}{\partial t}\big|_\mathbf{r}$ and $\frac{\partial}{\partial t}\big|_\mathbf{x}$. Furthermore, you need to apply the background relations:

$$\dot{\bar{\rho}}_i + 3H\bar{\rho}_i = 0\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_j \bar{\rho}_j\,. \tag{8}$$

Exercise 2

In this exercise, we consider equations (4)–(6) for a one-component, pressureless medium (i.e., $\delta_i = \delta$, $\bar{\rho}_i = \bar{\rho}_m$, $\mathbf{u}_i = \mathbf{u}$, $\delta p_i = 0$). We split the velocity field $\mathbf{u}$ into its divergence and curl:

$$\theta := -\frac{1}{a}\nabla \cdot \mathbf{u}\,, \qquad \boldsymbol{\omega} := \frac{1}{a}\nabla \times \mathbf{u}\,. \tag{9}$$

a) Use equation (5) to show that the linear order solution $\boldsymbol{\omega}^{(1)}$ for the curl is decaying as the universe expands.

b) Starting from the equations (4)–(6), show that the linear order solutions $\delta^{(1)}(t,\mathbf{k})$ and $\theta^{(1)}(t,\mathbf{k})$ for the density perturbation and the velocity divergence are given by:

$$\delta^{(1)}(t,\mathbf{k}) = D(t)\hat{\delta}(\mathbf{k})\,, \qquad \theta^{(1)}(t,\mathbf{k}) = HfD(t)\hat{\delta}(\mathbf{k})\,, \qquad f := \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\,, \tag{10}$$

where the *growth rate* D satisfies the differential equation

$$\ddot{D} + 2H\dot{D} - 4\pi G\bar{\rho}_m D = 0\,, \tag{11}$$

and is normalized to unity at some early epoch t_0 (i.e., $D(t_0) = 1$ and $\delta(t_0,\mathbf{k}) = \hat{\delta}(\mathbf{k})$).

Hint: Rewrite equation (4) and the divergence of (5) using θ, and linearize the resulting equations.

c) Show that in a matter-dominated universe, the density fluctuation grow as the scale factor, $D(t) = a(t)$.

Hint: Recall that in a matter-dominated universe, we have $a(t) = (2/3H_0t)^{2/3}$. You can neglect decaying solutions.

Exercise 3: Redshift-Space Distortion

Starting from the relations

$$\delta_s \approx \delta_g - \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}r}\,, \qquad \mathcal{V} = -f\frac{\partial}{\partial r}\Delta^{-1}\delta\,, \qquad \delta_g = b\delta_m\,, \tag{12}$$

prove that the redshift-space galaxy fluctuation is given by

$$\delta_s(\mathbf{s}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{s}} \left(b + f\mu_k^2\right)\delta_m(\mathbf{k})\,, \tag{13}$$

and hence, the *Kaiser formula* for the galaxy power spectrum in redshift-space holds true:

$$P_s(k,\mu_k) = (b + f\mu_k^2)^2 P_m(k)\,. \tag{14}$$

where b is the galaxy bias, $\delta_g = b\delta_m$, and μ_k is the cosine angle between the Fourier mode and the line-of-sight direction, $\mu_k = \hat{s}\cdot\hat{k}$. For the calculations, only keep the linear order terms and neglect higher orders.