$$\delta^0_0 = 1 = g^{0\mu} g_{\mu 0} = g^{00} g_{00} + g^{0\alpha} g_{\alpha 0} = g^{00}\left[-a^2(1+2A)\right] + \cancel{g^{0\alpha}\left[-a^2 B_\alpha\right]}$$

$O(0)$ $O(1)$ $O(1)$ $O(1)$ $O(2)$

$$g^{00} = -\frac{1}{a^2(1+2A)} = -\frac{1}{a^2}(1-2A) + O(2)$$

$$\bar{g}^{\alpha\beta}\bar{g}_{\beta\gamma} = \delta^\alpha_\gamma$$

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu \qquad g^{0\alpha}, \quad g^{\alpha\beta}$$

$$0 = \delta^\alpha_0 = g^{\alpha\mu} g_{\mu 0} = \underbrace{g^{\alpha 0}}_{O(1)} \underbrace{g_{00}}_{O(0)} + \underbrace{g^{\alpha\beta}}_{O(0)} \underbrace{g_{\beta 0}}_{O(1)} = g^{\alpha 0}(-a^2) + \frac{\bar{g}^{\alpha\beta}}{\cancel{a^2}}(-\cancel{a^2} B_\beta)$$

$$C^{\alpha\beta} = \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta} C_{\gamma\delta}$$

$$g^{0\alpha} = g^{\alpha 0} = -\frac{1}{a^2}\bar{g}^{\alpha\beta} B_\beta = -\frac{1}{a^2} B^\alpha$$

$$B^\alpha = \bar{g}^{\alpha\beta} B_\beta$$

$$A_\mu \qquad A^\mu = g^{\mu\nu} A_\nu$$

$$g^{\alpha 0} = g^{\alpha\mu} g^{0\nu} \qquad g_{\mu 0} = -\frac{1}{a^2} B^\alpha$$

$$B^\alpha \neq g^{\alpha\mu} B_\mu \qquad B^\alpha \equiv \bar{g}^{\alpha\beta} B_\beta$$

$$g^{\alpha\beta} = \frac{1}{a^2}\left(\bar{g}^{\alpha\beta} - 2C^{\alpha\beta}\right)$$

4-velocity : observer, fluids u^μ $\quad -1 = u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu$ timelike

$\bar{u}^\mu = \frac{1}{a}(1, \vec{0})$ $\quad (g_{\mu\nu} u^\mu u^\nu)^{(0)} = -1$ $\quad 1 = s^\mu s_\mu$ spacelike

$u^\mu = \frac{1}{a}(1-A, U^\alpha)$ $\quad 0 = n^\mu n_\mu$ null, light path

$u_\mu = g_{\mu\nu} u^\nu$ $\quad U^\alpha$ 3-vector $\quad U_\alpha \equiv \bar{g}_{\alpha\beta} U^\beta$ $\quad A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$

$A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$

* Scalar-Vector-Tensor decomposition of $g_{\mu\nu}$

$A \equiv \alpha$ (scalar). $\quad B_\alpha = \beta_{,\alpha} + B_\alpha^{(V)}$, $\quad C_{\alpha\beta} = \varphi \bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(V)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}$

(counts: 1, 1; 3 = 1 + 2; 6 = 1 + 1 + 2 + 2)

$0 = \nabla^\alpha B_\alpha^{(V)} = B^{(V)|\alpha}_\alpha$ $\quad$ transverse (divergenceless) $C^{(V)}_{\alpha|\alpha} = 0$

covariant derivative $|$ w.r.t. $\bar{g}_{\alpha\beta}$ $\quad$ transverse, traceless $C^{(t)}_{\alpha\beta|\alpha} = 0 = C^{(t)}_{\alpha\alpha}$

comma $\quad \gamma$, $\quad \gamma_{,\alpha} = \frac{\partial \gamma}{\partial x^\alpha}$ 3-vector $\quad A^\alpha{}_{,\beta} = \frac{\partial A^\alpha}{\partial x^\beta}$ $\quad A^\alpha{}_{|\beta}$

$\sim \gamma$

4D $\quad g_{\mu\nu}$, $\quad \Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}(g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu})$, 4D covariant derivative ; $\quad$ 4D Riemann tensor

$X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} X^\rho$, $\quad X_{\mu;\nu} = X_{\mu,\nu} - \Gamma^\rho_{\mu\nu} X_\rho$

3D $\quad \bar{g}_{\alpha\beta}$, $\quad \bar{\Gamma}^\alpha_{\beta\gamma}$ (same w/ $\bar{g}_{\alpha\beta}$), 3D covariant derivative $|$, $\quad$ 3D Riemann tensor

3D

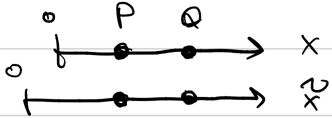
$X^\alpha{}_{|\beta} = X^\alpha{}_{,\beta} + \bar{\Gamma}^\alpha_{\beta\gamma} X^\gamma$

4D tensor $\tilde{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}$ scalar $\tilde{x} = x \leftarrow$ $\alpha, \beta, \gamma, \varphi$: scalar

3D tensor $\tilde{C}_{\alpha\beta} = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} C_{\gamma\delta}$ under 3D transformation

$B_\alpha \equiv \beta_{,\alpha} + B^{(V)}_\alpha$ $B_\alpha{}^{|\alpha} = \beta_{,\alpha}{}^{|\alpha} + \cancel{B^{(V)}_\alpha{}^{|\alpha}} = \Delta\beta$ $\beta = \Delta^{-1} B_\alpha{}^{|\alpha}$

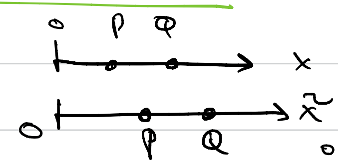
* Gauge Transformation . Coordinate transformation $x \leftrightarrow \tilde{x}$



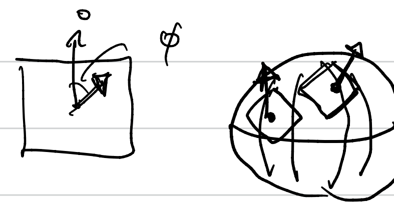
passive

$\phi_p(x_p) = \tilde{\phi}_p(\tilde{x}_p)$

Scalar



active



gauge transformation : gauge

QFT   GR $\phi\,(x)$ $\uparrow$ $\uparrow$

U(1), SU(2), SU(3)   diffeomorphism

gauge choice   gauge choice

$\tilde{\phi}(x) = R(x)\,\phi(x)$ $\tilde{\alpha}(x) = R(x)\,\alpha(x)$

$\mathcal{O}(1)$ $\qquad$ $\mathcal{O}(2)$

$\tilde{x}^\mu_p = x^\mu_p + \xi^\mu + \cdots$ $\qquad$ linear order

$\tilde{T}(x) = T(x) - \mathcal{L}_\xi T + \cdots$ $\qquad$ " $\qquad$ $\tilde{x}^\mu_p = x^\mu_p + \xi^\mu$

$\tilde{\phi}_p = \phi_p \rightarrow \tilde{\phi}(\tilde{x}_p) = \phi(x_p) \rightarrow \tilde{\phi}(x^\mu) = \phi(x^\mu - \xi^\mu) = \phi(x^\mu) - \xi^\mu \partial_\mu \phi + \cdots$

$\tilde{A}^\mu_{(\tilde{x}_p)} = \dfrac{\partial \tilde{x}^\mu}{\partial x^\nu} A^\nu(x_p)$

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

$\tilde{\phi}(x) = \phi(x) - \xi^\mu \partial_\mu \phi$

$\Delta_{\mu\nu}(x) \equiv \delta_\xi g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x)$

$= -\mathcal{L}_\xi g_{\mu\nu}(x) + \cdots$

$= -\mathcal{L}_\xi \bar{g}_{\mu\nu}(x)$ $\quad \mathcal{O}(1)$

$-\mathcal{L}_\xi \delta g_{\mu\nu} \sim \mathcal{O}(2)$

$\tilde{g}_{\mu\nu}(\tilde{x}_p) = \dfrac{\partial x}{\partial \tilde{x}} \dfrac{\partial x}{\partial \tilde{x}} g_{\rho\sigma}(x_p)$ $\qquad$ coord.

$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \Delta_{\mu\nu}(x) + \mathcal{O}(2)$ $\qquad$ gauge trans.

$g_{\mu\nu}$, $\rightarrow$ SVT based $g_{\alpha\beta}$ $\rightarrow \Gamma^\mu_{\rho\sigma}$, $R^\mu{}_{\nu\rho\sigma}$, $\ldots$, $G_{\mu\nu}$

$\alpha, \beta, \gamma, \varphi$

$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\varphi}$

EMT $\quad T_{\mu\nu} = \rho u_\mu u_\nu + \cdots$ $\qquad \rightarrow$ Einstein equation

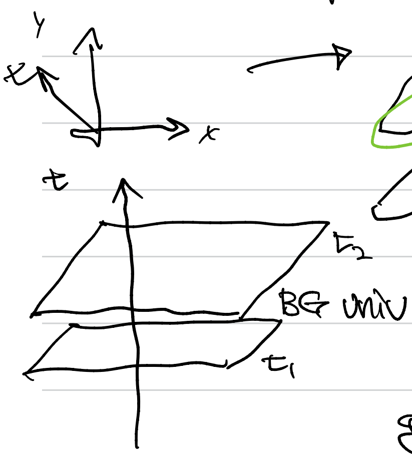
$\delta, v, \Phi, \ldots \rightarrow \tilde{\delta}, \tilde{v}, \tilde{\Phi}$

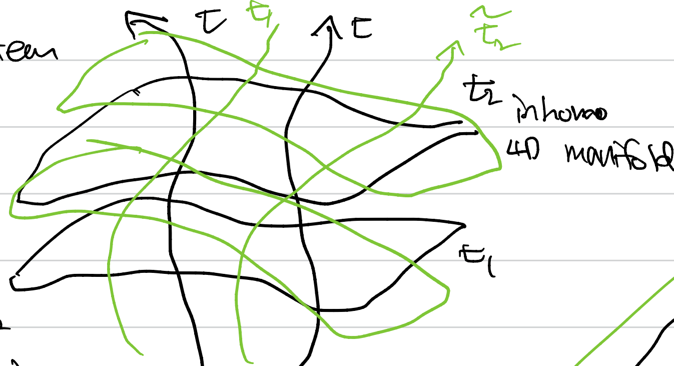
A. Bg. Cap

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

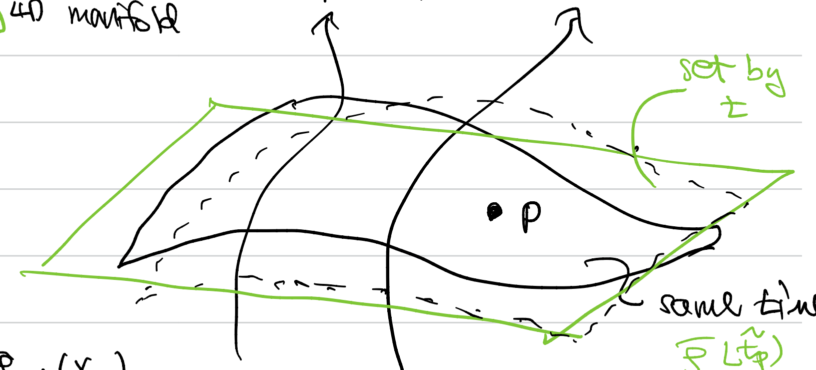
↳ RW ~ F(t) $\quad \bar{g}_{00} = -a^2(t)$, $\bar{g}_{0\alpha} = 0$, $\bar{g}^{(4)}_{\alpha\beta} = a^2(t)\,\bar{\bar{g}}_{\alpha\beta}$ $\simeq \delta_{\alpha\beta}$

coordinate system





$\tilde{t}_p = t_p + T$



$g_{\mu\nu}(x_p) = \bar{g}_{\mu\nu}(t_p) + \delta g_{\mu\nu}(x_p)$

$\tilde{g}_{\mu\nu}(\tilde{x}_p) = \bar{g}_{\mu\nu}(\tilde{t}_p) + \delta\tilde{g}_{\mu\nu}(\tilde{x}_p)$

$\rho(x_p) = \bar{\rho}(t_p) + \delta\rho(x_p)$

BG ~ coordinate

$\bar{\rho}(t)$

two $\tilde{x}$, x

$\tilde{t} = t + T$

$\mathcal{H} = aH = \frac{a'}{a}$

$+O(2)$

$= -(\xi_{\mu;\nu} + \xi_{\nu;\mu})$

$$\delta_\xi g_{\mu\nu} \equiv \tilde g_{\mu\nu}(x) - g_{\mu\nu}(x) = -\pounds_\xi \bar g_{\mu\nu} = -(\bar g_{\mu\nu,\rho}\xi^\rho + \bar g_{\rho\nu}\xi^\rho{}_{,\mu} + \bar g_{\rho\mu}\xi^\rho{}_{,\nu})$$

$$\delta_\xi g_{00} = \tilde g_{00}(x) - g_{00}(x) = -2\xi_{0;0} = -2(\xi_{0,0} - \Gamma^\mu_{00}\xi_\mu) = -2(\xi_0' - \underbrace{\Gamma^0_{00}}_{=\mathcal{H}}\xi_0 - \cancel{\Gamma^\alpha_{00}\xi_\alpha})$$

Γ^α_{00}: $0 + O(1)$

$$\xi_\mu = \bar g_{\mu\nu}\xi^\nu \qquad \xi_0 = \bar g_{0\mu}\xi^\mu = \bar g_{00}T + \cancel{\bar g_{0\alpha}\xi^\alpha} = -a^2T$$

$$\tilde x^\mu = x^\mu + \xi^\mu \to \tilde x^\alpha = x^\alpha + L^\alpha, \quad \tilde\eta = \eta + T, \quad \xi^\mu \equiv (T, L^\alpha), \quad L^\alpha = (L^{,\alpha} + L^{(v)\alpha})$$

$$= -2(T' + \mathcal{H}a^2T) = -a^2(1+2\tilde A) + a^2(1+2A) = -2a^2(\tilde A - A)$$

$$\xi_0' = (-a^2T)' = -2aa'T - a^2T' = -2a^2\mathcal{H}T - a^2T', \qquad \Gamma^0_{00}\xi_0 = -a^2\mathcal{H}T$$

$$\therefore \delta_\xi g_{00} = \tilde g_{00}(x) - g_{00}(x) = -2(-2a^2\mathcal{H}T - a^2T' + a^2\mathcal{H}T) = 2a^2(T' + \mathcal{H}T)$$

$$= -a^2(1+2\tilde A) + a^2(1+2A) = -2a^2(\tilde A - A) \qquad \therefore \tilde A = A - T' - \mathcal{H}T$$

$$\tilde C^{(t)}_{\alpha\beta} = C^{(t)}_{\alpha\beta}$$

Do it yourself!

$$\tilde\beta = \beta - T + L', \quad \tilde\varphi = \varphi - \mathcal{H}T, \quad \tilde\gamma = \gamma - L, \quad \tilde B^{(v)}_\alpha = B^{(v)}_\alpha + L^{(v)\prime}_\alpha, \quad \tilde C^{(v)}_\alpha = C^{(v)}_\alpha - L^{(v)}_\alpha$$

$$\chi \equiv a(\beta + \gamma') \qquad \tilde\chi = a(\tilde\beta + \tilde\gamma') = a(\beta - T + \cancel{L'} + \gamma' - \cancel{L'}) = \chi - aT$$

$$\varphi_\chi \equiv \varphi - H\chi \qquad \tilde\varphi_\chi = \tilde\varphi - H\tilde\chi = \varphi - \mathcal{H}T - H(\chi - aT) = \varphi - H\chi = \varphi_\chi$$

gauge $\chi \equiv 0$ $\quad \varphi_\chi = \varphi$ $\qquad \chi_\varphi \to \chi$ in gauge $\varphi = 0$

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - a^2\beta_{,\alpha}dx^\alpha d\eta + a^2(\bar{g}_{\alpha\beta} + 2\varphi\bar{g}_{\alpha\beta} + 2\gamma_{,\alpha|\beta})dx^\alpha dx^\beta$$

* Newtonian gauge = longitudinal gauge $\Rightarrow \chi = 0$ zero shear gauge

$\gamma = 0$, $\beta = 0$. gauge condition

diffeo symm $x^\mu \to \tilde{x}^\mu$ (T, L) $L^{,\alpha}$

$\tilde{\gamma} = \gamma - L$ $\gamma = 0 = \tilde{\gamma} \to L = 0$ $\chi = a(\beta + \gamma') = 0$

$\tilde{\beta} = \beta - T + L'$ $\beta = 0 = \tilde{\beta} \to T = 0$

$$ds^2 = -a^2(1+2A)d\eta^2 + 0 + a^2(\bar{g}_{\alpha\beta} + 2\varphi\bar{g}_{\alpha\beta} + V.T)dx^\alpha dx^\beta$$

* synchronous gauge $\tilde{\alpha} = \alpha = 0$. $\tilde{\beta} = \beta = 0$.

$$ds^2 = -a^2 d\eta^2 + 0 + a^2(\bar{g}_{\alpha\beta} + 2\varphi\bar{g}_{\alpha\beta} + 2\gamma_{,\alpha|\beta} + V.T)dx^\alpha dx^\beta$$

$\tilde{\alpha} = \alpha - \frac{1}{a}(aT)' \to (aT)' = 0$ $T = \frac{1}{a}C(x) \to 0$

$\tilde{\beta} = \beta - T + L' \to L' = T = \frac{1}{a}C(x)$ $L = C(x)\int\frac{d\eta}{a} + D(x) \to 0$

$\tilde{\varphi} = \varphi - HT = \varphi - HC(x)$ requires extra condition

comoving gauge condition $v = 0 \to \alpha = 0$

$G_{\mu\nu} = 8\pi G T_{\mu\nu}$ $g_{\mu\nu}$ & GT. $S = \int \sqrt{-g}\, d^4x \left[\frac{R}{16\pi G} + \mathcal{L}_m \right]$ scalar. spinor. ten. . . .

$T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m - 2 \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}$ $T_{\mu\nu;\mu} = 0$

$T_{\mu\nu} = \rho u_\mu u_\nu + p H_{\mu\nu} + \Pi_{\mu\nu}$ $H_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \rightarrow \eta_{ab} + u_a u_b$

fluid. u^μ_p $\tilde{u}^\mu = \frac{1}{a}(1, \vec{0})$

observer : measure ρ, p $u^\mu \xrightarrow{\text{rest}} u^a = (1, 0)$ $u_a = (-1, 0)$

u^μ_{obs} $g_{\mu\nu} \rightarrow \eta_{ab} = (-1, 111)$

$T_{\mu\nu} \xrightarrow{\text{rest}} \left(\begin{array}{c|ccc} \rho & & & \\ \hline & p & & \\ & & p & \\ & & & p \end{array} \right)$ $H_{\mu\nu} \xrightarrow{\text{rest}} \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{array} \right)$

$\rho_{obs} = \rho_{F.rest} + O(2)$. $P_{obs} = P_{F.rest} + O(2)$, $\Pi^{obs}_{\mu\nu} = \Pi^F_{\mu\nu}$ $\hat{E} = \gamma E$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$T^\mu{}_\nu \equiv g^{\mu\rho} T_{\rho\nu} = \rho u^\mu u_\nu + p(\delta^\mu_\nu + u^\mu u_\nu) + g^{\mu\rho}\Pi_{\rho\nu}$

$\left(\begin{array}{c|c} & \\ \hline & \end{array} \right)$

$\rho = \tilde{\rho}(1+\delta)$ $\qquad \tilde{\delta} = \delta + \dots$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$\bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}$$
$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

<u>symmetry</u> : two physically differen state

diffeo . $U(1)$. $SU(2)$: redundancies

→ same physical state

$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu} + \cdots$

Fluid quantities : $\rho,\ p,\ v,\ \pi_{\alpha\beta}$. $\supset$ dark matter, baryons, ...

Photons, neutrinos : we need distribution fct $f(\vec{x}, \vec{p}; t)$. $dN_{x,p} = f\, d^3x\, d^3p$

$\rho = \int d^3p \cdot E \cdot F$ $\qquad q^i = \int d^3p \cdot p^i \cdot F$ $\qquad F = \bar{f} + f$

$A^{ij} \sim \int d^3p\, p^i p^j F \sim p\, \delta^{ij} + \pi^{ij}$

Einstein Equation $\qquad G_{\mu\nu} = 8\pi G T_{\mu\nu}$ $\qquad g_{\mu\nu} = g^{bg}_{\mu\nu} + \delta g_{\mu\nu}$

$g_{\mu\nu}$, covariant derivative $A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} A^\rho$

$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} (g_{\rho\nu,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu})$ Christoffel symbol $\qquad$ extrinsic curvature vs intrinsic curvature

4D $\quad R^\mu{}_{\nu\rho\sigma} \equiv \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} - \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon}$ 20 Riemann tensor

$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$ 10 Ricci tensor $\qquad R \equiv R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$ Ricci scalar

$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ $\qquad R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu}$

for 4 vector u^μ $\qquad 2 u_{\mu;[\nu\rho]} = u_{\mu;\nu\rho} - u_{\mu;\rho\nu} = u_\sigma R^\sigma{}_{\mu\nu\rho}$

3D - space $\quad \bar{g}_{\alpha\beta}, \quad \bar{\Gamma}^\alpha_{\beta\gamma}, \quad \bar{R}^\alpha{}_{\beta\gamma\delta}$

$$-\frac{1}{2} g^{00} g_{00,0} - \cancel{\frac{1}{2} g^{0\alpha} g_{00,\alpha}}$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\mu}\left(g_{\mu 0,0} + g_{0\mu,0} - g_{00,\mu}\right) = g^{0\mu} g_{\mu 0,0} - \frac{1}{2} g^{0\mu} g_{00,\mu} = g^{00} g_{00,0} + \cancel{g^{0\alpha} g_{\alpha 0,0}}$$

$$= \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2}\,\frac{1-2A}{a^2}\left[a^2(1+2A)\right]' = (1-2A)\left[\mathcal{H}(1+2A) + A'\right] = \mathcal{H} + A'$$

$$\hookrightarrow\ 2aa'(1+2A) + 2a^2 A' = 2a^2\left[\mathcal{H}(1+2A) + A'\right]$$

Γ^α_{00}, $\Gamma^0_{\alpha 0}$, $\Gamma^0_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma} = \bar\Gamma^\alpha_{\beta\gamma} + \mathcal{H}\,\bar g_{\beta\gamma} B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\alpha}$ ← covariant derivative

↳ Christoffel symbol based $\bar g_{\alpha\beta}$

Background

$\bar R^\alpha_{\beta\gamma\delta} = 2K\,\delta^\alpha_{[\gamma}\,\bar g_{\delta]\beta}$ ← Riemann tensor based on $\bar g_{\alpha\beta}$

$R^\alpha_{\beta\gamma\delta} = 2(K+\mathcal{H}^2)\,\delta^\alpha_{[\gamma}\,\bar g_{\delta]\beta}$ ← spatial part of $R^\mu_{\nu\rho\sigma}$ based on $g^{bg}_{\mu\nu}$

$$R_{\alpha\beta} = 2K\,\bar g_{\alpha\beta} + (\mathcal{H}' + 2\mathcal{H}^2)\,\bar g_{\alpha\beta}(1-2A) + \mathcal{O}(1)$$

$$R = \frac{1}{a^2}\left[6(\mathcal{H}' + \mathcal{H}^2 + K) + \cdots\right]$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ → BG: Friedmann eq. $\bar G_{\mu\nu} = 8\pi G\,\bar T_{\mu\nu}$

→ $\delta G_{\mu\nu} = G_{\mu\nu} - \bar G_{\mu\nu} = 8\pi G\,(T_{\mu\nu} - \bar T_{\mu\nu}) = 8\pi G\,\delta T_{\mu\nu}$ $\quad T_{\mu\nu;\mu} = 0$

$\delta G_{00} = 8\pi G\,\delta T_{00}$, $\delta G_{0\alpha} = 8\pi G\,\delta T_{0\alpha}$, $\delta G_{\alpha\beta} = 8\pi G\,\delta T_{\alpha\beta}$

S   S, V   S, V, T

traceless transverse tensor : $\ddot{C}^{(t)}_{\alpha\beta} + 3H\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta - 2K}{a^2} C^{(t)}_{\alpha\beta} = \cancel{8\pi G \Pi^{(t)}_{\alpha\beta}}$ $\sim \cancel{\Pi_{\alpha\beta}}$

scalar part : $\dot\chi + H\chi - \varphi - \alpha = 8\pi G \Pi \simeq 0$

$\tilde\chi = \chi - aT$, $\tilde\varphi = \varphi - aHT$. $\tilde\alpha = \alpha - \frac{1}{a}(aT)^{\cdot}$

$$\dot{\tilde\chi} + H\tilde\chi - \tilde\varphi - \tilde\alpha = \dot\chi - (aT)^{\cdot} + H\chi - aHT - \varphi + aHT - \alpha + \frac{1}{a}(aT)^{\cdot}$$

$$= \dot\chi + H\chi - \varphi - \alpha$$

gauge-invariant

Newtonian gauge $\chi = 0 = \gamma = \beta$, $\chi = a(\beta + \gamma')$, $K = 0$ spatial curvature

$$H\kappa + \frac{1}{a^2}\Delta\varphi = -4\pi G\bar\rho\cdot\delta \qquad \kappa = 3H\alpha - 3\dot\varphi - \frac{\Delta}{a^2}\chi.$$

$$\kappa + \frac{\Delta}{a^2}\chi = 12\pi G(\bar\rho + \bar p)aV \qquad K = -3H + \kappa$$

$$\dot\kappa + 2H\kappa + \left(3\dot H + \frac{\Delta}{a^2}\right)\alpha = 4\pi G(\delta\rho + 3\delta p)$$

$$ds^2 = -a^2(1+2\alpha)d\eta^2 + a^2[\bar g_{\alpha\beta}(1+2\varphi)]dx^\alpha dx^\beta$$

$$\dot\chi + H\chi - \varphi - \alpha = 0 \qquad \alpha = -\varphi \quad \text{(pressureless)}$$

$$-\Delta\varphi = 4\pi G\bar\rho a^2\cdot\delta + Ha^2\kappa \qquad \kappa = 12\pi G\bar\rho aV$$

$$= \Delta\alpha \quad = 4\pi G\bar\rho a^2\delta_v \qquad \delta_v = \delta + aHV$$

$\boxed{\text{Question } V = 0\,?}$

Einstein Eqs w/ $K=0$

$$H\kappa + \frac{1}{a^2}\Delta\varphi = -4\pi G\bar\rho\cdot\delta \qquad \kappa = 3H\alpha - 3\dot\varphi - \frac{\Delta}{a^2}\chi\,, \quad \chi = a(\beta+\gamma')$$

$$\kappa + \frac{\Delta}{a^2}\chi = 12\pi G(\bar\rho+\bar p)av \qquad K = -3H + \kappa$$

$$\dot\kappa + 2H\kappa + \left(3\dot H + \frac{\Delta}{a^2}\right)\alpha = 4\pi G(\delta\rho + 3\delta p)$$

$$\dot\chi + H\chi - \varphi - \alpha = 8\pi G\Pi\,, \qquad \frac{[a^4(\bar\rho+\bar p)v]^{\cdot}}{a^4(\bar\rho+\bar p)} - \frac{1}{a}\alpha - \frac{1}{a(\bar\rho+\bar p)}\left(\delta p + \frac{2}{3}\frac{\Delta\Pi}{a^2}\right) = 0$$

Newtonian gauge

$\beta=\gamma=0 \quad \therefore\ \chi=0 \qquad L=T=0 \qquad v\to U$

$$ds^2 = -a^2(1+2\psi)d\eta^2 + 0 + a^2 g_{\alpha\beta}(1+2\phi)dx^\alpha dx^\beta \qquad \alpha\to\psi\,,\ \varphi\to\phi$$

pressureless fluids $\quad p\equiv 0 \qquad -\phi-\psi=0 \quad \therefore\ \psi=-\phi$

$$\kappa = 3H\alpha - 3\dot\varphi - \frac{\Delta}{a^2}\chi \xrightarrow{NG} \kappa = 3H\psi - 3\phi' = 3H\psi + 3\psi'$$

$$-\Delta\phi = 4\pi G\bar\rho\cdot\delta\cdot a^2 + H\kappa a^2 \qquad \kappa + 0 = 12\pi G\bar\rho a U$$

$$= \Delta\psi \quad = 4\pi G\bar\rho a^2(\delta_N + 3HU) \qquad H\kappa a^2 = 12\pi G\bar\rho\cdot a^2\cdot HU$$

$$= 4\pi G\bar\rho a^2\cdot\delta_v \qquad \delta_v = \delta + 3HU$$

$$\frac{[a^4\bar\rho U]^{\cdot}}{a^4\bar\rho} = \frac{1}{a}\psi \qquad \bar\rho_m \sim \frac{1}{a^3} \qquad a^3\bar\rho_m \sim \text{const.} \qquad \frac{(aU)^{\cdot}}{a} = \frac{\dot a}{a}U + \dot U = HU + \dot U = \frac{1}{a}\psi$$

$$\to U' + HU = \psi$$

$= \zeta = \mathcal{R}$

$\varphi_v = \varphi - Hv \quad = \phi - \mathcal{H}u$ comoving gauge curvature perturbation

→ conserved on super horizon scales all the time

$= \phi - \frac{\mathcal{H}(3\mathcal{H}\phi - 3\phi')}{12\pi G(\rho + P)a} \simeq \phi - \frac{2}{3}\frac{\phi}{1+w}$ $\qquad w = \frac{\bar{P}}{\bar{\rho}}$

RDE $\quad w = 1/3 \qquad \varphi_v = \text{constant} = \phi_{RDE} + \frac{2}{3}\frac{\phi_{RDE}}{1+1/3} = \frac{3}{2}\phi_{RDE}$

MDE $\quad w = 0 \qquad\qquad = \text{''} \qquad = \phi_{MDE} + \frac{2}{3}\frac{\phi_{MDE}}{1+0} = \frac{5}{3}\phi_{MDE}$

$\phi_{RDE} = \frac{2}{3}\varphi_v$. $\qquad \phi_{MDE} = \frac{3}{5}\varphi_v$

$M_{pl}^2 = \frac{1}{8\pi G}$. $m_{pl}^2 = \frac{1}{G}$

Standard Inflationary Model

$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right]$ ← $\frac{1}{2}(\partial_t\phi)^2 - (\nabla\phi)^2$

$-+++$ sign convention

slow-roll parameter $\epsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right)$ deviation from de-Sitter spacetime

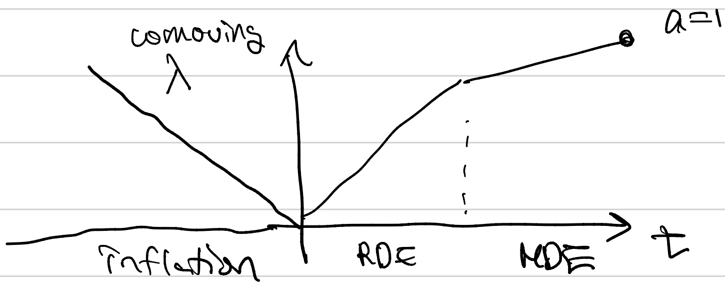
Goal : $\zeta \equiv \varphi_V$

$\Delta_\zeta^2(k) = \frac{k^3 P_\zeta(k)}{2\pi^2} = A_s\left(\frac{k}{k_0}\right)^{n_s-1}$ ($n_s - 1 \simeq 0$ scale-invariant) , $A_s = \frac{H^2}{8\pi^2\epsilon M_{pl}^2} \simeq 2\times 10^{-9}$

$T \sim 10^{14}$ GeV

$n_s - 1 \equiv \frac{d\ln\Delta_\zeta^2}{d\ln k} \simeq -2\epsilon \simeq 0$, $A_T = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2} \simeq 16\epsilon\cdot A_s$

$n_T \equiv \frac{d\ln\Delta_T^2}{d\ln k} \simeq -2\epsilon \simeq 0$, $\epsilon \simeq 0.01$ $\Delta_T^2(k) = A_T\left(\frac{k}{k_0}\right)^{n_T}$



$\lambda \sim \frac{1}{k}$ comoving Horizon $\sim \eta$

$\frac{1}{H}$

pc $\sim 10^{18}$ cm

$\frac{d}{dt}\left(\frac{1}{H}\right) < 0$ Inflation condition

e-folding $N_k = \ln\frac{a_{end}(k)}{a_{begin}(k)}$

$\sim$ 40 - 60 needed for homogeneity

$\lambda_{today} \sim 10$ Gpc → 1cm

$\frac{a_{end}}{a_0} = \frac{T_{cmb}}{10^{14}\,\text{GeV}} \sim \frac{\text{meV}}{10^{14}\,\text{GeV}} \sim 10^{-26}$

$V(\phi) = \frac{\lambda}{4!}\phi^4$ example.

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right]$$

slow-roll parameter $\epsilon = \frac{d}{dt}\left(\frac{1}{H}\right)$ deviation from de-Sitter spacetime

$$\Box\phi - V_{,\phi} = 0 \qquad T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu}$$

$$= \rho_\phi u^\phi_\mu u^\phi_\nu + P_\phi H_{\mu\nu} + \cdots$$

BG:

$$\bar{\rho}_\phi = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) \, , \quad \bar{P}_\phi = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}) \qquad \phi = \bar{\phi}(t) + \delta\phi(x,t)$$

$$H^2 = \frac{8\pi G}{3}\bar{\rho}_\phi \simeq \frac{8\pi G}{3}V(\bar{\phi}) \qquad H = \text{const} = \frac{d\ln a}{dt} \qquad a \sim e^{Ht}$$

Linear order:

$$\delta\rho_\phi = \dot{\bar{\phi}}\,\delta\dot{\phi} - \dot{\bar{\phi}}^2\alpha \cdots \, , \quad \delta P_\phi = \cdots$$

$$v_\phi = \delta\phi / \bar{\phi}' \, , \quad \Pi^\phi_{\alpha\beta} = 0$$

de Sitter $\quad H^2 = \frac{\Lambda}{3} = \text{constant} \qquad a(t) = e^{Ht} \, , \quad \epsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = 0$

$$a = (0, \infty) \qquad t = (-\infty, \infty) \, . \qquad \eta = (-\infty, 0)$$

We'll come back here

for dS