$M_{pl}^2 = \frac{1}{8\pi G}$ . $m_{pl}^2 = \frac{1}{G}$

# Standard Inflationary Model

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right] \leftarrow \frac{1}{2}(\partial_t\phi)^2 - (\nabla\phi)^2$$

$-+++$ sign convention

slow-roll parameter $\epsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right)$ deviation from de-Sitter spacetime

Goal : $\zeta \equiv \varphi_V$

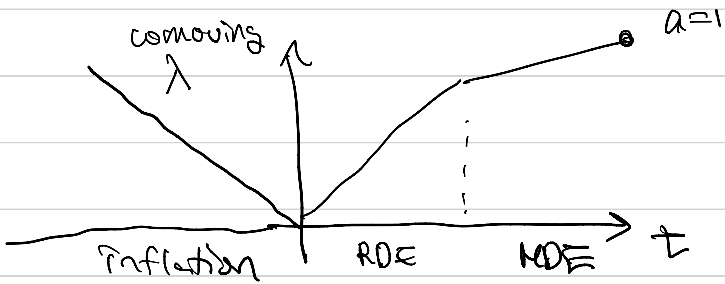
$$\Delta_\zeta^2(k) = \frac{k^3 P_\zeta(k)}{2\pi^2} = A_s\left(\frac{k}{k_0}\right)^{n_s-1}, \quad A_s = \frac{H^2}{8\pi^2\epsilon M_{pl}^2} \simeq 2\times 10^{-9}$$

$n_s - 1 \simeq 0$ scale-invariant

$T \sim 10^{14}$ GeV

$$n_s - 1 \equiv \frac{d\ln\Delta_\zeta^2}{d\ln k} \simeq -2\epsilon \simeq 0, \quad A_T = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2} \simeq 16\epsilon\cdot A_s$$

$$n_T \equiv \frac{d\ln\Delta_T^2}{d\ln k} \simeq -2\epsilon \simeq 0, \quad \epsilon \simeq 0.01, \quad \Delta_T^2(k) = A_T\left(\frac{k}{k_0}\right)^{n_T}$$




$\lambda \sim \frac{1}{k}$ comoving Horizon $\sim \eta$

$\frac{1}{H}$

pc $\sim 10^{18}$ cm

$\frac{d}{dt}\left(\frac{1}{H}\right) < 0$ Inflation condition

e-folding $N_k = \ln\frac{a_{end}(k)}{a_{begin}(k)}$

$\sim$ 40 - 60 needed for homogeneity

$\lambda_{today} \sim 10$ Gpc $\rightarrow$ 10 cm

$$\frac{a_{end}}{a_0} = \frac{T_{cmb}}{10^{14}\,\text{GeV}} \sim \frac{\text{meV}}{10^{14}\,\text{GeV}} \sim 10^{-26}$$

$V(\phi) = \frac{\lambda}{4!}\phi^4$ example.

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right]$$

slow-roll parameter $\boxed{\epsilon = \frac{d}{dt}\left(\frac{1}{H}\right)}$ deviation from de-Sitter spacetime

$$\Box\phi - V_{,\phi} = 0 \qquad T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu}$$

$$= \rho_\phi u^\phi_\mu u^\phi_\nu + P_\phi H_{\mu\nu} + \cdots$$

BG:

$$\bar{\rho}_\phi = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) \quad , \quad \bar{P}_\phi = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}) \qquad \phi = \bar{\phi}(t) + \delta\phi(x,t)$$

$$H^2 = \frac{8\pi G}{3}\bar{\rho}_\phi \simeq \frac{8\pi G}{3}V(\bar{\phi}) \qquad H = \text{const} = \frac{d\ln a}{dt} \qquad a \sim e^{Ht}$$

Linear order:

$$\delta\rho_\phi = \dot{\bar{\phi}}\,\delta\dot{\phi} - \dot{\bar{\phi}}^2\alpha \cdots \quad , \quad \delta P_\phi = \cdots$$

$$u_\phi = \delta\phi / \bar{\phi}' \quad , \quad \Pi^\phi_{\alpha\beta} = 0$$

de Sitter $\quad H^2 = \frac{\Lambda}{3} = \text{constant} \qquad H = \frac{d\ln a}{dt} \qquad \therefore a(t) = e^{Ht}$ with $a=1$ at $t=0$

$$a = (0, \infty) \qquad t = (-\infty, \infty). \qquad \eta = (-\infty, 0) \qquad \text{set } \eta_* = 0 \text{ at } t_* = \infty$$

$$dt = a\,d\eta \rightarrow d\eta = e^{-Ht}dt \qquad \therefore \eta_* - \eta = -\frac{1}{H}e^{-Ht}\Big|_t^{t_*} \qquad \therefore \eta = \frac{-1}{H}e^{-Ht}$$

$$a = e^{Ht} = -\frac{1}{H\eta} \quad , \quad \epsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = 0$$

inflation condition $\quad 0 > \frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) = -\frac{1-\epsilon}{a}$

* $\Phi = \varphi_v - \frac{K/a^2}{4\pi G(\bar{\rho}+\bar{p})}\varphi_x \rightarrow \varphi_v$ if $K=0$

using Einstein equation $\dot{\Phi} = \Xi - \frac{H}{\bar{\rho}+\bar{p}}\frac{k^2}{a^2}\left(\frac{c_s^2}{4\pi G}\varphi_x - \frac{2}{3}\Pi\right) \rightarrow \Xi$ as $k \rightarrow 0$

where $\Xi = (\dot{\bar{\rho}}\,\delta p - \dot{\bar{p}}\,\delta\rho) / 3(\bar{\rho}+\bar{p})^2 \rightarrow 0$ for adiabatic condition

* Inflaton field ? stability

# Quadratic Action : Quantum fluctuations

scalars : $\alpha$, $\beta$, $\gamma$, $\varphi$, $\delta\phi$, $\delta\rho_\phi$, $\delta P_\phi$, $v_\phi$, $\cancel{\Pi_\phi}$ $\xrightarrow{\text{real dof}}$ one

* gauge choice
  * comoving gauge $v_\phi = 0$ $\rightarrow$ $\varphi \rightarrow \varphi_v = \zeta$
  * $\Leftrightarrow$ uniform field $\delta\phi = 0$ $\qquad \phi = \bar{\phi}(t) + \underbrace{\delta\phi}_{=0}$
  * uniform curvature $\varphi = 0$

* Action :

$$S = \overbrace{\int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right]}^{\text{full theory : GR + }\phi} = S[g_{\mu\nu}, \phi] \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \phi = \bar{\phi} + \delta\phi$$

$$= \underbrace{S_0[\bar{g}_{\mu\nu}, \bar{\phi}]}_{\text{BG dynamics}} + \underbrace{S_1[\bar{g}_{\mu\nu}, \bar{\phi}]\,\zeta}_{\rightarrow \frac{\delta S}{\delta\zeta} = 0} + \underbrace{S_2[\bar{g}_{\mu\nu}, \bar{\phi}]\,\zeta^2}_{\text{Quadratic action}} + \underbrace{S_3[\bar{g}_{\mu\nu}, \bar{\phi}]\,\zeta^3}_{\text{cubic action}} + \underbrace{S_4 \cdots}_{\text{quartic action}}$$

quadratic action : free field for $\zeta$ $\sim (\partial\phi)^2 + m^2\phi^2$ SHO

cubic & higher action : interactive field $\qquad H_{(n>2)} = H_0 + H_{int}$

$$z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2 \varepsilon$$

$a z^2 \sim \varepsilon$

$$S_{(2)} = \frac{1}{2} \int dt\, d^3x\, a^3 \frac{\dot{\phi}^2}{H^2} \left[ \dot{\zeta}^2 - \frac{1}{a^2} (\partial \zeta)^2 \right] = \frac{1}{2} \int d\eta\, d^3x \left[ (v')^2 - (\partial v)^2 + \frac{z''}{z} v^2 \right]$$

* new field $v \equiv z \zeta$, canonical momentum $\pi_v \equiv \frac{\delta \mathcal{L}}{\delta v'} = v'$

* Hamiltonian: $\mathcal{H} \equiv v' \pi_v - \mathcal{L} = \frac{1}{2} \left[ (v')^2 + (\nabla v)^2 + m^2 v^2 \right]$

time-dependent massive free scalar field action $v$

$$m^2 \equiv -\frac{z''}{z} \xrightarrow{dS} -\frac{a''}{a} = -\frac{2}{\eta^2} \longrightarrow 0$$ massless in infinite past $\eta \to -\infty$

two mode function $v_k^{\pm}$

* EoM for $v$ : $(\Box - m^2) v = 0 \rightarrow v_k'' + \omega_k^2 v_k = 0 \qquad \omega_k^2 = k^2 + m^2$

$$v(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( v_k^+ e^{i\omega_k \eta} + v_k^- e^{-i\omega_k \eta} \right) \qquad (v, \pi_v) \leftrightarrow (q, p)$$

classical quantum

* quantization $v \to \hat{v}$, $\pi_v \to \hat{\pi}_v$, $[\hat{v}_x, \hat{\pi}_y] \sim \hbar\, \delta^D(x - y)$ $(q, p) \leftrightarrow (\hat{q}, \hat{p})$

creation $\hat{a}_k^+$, annihilation $\hat{a}_k$ : $v_k^+ \to \hat{v}_k^+ = \hat{a}_k^+ v_k^+$, $v_k^- \to \hat{v}_k^- = \hat{a}_k v_k^-$

* VEV : $\langle 0 | \hat{v}_k^+ \hat{v}_{k'} | 0 \rangle = (2\pi)^3 \delta^D(k - k') P_v(k)$ at $k = H$ horizon crossing

* solutions in de-Sitter bg

$m^2 = -\frac{2}{\eta^2}$ , $\omega_k^2 = k^2 - \frac{2}{\eta^2}$ , $v_k^{\pm} = \frac{1}{\sqrt{2k}} e^{\pm ik\eta} \left(1 \pm \frac{i}{k\eta}\right)$

infinite past $\eta \to -\infty$ , $m=0$ . $v_k^{\pm} = \frac{1}{\sqrt{2k}} e^{\pm ik\eta}$ $E = \omega_k = k$ free-ptl

horizon crossing $k\eta \to 0$ , $v_k^{\pm} \to \pm \frac{1}{\sqrt{2k}} \frac{i}{k\eta}$ $\frac{d^3x}{\sqrt{2E}}$ : Lorentz inv

$P_v(k) = \frac{1}{2k^3} \frac{1}{\eta^2} = \frac{a^2H^2}{2k^3}$ $v = z\zeta$ . $z^2 = 2a^2\epsilon$

$\therefore \Delta_\zeta^2(k) = \frac{k^3 P_\zeta}{2\pi^2} = \frac{k^3}{2\pi^2}\left(\frac{a^2H^2}{2k^3}\right)\left(\frac{1}{2a^2\epsilon}\right) = \frac{H^2}{8\pi^2\epsilon} \longleftarrow A_s = \Delta_\zeta^2$ at $k_0$

$= A_s\left(\frac{k}{k_0}\right)^{n_s-1}$

slope : $n_s - 1 \equiv \frac{d\ln\Delta^2}{d\ln k} = \frac{d\ln H^2}{d\ln k} - \frac{d\ln\epsilon}{d\ln k} \simeq 2\frac{d\ln H}{d\ln k} = 2\frac{\dot H}{H^2} = -2\epsilon$

horizon crossing $k = aH$ $\therefore d\ln k = d\ln a + d\ln H \simeq H dt + 0$

$\star$ Tensor fluctuations : cosmological gravitational waves

$$S_{(2)} = \frac{M_p^2}{8} \int d\eta\, d^3x\; a^2 \left[ (h_{ij}')^2 - (\nabla h_{ij})^2 \right] = \sum_{s=\pm 2} \int d\eta\, d^3k\; \underbrace{\frac{a^2}{4} M_p^2}_{} \left[ (h_k^{s\,\prime})^2 - k^2 (h_k^s)^2 \right]$$

$$h_{ij} \equiv 2 C_{ij}^{(t)} \equiv 2\, h^{(\pm 2)} Q_{ij}^{(\pm 2)}$$

no $\varepsilon$ , no scalar field

new variable $\quad v_k^s = \frac{1}{2} a M_p h_k^s$ , $\quad m = 0 \quad$ vs $\quad v = \sqrt{2a^2 \varepsilon}\, \zeta$

$$P_T = 2 P_{h^s} , \qquad \Delta_T^2 = \frac{k^3 P_T}{2\pi^2} = \frac{2}{\pi^2} \frac{H^2}{M_p^2} \equiv A_t \left( \frac{k}{k_0} \right)^{n_T}$$

$$n_T = \frac{d \ln \Delta_T^2}{d \ln k} = \frac{d \ln H^2}{d \ln k} \simeq -2\varepsilon$$

$$r = \Delta_T^2 / \Delta_s^2 = 16\, \varepsilon$$

very small

$\star$ Issues w/ standard inflationary model

* stability ( $\varepsilon \to \infty$ ) , no inflaton found
* $N_k = \ln \frac{a_{end}}{a(\phi_c)} = \int dt\, H = \int d\phi \frac{H}{\dot\phi} = \int \frac{d\phi}{\sqrt{2\varepsilon}} \quad \therefore \quad \frac{\Delta\phi_k}{M_{pl}} = \int dN \sqrt{\frac{r}{8}}$ $\quad > 1$

40 - 60 $\quad$ 0.1

* going beyond free - field $\quad$ trans - Planckian
* vacuum : non - local $\quad a_k |0\rangle = 0 \quad \forall k$

invariant under Poincare transformation