4 Standard Inflationary Models

Standard single field inflationary models provide a mechanism for the inflationary expansion (horizon problem) and the perturbation generation (initial condition) by a single scalar field, called inflaton. The scalar field Lagrangian has the canonical kinetic term, but various single field models differ in the scalar field potential, according to which the inflaton rolls over. In most cases, the slow-roll condition is adopted, such that the scalar field dynamics is insensitive to the details of the scalar field potential.

The outcome of the standard model predictions is as follows: The curvature fluctuations are scale-invariant ($n_s \simeq 1$) and highly Gaussian. The tensor fluctuations are also scale-invariant, but its amplitude is very small compared to the scalar fluctuations. The running of the indices is very small. Recent observations confirm these predictions and constrain the parameters with high precision. However, beyond these basic features/constraints, we do not have a solid model for inflation. Note that the energy scale of inflation is beyond the validity of the standard model physics, and most inflationary models have many theoretical issues, when quantum corrections are considered.

4.1 Single Scalar Field

4.1.1 Scalar Field Action

In addition to the Einstein-Hilbert action for gravity, we consider the action for a scalar field with canonical kinetic term and the potential V:

$$S = \int \sqrt{-g}\, d^4x \left[\frac{c^4}{16\pi G}\, R - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi)\right] , \tag{4.1}$$

where the kinetic term in the Minkowski spacetime reduces to the standard form

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi = \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2\right] . \tag{4.2}$$

The Euler-Lagrange equation yields the equation of motion for the scalar field

$$\Box\phi - V_{,\phi} = 0 , \qquad \Box := g^{\mu\nu}\nabla_\mu\nabla_\nu , \tag{4.3}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\,\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\,\phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu} . \tag{4.4}$$

It is often in literature that the Planck unit is adopted, and there exist two different conventions:

$$M_{\rm pl}^2 := \frac{1}{8\pi G} , \qquad m_{\rm pl}^2 := \frac{1}{G} . \tag{4.5}$$

4.1.2 Background Relation and Evolution Equations

In the background, the non-vanishing fluid quantities for a scalar field are the energy density and the pressure

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi) , \qquad p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi) , \tag{4.6}$$

and the equation of motion becomes

$$\ddot\phi + 3H\dot\phi + V_{,\phi} = 0 , \qquad \phi'' + 2\mathcal{H}\phi' + a^2V_{,\phi} = 0 . \tag{4.7}$$

The Friedmann equation for a scalar field is

$$H^2 = \frac{\rho_\phi}{3M_{\rm pl}^2} , \qquad \dot H = -\frac{\rho_\phi + p_\phi}{2M_{\rm pl}^2} = -\frac{\dot\phi^2}{2M_{\rm pl}^2} , \qquad H^2 + \dot H = \frac{\ddot a}{a} = \frac{1}{3M_{\rm pl}^2}\left(V - \dot\phi^2\right) , \tag{4.8}$$

where we assumed a flat universe and no cosmological constant. If the potential energy of the scalar field is the dominant energy component of the Universe or the kinetic energy is smaller than the potential energy (slow-roll), the expansion of the Universe is accelerating $\ddot a > 0$. Various inflationary models with slow-roll condition state that the potential is sufficiently flat, such that $V(\phi)$ is nearly constant during the inflationary period and ϕ slowly evolves (rolls over V).

4.1.3 de-Sitter Spacetime

The de-Sitter universe is a highly symmetric spacetime, defined as a background FRW universe with no matter and constant Hubble parameter. A constant Hubble parameter leads to an exponential expansion, and we parametrize the de-Sitter solution as

$$H^2 := \frac{\Lambda}{3} , \qquad a(t) = e^{Ht} = -\frac{1}{H\eta} , \qquad a = (0,\infty) , \qquad t = (-\infty,\infty) , \qquad \eta = (-\infty, 0) , \tag{4.9}$$

where the scale factor is normalized at $t = 0$. The slow-roll parameter for the de-Sitter spacetime is

$$\varepsilon := -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right) = 0 . \tag{4.10}$$

4.1.4 Slow-Roll Parameters

In general, inflationary models slightly deviate from the de-Sitter phase ($\varepsilon \neq 0$), and its deviation is captured by the slow-roll parameter:

$$\varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot{H}}{H^2} , \qquad \dot{H} = -H^2\varepsilon , \qquad \frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1-\varepsilon) , \qquad (3-\varepsilon)H^2 = \frac{V}{M_{\rm pl}^2} . \tag{4.11}$$

To solve the horizon problem, we know that the comoving horizon has to decrease in time

$$0 > \frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) = -\frac{\ddot{a}}{a^2H^2} = -\frac{1-\varepsilon}{a} . \tag{4.12}$$

The background evolution of a scalar field can be re-phrased in terms of the slow-roll parameters as

$$\varepsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2M_{\rm pl}^2} = \frac{3}{2}(1+w) , \qquad \dot{\phi}^2 = \rho_\phi + p_\phi . \tag{4.13}$$

If we ignore the second derivative of the field ($\ddot{\phi} \simeq 0$) in the equation of motion,

$$3H\dot{\phi} \simeq -V_{,\phi} , \qquad \rho_\phi + p_\phi \simeq \left(\frac{V_{,\phi}}{3H}\right)^2 , \tag{4.14}$$

the slow-roll parameters are then further related to the slow-roll parameters defined in terms of the derivatives of the potential also used below)

$$\varepsilon_V := \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \simeq \varepsilon , \qquad \eta_V := M_{\rm pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) \simeq \varepsilon + \eta , \qquad \xi_V := \frac{M_{\rm pl}^4 V_{,\phi}V_{,\phi\phi\phi}}{V^2} , \tag{4.15}$$

where we used the second slow-roll parameter

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} . \tag{4.16}$$

In fact, one can show the exact relation

$$\varepsilon = \varepsilon_V\left(1 - \frac{4}{3}\varepsilon_V + \frac{2}{3}\eta_V\right) . \tag{4.17}$$

In literature, different convention for slow-roll parameters are often used, in particular, in terms of Hubble flow:

$$\varepsilon_1 := \varepsilon , \qquad \varepsilon_2 := \frac{1}{H}\frac{d\ln\varepsilon}{dt} = 2(\varepsilon - \eta) , \qquad \varepsilon_{i+1} := \frac{1}{H}\frac{d\ln\varepsilon_i}{dt} . \tag{4.18}$$

Furthermore, the inflation has to last for some time, such that the modes we measure in CMB have to expand at least by $40-60$ e-folds. So it is convenient to define the number of e-folding for a given mode as the number of e-folds the mode k expanded from the horizon crossing until the end of inflation,[1]

$$N(\phi_k) := \ln\frac{a_{\rm end}}{a(\phi_k)} = \int_{t_k}^{t_{\rm end}} H\, dt , \qquad k = aH , \tag{4.19}$$

[1]The end of inflation is a bit ill-defined, as we do not have a concrete model. However, in terms of N we can safely use the condition that the slow-roll parameter becomes order unity $\varepsilon \simeq 1$.

where t_k is the time the k-mode crosses the horizon. Using the e-folding number, we can express the slow-roll parameters as

$$dN = Hdt = d\ln a \,, \qquad \varepsilon = -\frac{d\ln H}{dN} \,, \qquad \varepsilon_{i+1} = \frac{d\ln\varepsilon_i}{dN} \,. \tag{4.20}$$

4.1.5 Linear-Order Evolution

Given the energy momentum tensor, we can derive the fluid quantities for a scalar field:

$$\delta\rho_\phi = \dot\phi\dot{\delta\phi} - \dot\phi^2\alpha + V_{,\phi}\delta\phi = \delta\rho_v - 3H\dot\phi\,\delta\phi \,, \qquad \delta\rho_v := \delta\rho - \rho' v \,, \tag{4.21}$$

$$\delta p_\phi = \dot\phi\dot{\delta\phi} - \dot\phi^2\alpha - V_{,\phi}\delta\phi = \delta\rho_v - 3c_s^2 H\dot\phi\,\delta\phi \,, \qquad v_\phi = \frac{\delta\phi}{\phi'} \,, \tag{4.22}$$

$$e := \delta p - c_s^2\,\delta\rho = (1-c_s^2)\delta\rho_v \,, \qquad \pi^\phi_{\alpha\beta} = q^\phi_\alpha = 0 \,, \tag{4.23}$$

where we used the following relation and the sound speed is defined as

$$\dot\rho_\phi = \dot\phi(\ddot\phi + V_{,\phi}) = -3H\dot\phi^2 \,, \qquad \dot p_\phi = \dot\phi(\ddot\phi - V_{,\phi}) = \dot\phi(2\ddot\phi + 3H\dot\phi) \,, \qquad c_s^2 := \frac{\dot p_\phi}{\dot\rho_\phi} = -1 - \frac{2\ddot\phi}{3H\dot\phi} \,. \tag{4.24}$$

Therefore, the comoving gauge corresponds to the uniform field gauge for the single-field models:

$$\varphi_v = \varphi - \mathcal{H}v = \varphi - H\frac{\delta\phi}{\dot\phi} = \varphi_{\delta\phi} \,. \tag{4.25}$$

The equation of motion for a scalar field is then

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(V_{,\phi\phi} + \frac{k^2}{a^2}\right)\delta\phi = \dot\phi(\dot\alpha + \kappa) + (2\ddot\phi + 3H\dot\phi)\alpha \,. \tag{4.26}$$

Using the Einstein equations, we derive the governing equation for Mukhanov variable Φ (which is the comoving-gauge curvature)

$$\Phi := \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)}\,\varphi_\chi = \frac{H^2}{4\pi G(\rho+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{2H^2\Pi}{\rho+p} \,, \tag{4.27}$$

$$\dot\Phi = -\frac{H}{4\pi G(\rho+p)}\frac{k^2 c_s^2}{a^2}\varphi_\chi - \frac{H}{\rho+p}\left(e - \frac{2k^2}{3a^2}\Pi\right) \equiv -\frac{Hc_A^2}{4\pi G(\rho+p)}\frac{k^2}{a^2}\varphi_\chi \,, \tag{4.28}$$

where the derivation is fully general and we defined the physical sound speed c_A for inflaton

$$c_A^2 := c_s^2 + 4\pi G\frac{a^2}{k^2}\frac{e}{\varphi_\chi} \equiv 1 \,. \tag{4.29}$$

It is clear that the comoving-gauge curvature is conserved on super horizon scales.

4.2 Quantum Fluctuations in Quadratic Action

The background relation describes the inflationary expansion, and the equation of motion we derived describes the evolution of the perturbations at the linear order. Here we will derive their statistical properties. However, before we proceed, we need to better understand the structure of the theory. Even for the standard inflationary models of a single field, the theory is not a free-field, but an interacting field theory.

This can be illustrated as follows. To simplify the calculations, we choose the comoving gauge

$$0 = v_\phi = \frac{\delta\phi}{\phi'} \,, \qquad \phi(x) = \bar\phi(t) \,, \qquad \zeta := \varphi_v = \varphi_{\delta\phi} \,, \tag{4.30}$$

and it coincides with the uniform field gauge. Our main variable for scalar fluctuation is then the comoving gauge curvature ζ, as the scalar field is uniform. We can expand the action perturbatively to give

$$S = S_0[\bar\phi, \bar g_{ab}] + S_2[\zeta^2] + S_3[\zeta^3] + \cdots \,, \qquad H = H_0 + H_{\rm int} \,, \qquad H_{\rm int} = \sum_i F_i(\varepsilon, \eta, \cdots)\zeta^3(\tau) + \cdots \,, \tag{4.31}$$

where the background action S_0 defines the background evolution and its slow-roll parameters. Here we will study the quadratic action S_2 in great detail to derive the power spectrum of the scalar and tensor fluctuations, and the quadratic action is indeed a free-field action in the de-Sitter background (or with small deviations around it). However, remember that the full theory is interacting, and we cannot use the free-field theory to quantize the fluctuations, if we go beyond the quadratic action or compute the high-order correlation functions.

4.2.1 Quadratic Action for Scalars

To derive the linear-order equation of motion, we need to expand the action to the quadratic in perturbations. To simplify the calculations, we choose the comoving gauge. After some integrations by part of the quadratic action, the quadratic action for scalars in the comoving gauge becomes[2]

$$S_{(2)} = \frac{1}{2}\int dt\, d^3\mathbf{x}\ a^3\frac{\dot\phi^2}{H^2}\left[\dot\zeta^2 - \frac{1}{a^2}(\nabla\zeta)^2\right] = \frac{1}{2}\int d\eta\, d^3\mathbf{x}\ \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right] , \tag{4.32}$$

where we assume $M_{\rm pl} = 1$ and we defined the canonically-normalized (Mukhanov-Sasaki) variable

$$v := z\zeta\ , \qquad \zeta := \varphi_v\ , \qquad z^2 := a^2\frac{\dot\phi^2}{H^2} = 2a^2\varepsilon\ . \tag{4.33}$$

The Lagrangian now takes the form of the simple harmonic oscillator, but with time-dependent mass term

$$m^2(\eta) := -\frac{z''}{z} \underset{\rm dS}{\longrightarrow} -\frac{a''}{a} = -\frac{2}{\eta^2}\ . \tag{4.34}$$

where we took the de-Sitter limit ($\varepsilon = z = 0$). The canonical momentum and the Hamiltonian are then

$$\pi = \frac{\delta\mathcal{L}}{\delta v'} = v'\ , \qquad \mathcal{H} = \pi v' - \mathcal{L} = \frac{1}{2}\left[(v')^2 + (\nabla v)^2 + m^2v^2\right]\ . \tag{4.35}$$

The equation of motion for the Mukhanov-Sasaki variable is the Klein-Gordon equation:

$$\left(\Box - m^2\right)v = 0\ , \qquad v''_\mathbf{k} + \omega_k^2 v_\mathbf{k} = 0\ , \qquad \omega_k^2 := k^2 + m^2\ , \qquad v_\mathbf{k} = v^*_{-\mathbf{k}}\ . \tag{4.36}$$

The mode functions take the simple solution for the time-dependence under the assumption that $\omega_k \simeq k$ is time-independent in the limit $\eta \to -\infty$:

$$v_\mathbf{k}(\eta) \equiv v^+_\mathbf{k}e^{i\omega_k\eta} + v^-_\mathbf{k}e^{-i\omega_k\eta} := v^+_\mathbf{k}(\eta) + v^-_\mathbf{k}(\eta)\ , \qquad v^+_\mathbf{k} = (v^-_{-\mathbf{k}})^\dagger\ , \tag{4.37}$$

where the amplitude of the mode functions are undetermined. Therefore, the general solution can be written as

$$v(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_\mathbf{k}e^{i\omega_k\eta} + v^-_\mathbf{k}e^{-i\omega_k\eta}\right)e^{i\mathbf{k}\cdot\mathbf{x}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_{-\mathbf{k}}\,e^{-ikx} + v^-_\mathbf{k}e^{ikx}\right)\ , \qquad k := (\omega_k, \mathbf{k})\ . \tag{4.38}$$

4.2.2 Canonical Quantization

So far, we have derived a classical solution of the quadratic action for scalars. By promoting the Mukhanov-Sasaki field v and its canonical momentum field π to quantum fields, we need to impose the canonical quantization relation ($\hbar = 1$)

$$[\hat v(\eta,\mathbf{x}), \hat\pi(\eta,\mathbf{y})] = i\delta^{\rm 3D}(\mathbf{x}-\mathbf{y})\ , \qquad [\hat v(\eta,\mathbf{x}), \hat v(\eta,\mathbf{y})] = [\hat\pi(\eta,\mathbf{x}), \hat\pi(\eta,\mathbf{y})] = 0\ , \tag{4.39}$$

where we work in the Heisenberg picture for the time-dependent operators. Apparent from the notation, we want to define the creation and annihilation operators as

$$v^-_\mathbf{k} := \hat a_\mathbf{k}v^-_k\ , \qquad \left(v^-_\mathbf{k}\right)^\dagger = v^+_{-\mathbf{k}} = \hat a^\dagger_\mathbf{k}v^+_k\ , \qquad (v^-_k)^* = v^+_k\ , \tag{4.40}$$

[2]Here, "scalars" are used to refer to the scalar fluctuations, not to be confused with the scalar field.

such that we derive

$$\hat{v}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\hat{a}^\dagger_\mathbf{k} v^+_k \, e^{-ikx} + \hat{a}_\mathbf{k} v^-_k \, e^{ikx} \right) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\hat{a}^\dagger_\mathbf{k} v^+_k(\eta) \, e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_\mathbf{k} v^-_k(\eta) \, e^{i\mathbf{k}\cdot\mathbf{x}} \right] , \tag{4.41}$$

where we defined

$$v^\pm_k(\eta) := v^\pm_k e^{\pm i\omega_k \eta} . \tag{4.42}$$

By substituting into the canonical quantization relation, we can derive that the ladder operators indeed satisfy the standard quantization relation at the equal time

$$[\hat{a}_\mathbf{k}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta^{3\mathrm{D}}(\mathbf{k} - \mathbf{k}') , \qquad [\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_\mathbf{k}, \hat{a}^\dagger_{\mathbf{k}'}] = 0 , \tag{4.43}$$

if the normalization for the mode functions is properly chosen

$$W[v^-_k, v^+_k] := v^-_k v^{+\prime}_k - v^{-\prime}_k v^+_k := i . \tag{4.44}$$

With the properly normalized operators, we obtain the usual relations

$$\hat{a}_\mathbf{k}|0\rangle = 0 , \qquad \langle 0|0\rangle = 1 , \qquad |n_\mathbf{k}\rangle = \sqrt{\frac{2E_k}{n!}} \, [(\hat{a}^\dagger_\mathbf{k})^n]|0\rangle , \tag{4.45}$$

where $\sqrt{2E}$ is put to make it Lorentz invariant. One can quantize the field, starting with the time-independent Harmonic oscillators, then applying the Heisenberg picture with the free-field Hamiltonian, as in Peskin & Schröder.

4.2.3 Vacuum Expectation Value

While we imposed the normalization condition for the mode functions in terms of their Wronskian, the physical vacuum is yet to be fully determined, due to the arbitrariness in the mode functions. Consider a different set of mode functions $u^\pm_k$ that are related to the original mode functions as

$$u^-_k(\eta) = \alpha_k v^-_k(\eta) + \beta_k v^+_k(\eta) , \tag{4.46}$$

and construct the creation and annihilation operators $\hat{b}^\pm_\mathbf{k}$ with $u^\pm_k$

$$u^-_\mathbf{k} := \hat{b}_\mathbf{k} u^-_k . \tag{4.47}$$

Using this relation, we can write the operator $\hat{v}$ and its canonical momentum $\hat{\pi}$ in terms of $\hat{b}_\mathbf{k}$ and $\hat{b}^\dagger_\mathbf{k}$. These two sets of quantum operators are then related as by, so called, the Bogolyubov transformation:

$$\hat{a}_\mathbf{k} = \alpha^*_k \hat{b}_\mathbf{k} + \beta_k \hat{b}^\dagger_{-\mathbf{k}} , \qquad \hat{a}^\dagger_\mathbf{k} = \alpha_k \hat{b}^\dagger_\mathbf{k} + \beta^*_k \hat{b}_{-\mathbf{k}} , \qquad |\alpha_k|^2 - |\beta_k|^2 = 1 , \tag{4.48}$$

where the normalization for the transformation coefficients is due to the Wronskian normalization. Note that the vacuum defined by one set of operators $\hat{a}_\mathbf{k}$ is not the vacuum with respect to the other set of operators $\hat{b}_\mathbf{k}$. To properly determine the physical vacuum, we need to fix the mode function completely.

In terms of the mode functions, the Hamiltonian in Minkowski spacetime is

$$\hat{H} = \int d^3\mathbf{x} \, \hat{\mathcal{H}} , \qquad \hat{\mathcal{H}} = \frac{1}{2} \left[\hat{\pi}^2 + (\nabla \hat{v})^2 \right] , \qquad m \to 0 . \tag{4.49}$$

Using the expression for the mode function in Eq. (4.41), we derive the Hamiltonian

$$\hat{H} = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\left((v^{+\prime}_k)^2 + k^2 (v^+_k)^2 \right) \hat{a}^\dagger_\mathbf{k} \hat{a}^\dagger_{-\mathbf{k}} + \left((v^{-\prime}_k)^2 + k^2 (v^-_k)^2 \right) \hat{a}_\mathbf{k} \hat{a}_{-\mathbf{k}} + \left(\hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} + \hat{a}_\mathbf{k} \hat{a}^\dagger_\mathbf{k} \right) \left(|v^{-\prime}_k|^2 + k^2 |v^-_k|^2 \right) \right] , \tag{4.50}$$

acting on the vacuum $|0\rangle$ as

$$\hat{H}|0\rangle = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\left((v^{+\prime}_k)^2 + k^2 (v^+_k)^2 \right) \hat{a}^+_\mathbf{k} \hat{a}^+_{-\mathbf{k}} + \left(|v^{-\prime}_k|^2 + k^2 |v^-_k|^2 \right) (2\pi)^3 \delta^{3\mathrm{D}}(0) \right] |0\rangle . \tag{4.51}$$

The vacuum $|0\rangle$ is an eigenstate of the Hamiltonian, and indeed the first round bracket vanishes. Given the normalization of the Wronskian, the physical mode function is then

$$W[v_k^-, v_k^+] = 2ik|v_k^-|^2 = i \,, \qquad v_k^-(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \,. \tag{4.52}$$

Therefore, we derive the vacuum expectation values

$$\left\langle 0|\hat{v}_{\mathbf{k}}^\dagger \hat{v}_{\mathbf{k}'}|0\right\rangle = (2\pi)^3\delta^{3D}(\mathbf{k}-\mathbf{k}')P_v(k) \,, \qquad P_v(k) = |v_k^-|^2 = \frac{1}{2k} \,. \tag{4.53}$$

4.2.4 Scalar Fluctuations

Now we consider the time-dependent mass term in the equation of motion, and following the same procedure we pick the vacuum that corresponds to the solution in the Minkowski spacetime as the modes were deep inside the horizon in the far past

$$v_k(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}}\, e^{-i\omega_k(\eta)\eta} \,, \qquad \lim_{\eta\to-\infty} v_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \,, \tag{4.54}$$

and this choice is called the Bunch-Davis vacuum. To the zero-th order in the slow-roll approximation ($\varepsilon = 0$), the inflationary period is the de-Sitter spacetime, in which

$$m^2(\eta) = -\frac{a''}{a} = -\frac{2}{\eta^2} \,, \qquad \omega_k^2 = k^2 - \frac{2}{\eta^2} \,, \tag{4.55}$$

and we can derive the exact solution for the mode functions:

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \,. \tag{4.56}$$

When the k-mode is stretched beyond the horizon, the amplitude of the mode function is

$$\lim_{k\eta\to0} v_k(\eta) = \frac{1}{i\sqrt{2}}\frac{1}{k^{3/2}\eta} \,, \qquad \lim_{k\eta\to0} k^3|v_k|^2 = \frac{1}{2\eta^2} = \frac{a^2H^2}{2} \,, \tag{4.57}$$

and the power spectra of the mode function and the comoving-gauge curvature are

$$P_v \equiv |v_k|^2 = \frac{a^2H^2}{2k^3} \,, \qquad \Delta_\zeta^2 := \frac{k^3}{2\pi^2}P_\zeta = \frac{1}{2a^2\varepsilon}\Delta_v^2 = \frac{H^2}{8\pi^2\varepsilon} \,. \tag{4.58}$$

4.2.5 Tensor Fluctuations: Gravity Waves

We can repeat the exercise for the scalar fluctuations to derive the tensor fluctuations. The quadratic action for tensor is

$$S_{(2)} = \frac{M_{\rm pl}^2}{8}\int d\eta\, d^3\mathbf{x}\, a^2\left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] = \sum_{s=\pm2}\int d\eta\, d^3\mathbf{k}\, \frac{a^2}{4}\, M_{\rm pl}^2\left[(h_{\mathbf{k}}^{s\prime})^2 - k^2(h_{\mathbf{k}}^s)^2\right] \,, \tag{4.59}$$

where we again decomposed the tensor in terms of two helicity eigenstates

$$h_{ij} := 2C_{ij}^{(t)} = 2h^{(\pm2)}Q_{ij}^{(\pm2)} \,. \tag{4.60}$$

From the action, the Mukhanov-Sasaki variable for tensor fluctuations is

$$v_{\mathbf{k}}^s := \frac{a}{2}M_{\rm pl}h_{\mathbf{k}}^s \,, \qquad m = 0 \,, \tag{4.61}$$

and we can readily derive the tensor power spectrum

$$P_v = \frac{(aH)^2}{2k^3} \,, \qquad P_T := 2P_{h_k^s} = 2\left(\frac{2}{aM_{\rm pl}}\right)^2 P_v = \frac{4}{k^3}\frac{H^2}{M_{\rm pl}^2} \,. \tag{4.62}$$

The amplitude of the tensor power spectrum is the energy scale of the inflation in the early Universe, and its ratio to the scalar power spectrum is

$$r := \frac{\Delta_t^2}{\Delta_s^2} = \frac{8}{M_{\rm pl}^2}\frac{\dot\phi^2}{H^2} = 16\varepsilon \,, \tag{4.63}$$

slow-roll suppressed.

4.3 Predictions of the Standard Inflationary Models

4.3.1 Consistency Relations

For the standard single field inflationary models with the slow-roll approximation, we summarize the predictions for scalar fluctuations

$$P_\zeta = \left(\frac{2\pi^2}{k^3}\right) A_s \,, \qquad\qquad A_s := \frac{H^2}{8\pi^2 \varepsilon M_{\rm pl}^2} = \frac{1}{24\pi^2\varepsilon}\frac{V}{M_{\rm pl}^4} \,, \tag{4.64}$$

$$n_s - 1 := \frac{d\ln k^3 P_\zeta}{d\ln k} = (-2\varepsilon - \varepsilon_2)(1-\varepsilon)^{-1} \simeq 2\eta_V - 6\varepsilon_V \,, \tag{4.65}$$

the predictions for tensor fluctuations

$$P_T = \frac{4}{k^3}\frac{H_*^2}{M_{\rm pl}^2} = \left(\frac{2\pi^2}{k^3}\right) A_T \,, \qquad\qquad A_T := \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2} = \frac{2V}{3\pi^2 M_{\rm pl}^4} \,, \qquad\qquad n_t := \frac{d\ln k^3 P_T}{d\ln k} \simeq -2\varepsilon \,, \tag{4.66}$$

and the consistency relations

$$r := \frac{A_T}{A_s} = \frac{8\dot\phi_*^2}{H_*^2} = 16\varepsilon = -8n_t \,. \tag{4.67}$$

By measuring the power spectrum amplitude and its slope for both scalar and tensor fluctuations, we can ensure that the fluctuations are indeed generated by a single field inflaton or rule out the standard inflationary models. There exist other predictions in the standard inflationary models (and of course, for the beyond the standard models) that can be used to test models, such as the primordial non-Gaussianity and so on.

4.3.2 Lyth Bound

Given the definition of the e-folds, we can further manipulate it by using the inflaton as a time clock:

$$N(\phi_k) = \int_{\phi_k}^{\phi_{\rm end}} d\phi\,\frac{H}{\dot\phi} = \int_{\phi_k}^{\phi_{\rm end}} \frac{d\phi}{M_{\rm pl}\sqrt{2\varepsilon}} \,, \qquad\qquad r = 16\varepsilon = \frac{8}{M_{\rm pl}^2}\left(\frac{d\phi}{dN}\right)^2 \,, \tag{4.68}$$

and this relation further implies that the excursion of the inflaton field is related to the tensor-to-scalar ratio as

$$\frac{\Delta\phi_k}{M_{\rm pl}} \simeq \int_{N_{\rm end}}^{N_{\rm cmb}} dN\,\sqrt{\frac{r}{8}} \,, \tag{4.69}$$

where $\varepsilon(\phi_{\rm end}) \equiv 1$. To solve the horizon problem, the mode k should have expanded at least 40−60 in e-folds. So, this consistency relation (Lyth, 1997) implies that an inflationary field variation of the order of the Planck mass is needed to produce $r > 0.01$. From the theoretical point of view, this sets the upper bound on the amplitude of gravitational waves. Indeed, the standard inflationary model predictions are very small.

Note that the uncertainty in e-folds N is due to our ignorance in the reheating era: After the inflationary period ends, the inflaton field decays into other particles and reheats the Universe. This period is expected to be described by a matter-dominated era, as the inflaton oscillates around the minimum of the potential, effectively acting as a matter. However, we know very little about this period.

The current observational constraint is

$$A_s \simeq 2.2\times 10^{-9} \,, \qquad\qquad n_s \simeq 0.96 \,, \qquad\qquad \varepsilon \simeq 0.01 \,. \tag{4.70}$$

indicating the energy scale of the inflation is

$$A_T = \frac{2V}{3\pi^2 M_{\rm pl}^4} = 16\varepsilon A_s \,, \qquad\qquad H^2 = \frac{V}{3M_{\rm pl}^2} = \varepsilon\left(2\times 10^{14}\ {\rm GeV}\right)^2 \,. \tag{4.71}$$

4.3.3 A Worked Example

Here we consider a very simple inflationary model with a power-law potential:

$$V = \frac{1}{2} m^{4-\alpha} \phi^\alpha \,, \tag{4.72}$$

where the mass m and the slope α are the free parameters of the model. It chaotically starts everywhere at any time in field configurations, and its predictions are then

$$\varepsilon_V = \frac{\alpha^2}{2} \left(\frac{M_{\rm pl}}{\phi} \right)^2 \,, \qquad \eta_V = \alpha(\alpha-1) \left(\frac{M_{\rm pl}}{\phi} \right)^2 \,, \tag{4.73}$$

$$N \simeq \int \frac{d\phi}{M_{\rm pl}^2} \frac{V}{V'} = \frac{\phi^2 - \phi^2_{\rm end}}{2M_{\rm pl}^2 \alpha} \,, \qquad r \simeq 16 \varepsilon_V \,, \qquad n_s - 1 \simeq 2\eta_V - 6\varepsilon_V \,. \tag{4.74}$$

Approximating $\phi_{\rm end} \simeq 0$, we further derive

$$N \simeq \frac{1}{2\alpha} \left(\frac{\phi}{M_{\rm pl}} \right)^2 \,, \qquad \varepsilon_V \simeq \frac{\alpha}{4N} \,, \qquad \eta_V = \frac{\alpha-1}{2N} \,, \qquad 1 - n_s \simeq \frac{\alpha+2}{2N} \,, \qquad r \simeq \frac{4\alpha}{N} \,. \tag{4.75}$$

4.4 Adiabatic Modes and Isocurvature Modes

• ***Adiabatic modes***.— Assuming a flat Universe, we can arrange Eq. (4.28) to show

$$\dot\Phi = \Xi - \frac{H}{\rho + p} \frac{k^2}{a^2} \left(\frac{c_s^2}{4\pi G} \varphi_\chi - \frac{2}{3} \Pi \right) \,, \qquad \Xi := \frac{\dot{\bar\rho}\, \delta p - \dot{\bar p}\, \delta\rho}{3(\bar\rho + \bar p)^2} \equiv -\frac{He}{\rho + p} \,. \tag{4.76}$$

Therefore, in the limit $k \to 0$, if $\Xi = 0$ vanishes, the comoving-gauge curvature perturbation is conserved, regardless of contents in the Universe. Indeed, $\Xi = 0$ if the pressure is a function of the density, and it holds true for the matter-dominated era, the radiation-dominated era, and for the single field inflation.[3] This condition is called *adiabatic*, because individual components fluctuate at the same rate at a given point:

$$\frac{\delta\rho_i}{\dot\rho_i} = \frac{\delta\rho_{\rm tot}}{\dot\rho_{\rm tot}} = \frac{\delta p_i}{\dot p_i} = \frac{\delta p_{\rm tot}}{\dot p_{\rm tot}} \equiv -\varphi_v \, \mathcal{I} \,, \qquad \frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \quad \text{for } \forall\, a, b \,. \tag{4.77}$$

Even for single-field inflationary scenarios, there should have existed many other matter fields, and some energy transfer to these fields are inevitable. However, these non-adiabatic perturbations decay fast as the inflation proceeds, and they become exponentially suppressed when these matter fields dominate the energy budget during the reheating era.

In the limit $k \to 0$, we can indeed derive the adiabatic condition

$$\mathcal{I} := \frac{1}{a} \int_{t_i}^t dt\, a(t) \,, \qquad v_\chi \equiv -\frac{1}{a} \mathcal{I}\, \varphi_v \,. \tag{4.78}$$

• ***Isocurvature mode***.— The evolution of isocurvature perturbations depends not only on inflationary dynamics, but also on post-inflationary evolution. For example, if all particles thermalize after inflation, all isocurvature perturbations become adiabatic perturbations eventually. The isocurvature perturbations and the entropy perturbations are interchangeably used, because they do represent the perturbations between species and it does conserve the curvature. In practice, the entropy perturbations are parametrized by two free parameters at some pivot scale k_0 (0.002/Mpc in WMAP), i.e., ratio α of the isocurvature to the adiabatic perturbations and their correlation β

$$\frac{P_{\mathcal{S}}}{P_\zeta} := \frac{\alpha}{1-\alpha} \,, \qquad \beta := \frac{P_{\mathcal{S}\zeta}}{\sqrt{P_{\mathcal{S}} P_\zeta}} \,, \tag{4.79}$$

where the relative entropy perturbation (or specific entropy) is defined as

$$\mathcal{S}_{XY} \equiv \delta \left(\frac{n_X}{n_Y} \right) \Big/ \left(\frac{n_X}{n_Y} \right) = \frac{\delta n_X}{n_X} - \frac{\delta n_Y}{n_Y} = \frac{\delta_X}{1+w_X} - \frac{\delta_Y}{1+w_Y} \,. \tag{4.80}$$

[3] It vanished only in the limit $k = 0$ for single field models.

By defining the gauge-invariant curvature perturbation in the uniform-density gauge

$$\varphi_\delta = \varphi - H\frac{\delta\rho}{\dot\rho} = \varphi + \frac{\delta}{3(1+w)} \ , \tag{4.81}$$

we can readily show that the entropy perturbation is gauge invariant

$$\mathcal{S}_{XY} = 3\left(\varphi_\delta^X - \varphi_\delta^Y\right) \ . \tag{4.82}$$

In literature, it is often the case that the species Y is reserved for photons.

• PNG, multi-field, delta-N formalism, curvaton