1 Newtonian Perturbation Theory

1.1 Standard Newtonian Perturbation Theory

1.1.1 Summary of the Governing Equations

In Newtonian dynamics, fully nonlinear equation pressureless fluid (CDM and baryons) can be written down:

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{v}=-\frac{1}{a}\nabla\cdot(\mathbf{v}\delta)\;,\qquad \nabla\cdot\dot{\mathbf{v}}+H\nabla\cdot\mathbf{v}+\frac{3H^2}{2}a\Omega_m\delta=-\frac{1}{a}\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\right]\;,\qquad \nabla^2\phi=4\pi G\rho\,. \tag{1.1}$$

The Euler equation can be split into one for divergence and one for vorticity. The vorticity vector $\nabla\times\mathbf{v}$ decays at the linear order. At nonlinear level, if no anisotropic pressure and no initial vorticity, the vorticity vanishes at all orders. However, in reality, the anisotropic pressure arises from shell crossing, generating vorticity on small scales, even in the absence of the initial vorticity. Of course, baryons are not exactly pressureless; they form galaxies, and their feedback effects are also important up to fairly large scales. These all modify the SPT equation.

• regime of validity, measurement precision, analytic vs numerical simulations, galaxy surveys

1.1.2 Basic Formalism

We consider multi-component fluids in the presence of isotropic pressure. In case of n-fluids with the mass densities ϱ_i, the pressures p_i, the velocities $\mathbf{v}_i$ ($i=1,2,\ldots n$), and the gravitational potential Φ, we have

$$\dot{\varrho}_i+\nabla\cdot(\varrho_i\mathbf{v}_i)=0\,,\qquad \dot{\mathbf{v}}_i+\mathbf{v}_i\cdot\nabla\mathbf{v}_i=-\frac{1}{\varrho_i}\nabla p_i-\nabla\Phi\,,\qquad \nabla^2\Phi=4\pi G\sum_{j=1}^{n}\varrho_j\,. \tag{1.2}$$

Assuming the presence of spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

$$\varrho_i=\bar{\varrho}_i+\delta\varrho_i,\qquad p_i=\bar{p}_i+\delta p_i\,,\qquad \mathbf{v}_i=H\mathbf{r}+\mathbf{u}_i\,,\qquad \Phi=\bar{\Phi}+\delta\Phi\,, \tag{1.3}$$

where $H:=\dot{a}/a$, and $a(t)$ is a cosmic scale factor. We move to the comoving coordinate $\mathbf{x}$ where

$$\mathbf{r}:=a(t)\mathbf{x}\,,\qquad \nabla=\nabla_{\mathbf{r}}=\frac{1}{a}\nabla_{\mathbf{x}}\,,\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}\Big|_{\mathbf{r}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}+\left(\frac{\partial}{\partial t}\Big|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}-H\mathbf{x}\cdot\nabla_{\mathbf{x}}\,. \tag{1.4}$$

In the following we neglect the subindex $\mathbf{x}$. To the background order we derive

$$\dot{\bar{\varrho}}_i+3H\bar{\varrho}_i=0\,,\qquad \dot{H}+H^2=\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\sum_j\bar{\varrho}_j\,,\qquad H^2=\frac{8\pi G}{3}\sum_j\bar{\varrho}_j+\frac{2E}{a^2}\,, \tag{1.5}$$

where the second equation is derived by taking the divergence of the Euler equation and for the third equation we used

$$\left(a^2H^2\right)^{\cdot}=2a^2H\left(H^2+\dot{H}\right)\,,\qquad \sum\left(a^2\bar{\varrho}\right)^{\cdot}=-a^2H\sum\bar{\varrho}\,. \tag{1.6}$$

The integration constant E can be interpreted s as the specific total energy in Newton's gravity; in Einstein's gravity we have $2E=-Kc^2$ where K can be normalized to be the sign of spatial curvature. Note the difference in the background equation in Newtonian cosmology. The nonlinear governing equations can be expressed in terms of the perturbation variables as

$$\dot{\delta}_i+\frac{1}{a}\nabla\cdot\mathbf{u}_i=-\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i)\;,\qquad \frac{1}{a^2}\nabla^2\delta\Phi=4\pi G\sum_j\bar{\varrho}_j\delta_j\,, \tag{1.7}$$

$$\dot{\mathbf{u}}_i+H\mathbf{u}_i+\frac{1}{a}\mathbf{u}_i\cdot\nabla\mathbf{u}_i=-\frac{1}{a\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i}-\frac{1}{a}\nabla\delta\Phi\,. \tag{1.8}$$

By introducing the expansion θ_i and the rotation $\overrightarrow{\omega}_i$ of each component as

$$\theta_i := -\frac{1}{a}\nabla\cdot\mathbf{u}_i \, , \qquad \overrightarrow{\omega}_i := \frac{1}{a}\nabla\times\mathbf{u}_i \, , \tag{1.9}$$

we derive

$$\dot{\theta}_i + 2H\theta_i - 4\pi G\sum_j \bar{\varrho}_j\delta_j = \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar{\varrho}_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right) \, , \tag{1.10}$$

$$\dot{\overrightarrow{\omega}}_i + 2H\overrightarrow{\omega}_i = -\frac{1}{a^2}\nabla\times(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar{\varrho}_i}\frac{(\nabla\delta_i)\times\nabla\delta p_i}{(1+\delta_i)^2} \, . \tag{1.11}$$

By introducing decomposition of perturbed velocity into the potential- and transverse parts as

$$\mathbf{u}_i := -\nabla U_i + \mathbf{u}_i^{(v)} \, , \qquad \nabla\cdot\mathbf{u}_i^{(v)} \equiv 0 \, , \qquad \theta_i = \frac{\Delta}{a}U_i \, , \quad \overrightarrow{\omega}_i = \frac{1}{a}\nabla\times\mathbf{u}_i^{(v)} \, , \tag{1.12}$$

we have

$$\dot{\mathbf{u}}_i^{(v)} + H\mathbf{u}_i^{(v)} = -\frac{1}{a}\left[\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i} - \nabla\Delta^{-1}\nabla\cdot\left(\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i}\right)\right] \, . \tag{1.13}$$

Combining equations above, we can derive

$$\ddot{\delta}_i + 2H\dot{\delta}_i - 4\pi G\sum_j \bar{\varrho}_j\delta_j = -\frac{1}{a^2}\left[a\nabla\cdot(\delta_i\mathbf{u}_i)\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar{\varrho}_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right) \, . \tag{1.14}$$

These equations are valid to fully nonlinear order. The density fluctuation grows against the Hubble friction. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.

• numerical simulations, baryons

1.1.3 Linear-Order and Second-Order Solutions

We will derive the solutions for a single pressureless medium (now we change notation $\boldsymbol{u}_i \to \mathbf{v}$)

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{v} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{v}) \, , \qquad \dot{\theta} + 2H\theta - 4\pi G\bar{\varrho}\delta = \frac{1}{a^2}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) \, , \tag{1.15}$$

where we now use $\mathbf{v}$ to represent the velocity perturbation. These are the governing equation for the cosmological N-body simulations. The calculations are greatly simplified in Fourier space, and our convention is

$$A(\mathbf{x}) \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, A(\mathbf{k}) \, , \qquad A(\mathbf{k}) \equiv \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, A(\mathbf{x}) \, , \tag{1.16}$$

and we often use the identity:

$$\delta^D(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \, . \tag{1.17}$$

First, we derive the linear-order solution. The conservation equation yields

$$\dot{\delta}^{(1)}(t,\mathbf{k}) = \theta^{(1)}(t,\mathbf{k}) \, . \tag{1.18}$$

At the linear order in perturbations, we can separate the time-dependence and the spatial-dependence, i.e., all different Fourier modes evolve at the same rate, and the growth rate D satisfies

$$\ddot{D} + 2H\dot{D} - 4\pi G\bar{\rho}_m D = 0 \, , \qquad D(t) \equiv \frac{D_1(t)}{D_1(t_0)} \, , \tag{1.19}$$

where the (dimensionless) growth factor $D(t)$ is normalized to unity at some early epoch t_0 when the nonlinearities are ignored $\delta(t_0,\mathbf{k}):=\delta^{(1)}_1(t_0,\mathbf{k})\equiv\hat\delta(\mathbf{k})$. The linear-order solution for the density and the velocity divergence is then

$$\delta^{(1)}(t,\mathbf{k})=D(t)\hat\delta(\mathbf{k})~,\qquad \theta^{(1)}(t,\mathbf{k})=HfD(t)\hat\delta(\mathbf{k})~,\qquad f:=\frac{d\ln D}{d\ln a}~,\qquad \dot D\equiv HfD~,\tag{1.20}$$

where the superscript indicates the perturbation order, the logarithmic growth rate f is approximately time-independent and it is unity $f=1$ in the matter-dominated era.

To derive the second-order solution, we need to Fourier decompose the source terms in the right-hand side of the dynamical equation. At the second-order in perturbations, the quadratic terms represent the product of two linear-order terms. Furthermore, the quadratic product in configuration space becomes the convolution in Fourier space:

$$\left[-\frac{1}{a}\nabla\cdot(\delta\mathbf{v})\right]^{(2)}=HfD^2\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\left(1+\frac{\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{Q_1^2}\right)\hat\delta(\boldsymbol{Q}_1)\hat\delta(\boldsymbol{Q}_2)~,\tag{1.21}$$

$$\left\{\frac{1}{a^2}\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right\}^{(2)}=H^2f^2D^2\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\frac{|\boldsymbol{Q}_1+\boldsymbol{Q}_2|^2\boldsymbol{Q}_1\cdot\boldsymbol{Q}_2}{2Q_1^2Q_2^2}\hat\delta(\boldsymbol{Q}_1)\hat\delta(\boldsymbol{Q}_2)~,\tag{1.22}$$

where we defined $\boldsymbol{Q}_{12}=\boldsymbol{Q}_1+\boldsymbol{Q}_2$. Using the source functions in Fourier space, we can solve the governing equations for the density and the velocity divergence as

$$\frac{\delta^{(2)}(t,\mathbf{k})}{D^2}=\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12})\hat\delta(\mathbf{q}_1)\hat\delta(\mathbf{q}_2)\left[\frac{5}{7}+\frac{2}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2q_2^2}+\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)\right]~,\tag{1.23}$$

$$\frac{\theta^{(2)}(t,\mathbf{k})}{HfD^2}=\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12})\hat\delta(\mathbf{q}_1)\hat\delta(\mathbf{q}_2)\left[\frac{3}{7}+\frac{4}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2q_2^2}+\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)\right]~.\tag{1.24}$$

• HW: derive the second-order solutions

1.1.4 General Solution

Beyond the linear order, the density and the velocity divergence grows in a nonlinear fashion, i.e., different Fourier modes couple. By assuming the separability of the time and the spatial dependences, the standard perturbation theory (SPT) takes a perturbative approach to the nonlinear solution:

$$\delta(t,\mathbf{k}):=\sum_{n=1}^\infty D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat\delta(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})F_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n)\equiv\sum_{n=1}^\infty D^n(t)\delta^{(n)}(\mathbf{k})~,\tag{1.25}$$

$$\frac{\theta(t,\mathbf{k})}{Hf}:=\sum_{n=1}^\infty D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat\delta(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})G_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n)\equiv\sum_{n=1}^\infty D^n(t)\theta^{(n)}(\mathbf{k})~,\tag{1.26}$$

where $\mathbf{q}_{12\cdots n}\equiv\mathbf{q}_1+\cdots+\mathbf{q}_n$, $\delta^{(n)}(\mathbf{k})$ and $\theta^{(n)}(\mathbf{k})$ are time-independent n-th order perturbations, $F_n^{(s)}$ and $G_n^{(s)}$ are the SPT kernels symmetrized over its arguments. With these decompositions in Fourier space, the LHS of the Newtonian dynamical equations become

$$\dot\delta+\theta=Hf\sum_{n=1}^\infty D^n\left(n\delta^{(n)}-\theta^{(n)}\right)~,\qquad \dot\theta+2H\theta-4\pi G\bar\rho_m\delta=H^2f^2\sum\frac{D^n}{2}\left[(1+2n)\theta^{(n)}-3\delta^{(n)}\right]~,\tag{1.27}$$

where we utilized the relation between the growth factor and the growth rate $\dot D=HDf$. The RHS of the Newtonian dynamical equations are the convolution in Fourier space:

$$\left[-\frac{1}{a}\nabla\cdot(\delta\mathbf{v})\right](\mathbf{k})=\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\alpha_{12}\theta(\boldsymbol{Q}_1,t)\delta(\boldsymbol{Q}_2,t)\equiv Hf\sum_{n=1}^\infty D^nA_n(\mathbf{k})~,\tag{1.28}$$

$$\left\{\frac{1}{a^2}\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right\}(\mathbf{k})=\int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\beta_{12}\theta(\boldsymbol{Q}_1,t)\theta(\boldsymbol{Q}_2,t)\equiv H^2f^2\sum_{n=1}^\infty D^nB_n(\mathbf{k})~,\tag{1.29}$$

where the vertex functions are defined as

$$\alpha_{12} := \alpha(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \equiv 1 + \frac{\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2}{Q_1^2} , \qquad \beta_{12} := \beta(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \equiv \frac{|\boldsymbol{Q}_1 + \boldsymbol{Q}_2|^2 \boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2}{2Q_1^2 Q_2^2} , \tag{1.30}$$

and the n-th order perturbation kernels $A_n(\mathbf{k})$ and $B_n(\mathbf{k})$ are

$$A_n(\mathbf{k}) = \left[\prod_i^n \int \frac{d^3\mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i)\right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\cdots n}) \sum_{i=1}^{n-1} \alpha_{12} G_i(\mathbf{q}_1, \cdots, \mathbf{q}_i) F_{n-i}(\mathbf{q}_{i+1}, \cdots, \mathbf{q}_n) , \tag{1.31}$$

$$B_n(\mathbf{k}) = \left[\prod_i^n \int \frac{d^3\mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i)\right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\cdots n}) \sum_{i=1}^{n-1} \beta_{12} G_i(\mathbf{q}_1, \cdots, \mathbf{q}_i) G_{n-i}(\mathbf{q}_{i+1}, \cdots, \mathbf{q}_n) , \tag{1.32}$$

with $\boldsymbol{Q}_1 = \mathbf{q}_{1\cdots i}$ and $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \mathbf{k}$.

Therefore, the two Newtonian dynamical equations become algebraic equations with the time-dependence removed:

$$n\delta^{(n)} - \theta^{(n)} = A_n , \qquad (1+2n)\theta^{(n)} - 3\delta^{(n)} = 2B_n , \tag{1.33}$$

and the well-known recurrence formulas for the solutions are

$$\delta^{(n)} = \frac{(1+2n)A_n + 2B_n}{(2n+3)(n-1)} , \qquad \theta^{(n)} = \frac{3A_n + 2nB_n}{(2n+3)(n-1)} , \tag{1.34}$$

and similarly so for the SPT kernels

$$F_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n+3)(n-1)} \left[(1+2n)\alpha_{12} F_{n-i} + 2\beta_{12}\, G_{n-i}\right] , \tag{1.35}$$

$$G_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n+3)(n-1)} \left[3\alpha_{12} F_{n-i} + 2n\beta_{12} G_{n-i}\right] , \qquad F_1 = G_1 = 1 . \tag{1.36}$$