ETH
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Theoretical Astrophysics and Cosmology

Exercise Sheet 9

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Exercise 1: Linear-Order Gauge Transformations

a) Derive the gauge transformations of the scalar perturbations α, β, φ, γ, U and v given in Eq. (3.24).

b) Check that the scalar variables φ_v, φ_χ, v_χ, φ_δ, δ_v and α_χ in Eq. (3.26) are gauge-invariant.

Hint: You can solve a) by using that $\delta_\xi g_{\mu\nu}(x) = -\mathcal{L}_\xi g_{\mu\nu}$ and $\delta_\xi u^\mu = -\mathcal{L}_\xi u^\mu$ as described in the lecture notes (all necessary definitions are given in Sections 3.1.1 and 3.1.2). Having solved a), the fact that φ_v, φ_χ, v_χ and α_χ are gauge-invariant follows easily. To show that φ_δ and δ_v are gauge-invariant, you additionally need to consider the gauge-transformation of the stress-energy tensor, $\delta_\xi T_{\mu\nu}$, to determine how δp and δ transform. Also, recall that $\rho \propto a^{3(1+w)}$ and $\rho + p = \rho(1+w)$.

Exercise 2: Riemann Tensor

Assuming a flat universe with no vector and tensor perturbations and adopting the conformal Newtonian gauge,

$$\mathrm{d}s^2 = -a^2(1+2\psi)\mathrm{d}\eta^2 + a^2(1+2\phi)\bar{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \tag{1}$$

derive the Christoffel symbols in Eqs. (3.67)−(3.71), the Riemann tensor in Eqs. (3.79)−(3.85), and the Ricci tensor and scalar in Eqs. (3.86)−(3.89).

Exercise 3: Einstein Equations

With the same conditions as in Exercise 2, derive the Einstein equations given Section 3.3.5 of the lecture notes:

$$\kappa = 3H\psi - 2\dot{\phi}\,, \qquad \Delta\phi + a\mathcal{H}\left(3H\psi - 3\dot{\phi}\right) = -4\pi G a^2 \delta\rho\,, \qquad \phi + \psi = -8\pi G\,\Pi\,, \tag{2}$$

where $\kappa = 12\pi G(\rho + p)av$ and Π is the scalar part of the anisotropic stress tensor.

Exercise 4: Curvature Perturbation

Show that the comoving-gauge curvature perturbation $\varphi_v = \phi - \mathcal{H}v$ is related to the Newtonian potential ϕ as

$$\varphi_v = \frac{5+3w}{3(1+w)}\phi \tag{3}$$

in the limit $k \to 0$. Assume that $\Pi = 0$.

Hint: In the limit $k \to 0$, you can assume $\dot{\phi} = 0$.