AST802 Advanced Topics of Theoretical Cosmology

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The purpose of this intensive block course is to introduce active research fields and provide essential ingredients and tools for research in theoretical cosmology. The course will focus on large-scale structure probes of inflationary cosmology. The prerequisite for this course is AST513 Theoretical Cosmology with good understanding of general relativity.

Contents

1 Spherical Collapse Model in Cosmology **1**
1.1 Spherical Collapse Model . . . 1
1.2 Dark Matter Halo Mass Function . . . 2
1.2.1 Basic Idea . . . 2
1.2.2 Halo Mass Function . . . 3
1.2.3 Halo Mass Functions in Literature . . . 4
1.2.4 Astrophysical Applications . . . 4
1.3 From Probability Functional to Correlation Function . . . 4
1.3.1 Density Probability Functional . . . 4
1.3.2 Mildly Non-Gaussian Halo Mass Function . . . 6
1.3.3 Peak-Background Split and Statistics of Thresholded Region . . . 7
1.4 Beyond the Spherical Model: Peak Models . . . 9
1.4.1 Basic Idea of Peak Models . . . 9
1.4.2 Multivariate Gaussian Joint Probability Distribution . . . 9
1.4.3 Gaussian Peaks Models . . . 11
1.4.4 Conditional Probability for Ellipticity and Prolateness . . . 12

2 Newtonian Perturbation Theory and Galaxy Bias **13**
2.1 Standard Perturbation Theory . . . 13
2.1.1 Basic Formalism . . . 13
2.1.2 Recurrence Relation and Third Order Solution . . . 14
2.1.3 Asymptotic Behavior . . . 15
2.1.4 Unified Treatment of the Standard Perturbation Theory . . . 16
2.2 Galaxy Bias Primer . . . 17
2.2.1 Renormalizing the Bias Parameter . . . 17
2.2.2 Summary of Spherical Collapse Model . . . 18
2.2.3 Renormalized Galaxy Bias . . . 19
2.2.4 Gravitational Tidal Tensor Bias . . . 21
2.2.5 Halo Exclusion . . . 22
2.2.6 Miscellaneous . . . 23
2.3 Lagrangian Perturbation Theory . . . 23
2.3.1 Basic Idea: Zel'dovich Approximation . . . 23
2.3.2 Resummation in LPT: One-Loop Power Spectrum . . . 26
2.3.3 Zel'dovich Power Spectrum . . . 27
2.3.4 Galaxy Bias in the Lagrangian Frame . . . 27
2.3.5 Perturbation Theory and Nonlocal Bias . . . 28
2.3.6 Examples of Bias Models . . . 29
2.3.7 Multi-Point Propagator for Matter and Biased Tracers . . . 31
2.3.8 The Recursion Relation in Lagrangian Perturbation Theory . . . 31
2.4 Redshift-Space Power Spectrum . . . 32
2.4.1 Large-Scale Velocity Correlation . . . 33
2.4.2 Gaussian Streaming Model . . . 33
2.4.3 Complete Treatment of Redshift-space Distortion . . . 34
2.4.4 Improved Model of the Redshift-Space Distortion . . . 35
2.4.5 Summary of Other Phenomenoligical Models of FoG . . . 36
2.5 Effective Field Theory . . . 36
2.5.1 Basic Formalsim . . . 36
2.5.2 Effective Stress Tensor . . . 38
2.5.3 One-Loop Power Spectrum . . . 40

3 Nonlinear Relativistic Dynamics: ADM Formalism and its Cosmological Applications **43**
3.1 Arnowitt-Deser-Misner (ADM) Formalism . . . 43
3.1.1 Basics . . . 43
3.1.2 Hamiltonian Approach . . . 44
3.1.3 FRW Metric and its Connection to ADM variables up to 2nd order . . . 46
3.2 Energy-Momentum Tensor in ADM Formalism . . . 47
3.2.1 General Decomposition . . . 47
3.2.2 Normal Frame and Its Relation to Energy Frame . . . 48
3.3 Fully Nonlinear Einstein Equations . . . 49
3.3.1 ADM Equations . . . 49
3.3.2 Second-Order Equations . . . 50

4 Standard Inflationary Models **52**
4.1 Standard Inflationary Model . . . 52
4.1.1 Scalar Field Action . . . 52
4.1.2 Background Evolution and the Linear-Order Equations . . . 53
4.1.3 Quadratic Action . . . 55
4.1.4 Conservation on Super-Horizon Scales . . . 56
4.1.5 Quantum Fluctuations in Quadratic Action . . . 57
4.1.6 Worked Examples . . . 61
4.1.7 Hamiltonian Approach to the Quadratic Action . . . 62
4.1.8 Cubic Action . . . 62
4.1.9 In-In Formalism . . . 64
4.1.10 Bispectrum Computation in a Flat Gauge . . . 65
4.1.11 Bispectrum in a General Gauge . . . 67
4.1.12 Bispectrum in the Squeezed Limit . . . 68
4.1.13 Quantum Fluctuations in Cubic Action . . . 69
4.1.14 Single-Field Consistency Relation . . . 71
4.1.15 Lyth Bound . . . 73
4.2 Primordial Non-Gaussianity . . . 73
4.2.1 Definition . . . 73
4.2.2 Shapes of Non-Gaussianity . . . 74
4.3 δN-Formalism . . . 75
4.3.1 Basic formalism . . . 75
4.3.2 Applications . . . 76

5 Applications of the Effective Field Theory to Inflationary Models **79**
5.1 Basics of EFT . . . 79
5.1.1 Simple Toy Model and Guiding Example . . . 79
5.1.2 Perturbative Treatment of the EFT Action . . . 80
5.2 Effective Field Theory of Inflation . . . 81
5.2.1 Weinberg (2008): Effective Field Theory for Inflation . . . 81
5.2.2 General Effective Action in the Unitary Gauge . . . 83
5.2.3 Goldstone Action in the Effective Lagrangian . . . 83
5.2.4 Cheung, Creminelli, Fitzpatrick, Kaplan, Senatore 2008 . . . 84

6 Modification of Gravity and Dark Energy Models **86**
6.1 Brans-Dicke Scalar-Tensor Theory . . . 86
6.2 Faulkner et al. (2007): Scalar-Tensor $f(R)$ Theories . . . 87
6.3 Park et al. (2010): Weinberg Approach to Cosmic Acceleration . . . 89
6.4 Examples of Dark Energy Models . . . 90
6.5 Massive Gravity Theories . . . 91
6.5.1 Fierz & Pauli 1939: On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field 91
6.5.2 van Dam & Veltman 1970: Massive and Massless Yang-Mills and Gravitational Fields . . . 91
6.5.3 Vainshtein 1972: To the Problem of Nonvanishing Gravitation Mass . . . 91
6.5.4 Boulware & Deser 1972: Inconsistency of Finite Range Gravitation . . . 92
6.5.5 van Nieuwenhuizen 1973: On Ghost-Free Tensor Lagrangians and Linearized Gravitation . . . 93

6.6 Proca Theory . . . 94
6.6.1 A simple Generalized Proca model . . . 94

7 CMB Temperature Anisotropies **96**
7.1 Collisionless Boltzmann Equation . . . 96
7.1.1 Geodesic Equation . . . 96
7.1.2 Collisionless Boltzmann Equation for Massless Particles . . . 97
7.1.3 Convention for Multipole Decomposition . . . 98
7.1.4 Multipole Expansion of the Boltzmann Equation . . . 98
7.1.5 Massless Particles . . . 99
7.2 Collisions of the Baryon-Photon Fluid: Thompson Scattering . . . 100
7.2.1 Collisional Boltzmann Equation for Photons . . . 100
7.2.2 Collisional Boltzmann Equation for Baryons . . . 101
7.3 Initial Conditions for the Evolution . . . 101
7.4 Observed CMB Power Spectrum . . . 102
7.4.1 Acoustic Oscillation: Tight-Coupling Approximation . . . 102
7.4.2 Diffusion Damping . . . 103
7.4.3 Free Streaming: Line-of-Sight Integration . . . 104
7.4.4 CMB Power Spectrum . . . 105

1 Spherical Collapse Model in Cosmology

1.1 Spherical Collapse Model

A simple spherical collapse model was developed long time ago to serve as a toy model for dark matter halo formation (see Peebles (1980) for details). The idea is that a slightly overdense region in a flat universe evolves as if the region were a closed universe, such that it expands almost together with the background universe but eventually turns around and collapses. The overdense region described by the closed universe would collapse to a singularity, but in reality it virializes and stops contracting. By using the analytical solutions for the two universes, we can readily derive many useful relations about the evolution of such overdense regions.

Einstein-de Sitter Universe

A flat homogeneous universe dominated by pressureless matter is called the Einstein-de Sitter Universe:

$$H^2 = \frac{8\pi G}{3}\rho_m \,, \qquad \rho_m \propto \frac{1}{a^3} \,. \tag{1.1}$$

This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations are

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{\eta_0}\right)^2 \,, \qquad \frac{t}{t_0} = \left(\frac{\eta}{\eta_0}\right)^3 \,, \qquad \eta_0 = 3t_0 \,, \tag{1.2}$$

$$H = \frac{2}{3t} \,, \qquad \mathcal{H} = \frac{2}{\eta} \,, \qquad \rho_m = \frac{1}{6\pi G t^2} \,, \qquad r = \eta_0 - \eta = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \,, \tag{1.3}$$

where the reference point t_0 satisfies $a(t_0) = 1$, but it can be any time $t_0 \in (0, \infty)$. At a given epoch t_0, one can define a mass scale

$$M := \frac{4\pi}{3}\rho_0 = \frac{H_0^2}{2G} = \frac{2}{9Gt_0^2} \,, \qquad H_0 = \frac{2}{3t_0} \,, \tag{1.4}$$

Closed Homogeneous Universe

An analytic solution can be derived for a closed universe with again pressureless matter. The evolution equations for a closed universe are

$$\frac{\tilde{a}}{\tilde{a}_t} = \frac{1-\cos\theta}{2} \,, \qquad t = \frac{t_t}{\pi}(\theta - \sin\theta) = \frac{\tilde{a}_t^2(\theta-\sin\theta)}{2\sqrt{K}} \,, \qquad d\tilde{\eta} = \frac{\tilde{a}_t}{\sqrt{K}}d\theta \,, \tag{1.5}$$

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\rho}_m - \frac{K}{\tilde{a}^2} = \frac{K}{\tilde{a}^2}\left(\frac{\tilde{a}_t}{\tilde{a}} - 1\right) \,, \tag{1.6}$$

where we used tilde to distinguish quantities in the closed universe from the flat universe and the maximum expansion (or turn-around $\tilde{a}_t$) is reached at $\theta = \pi$ ($\tilde{H}_t = 0$). The density parameters are related to the curvature K of the universe as

$$\Omega_m - 1 = -\Omega_k = -\frac{K}{\tilde{a}^2\tilde{H}_0^2} \,, \qquad K = \frac{8\pi G}{3}\frac{\rho_0}{\tilde{a}_t} = \frac{H_0^2}{\tilde{a}_t} = \frac{2GM}{\tilde{a}_t} = \frac{\pi^2\tilde{a}_t^2}{4t_t^2} \,, \qquad \frac{16}{9}\left(\frac{t_t}{\pi t_0}\right)^2 = \tilde{a}_t^3 \,. \tag{1.7}$$

Spherical Collapse Model

Matching the density equal at some early time, say t_0 (i.e., $\delta_0 = 0$), the time evolution of the overdense region can be derived in a non-perturbative way as

$$1 + \delta = \frac{\tilde{\rho}_m}{\rho_m} = \left(\frac{a}{\tilde{a}}\right)^3 = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} \,, \tag{1.8}$$

where we used

$$a^3 = \left(\frac{t}{t_0}\right)^2 = \left(\frac{t_t}{\pi t_0}\right)^2(\theta-\sin\theta)^2 \,, \qquad \tilde{a}^3 = \left(\frac{\tilde{a}_t}{2}\right)^3(1-\cos\theta)^3 = \frac{2}{9}\left(\frac{t_t}{\pi t_0}\right)^2(1-\cos\theta)^3 \,. \tag{1.9}$$

The density contrast vanishes at $\theta \to 0$, which implies that $\rho_m = \tilde{\rho}_m$ only at $\theta \to 0$. Therefore, the density contrast δ_t at its maximum expansion

$$1 + \delta_t = \frac{9\pi^2}{16} \simeq 5.6 \,, \tag{1.10}$$

is about a few, while the density contrast δ_v at its virialization

$$1 + \delta_v = 18\pi^2 \simeq 177.7 \,, \tag{1.11}$$

is a few hundreds, under the assumption that the overdensity region virialized at the half of its maximum expansion. Note that the universe further expands and the background density is reduced by factor 4, until it collapses at $t_v = 2t_t$ (or $\theta = 2\pi$).

Finally, expanding the expressions to the linear order

$$a = \frac{1}{36^{1/3}} \left(\frac{t_t}{\pi t_0}\right)^{2/3} \theta^2 + \cdots \,, \qquad \delta = \frac{3}{20}\theta^2 + \cdots \,, \tag{1.12}$$

and evaluating the linear order expressions at θ_i for a_i and δ_i, we first compute

$$\frac{\delta_i}{a_i} = \frac{3}{20} 36^{1/3} \left(\frac{t_t}{\pi t_0}\right)^{-1/3} + \cdots \,, \tag{1.13}$$

and the density contrast linearly extrapolated to late time and its value at virialization are then derived as

$$\delta_L = \frac{D}{D_i}\delta_i = \frac{a}{a_i}\delta_i = \frac{3}{10}\left(\frac{9}{2}\right)^{1/3} (\theta - \sin\theta)^{2/3} \,, \qquad D \propto a \,. \tag{1.14}$$

This equation implies that at the time of collapse the density contrast δ_L is

$$\delta_v \simeq 1.686 \,. \tag{1.15}$$

For $|\delta| \ll 1$, we derive the relation

$$\delta = \delta_L + \frac{17}{21}\delta_L^2 + \frac{341}{567}\delta_L^3 + \frac{55805}{130977}\delta_L^4 + \cdots \,, \qquad \delta_L = \delta - \frac{17}{21}\delta^2 + \frac{2815}{3969}\delta^3 - \frac{590725}{916839}\delta^4 + \cdots \,. \tag{1.16}$$

Biased Tracer

For any biased tracer δ_X, the Eulerian and the Lagrangian bias parameters can be written in a series

$$\delta_X = \sum_{n=1}^{\infty} \frac{b_n}{n!}\delta^n \,, \qquad \delta_X^L = \sum_{n=1}^{\infty} \frac{b_n^L}{n!}\delta_L^n \,, \tag{1.17}$$

where the superscript L represents quantities in the Lagrangian space. Mind that the density contrast δ for the Eulerian is nonlinear and δ_L for the Lagrangian is linear. If the number density of the objects X is conserved

$$\rho_m \, d^3x = \bar{\rho}_m \, d^3q \,, \qquad \rho_X \, d^3x = \rho_X^L \, d^3q \,, \qquad \therefore \;\; 1 + \delta_X = (1+\delta_m)(1+\delta_X^L) \,, \tag{1.18}$$

the bias parameters are related as

$$b_1 = b_1^{\rm L} + 1 \,, \qquad b_2 = b_2^{\rm L} + \frac{8}{21}b_1^{\rm L} \,, \qquad b_3 = b_3^{\rm L} - \frac{13}{7}b_2^{\rm L} - \frac{796}{1323}b_1^{\rm L} \,, \qquad b_4 = b_4^{\rm L} - \frac{40}{7}b_3^{\rm L} + \frac{7220}{1323}b_2^{\rm L} + \frac{476320}{305613}b_1^{\rm L} \,. \tag{1.19}$$

This simple relation owes to the fact that the spherical collapse model is local in both Eulerian and Lagrangian spaces.

1.2 Dark Matter Halo Mass Function

1.2.1 Basic Idea

Given the simple spherical collapse model, we would like to associate the collapsed region with some virialized objects like dark matter halos.

1.2.2 Halo Mass Function

Given the simple spherical collapse model, we would like to associate the collapsed region with some virialized objects like massive galaxy clusters or dark matter halos. Of our main interest is then the number density of such objects in a mass range $M \sim M + dM$, and this is called the mass function. However, note that the spherical collapse model does not specify the mass of the collapsed object or the time of collapse.

A simple model called, the excursion set approach, was developed: One starts with a smoothing scale R and associates it with mass M. The density fluctuation δ_R after smoothing with R is very small ($\delta_R = 0$, if $R = \infty$), and this region has never reached the critical density threshold δ_c in its entire history. This implies that there is no virialized object associated with such mass. One then decreases the smoothing scale (or mass), and looks for the collapsed probability: Some overdense regions have at some point in the past reached the critical density, while some underdense regions have not. Therefore, the total fraction F_c of collapse can be obtained by using the survival probability P_s, and it is related to the mass function as

$$F_c(\geq M) = 1 - \int_{-\infty}^{\delta_c} d\delta\, P_s = \int_M^\infty dM\, \frac{dn}{dM}\, \frac{M}{\bar\rho_m}\,, \qquad \therefore\ \frac{dn}{dM} = \frac{\bar\rho_m}{M}\left(-\frac{\partial F_c}{\partial M}\right) =: \frac{\bar\rho_m}{M}\, f(\nu)\, \frac{d\ln\nu}{dM}\,, \tag{1.20}$$

where it is assumed that the mass function only depends on mass and we defined the multiplicity function f through the relation

$$\nu := \frac{\delta_c(z)}{\sigma(M)}\,, \qquad \int_0^\infty \frac{d\nu}{\nu}\, f = 1\,. \tag{1.21}$$

The task of obtaining the mass function boils down to computing the survival probability and expressing it in terms of the multiplicity function. The way to find the survival probability at a given mass scale M is to derive the evolution of the density fluctuation as we decrease the smoothing scale R. The reason is that the region may have already collapsed at a larger mass scale or smoothing scale, and this contribution should be removed in computing the survival probability at a lower mass scale. In a given time, the survival probability at n-th step depends on the entire history of the trajectory (non-Markovian process) as

$$P_s(\delta_n, \sigma_n) d\delta_n = d\delta_n \int_{-\infty}^{\delta_c} d\delta_{n-1} \cdots \int_{-\infty}^{\delta_c} d\delta_1\, P_s(\delta_1, \cdots \delta_n, \sigma_1, \cdots, \sigma_n)\,, \qquad M_n(\sigma_n) < \cdots < M_1(\sigma_1)\,, \tag{1.22}$$

it is notoriously difficult to solve, even numerically. However, once we assume that the fluctuations are independent at each smoothing and are Gaussian distributed (true only in Fourier space at linear order), the trajectory only depends on the previous step (Markovian process) and the survival probability becomes

$$P_s(\delta_n, \sigma_n) = \int_{-\infty}^{\delta_c} d\delta_{n-1} P_t(\delta_n, \sigma_n | \delta_{n-1}, \sigma_{n-1})\, P_s(\delta_{n-1}, \sigma_{n-1})\,, \tag{1.23}$$

where the transition probability P_t is nothing but a conditional probability for a Gaussian. With the boundary condition $P_s = 0$ at $\delta = \delta_c$, the solution is (derived by Chandrasekhar for other purposes)

$$P_s = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) - \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(2\delta_c - \delta)^2}{2\sigma^2}\right] \qquad \text{for } \delta \leq \delta_c\,. \tag{1.24}$$

Note that P_s becomes Gaussian with $\delta_c \to \infty$. The survival probability for its simplest case is described by a Gaussian distribution, but the second term reflects that there exist equally likely trajectories around the threshold that have reached the threshold in the past. The collapsed fraction is

$$F_c = 1 - \frac{1}{2}\text{erf}\left(\frac{\nu}{\sqrt{2}}\right) - \frac{1}{2}\text{erf}\left(\frac{\nu}{\sqrt{2}}\right) = \text{erfc}\left(\frac{\nu_c}{\sqrt{2}}\right)\,, \tag{1.25}$$

and the multiplicity function is

$$f(\nu) = \sqrt{\frac{2}{\pi}}\, \nu\, e^{-\nu^2/2}\,, \tag{1.26}$$

where the second erfc was missed in PS, resulting in a factor 2 difference in the multiplicity function. This part was resolved by BCEK in terms of cloud-in-cloud.

Of course, this model relies on many approximations, and it is not accurate. However, it provides physical intuitions, connecting the complicated formation of galaxy clusters and the dynamical evolution of the matter density fluctuations. In general, numerical N-body simulations are run, and dark matter halos are identified by using some algorithm such as the friends-of-friends method or its variants to derive the mass function from the simulations.

1.2.3 Halo Mass Functions in Literature

In general, numerical N-body simulations are run, and dark matter halos are identified by using some algorithm such as the friends-of-friends method or its variants. These halo mass functions differ from the simple analytic formula we derived for the Gaussian random field. However, the functional form is relatively resilient, such that an introduction of a few nuisance parameters to the mass function can provide a good fit to the simulation results. A lot of variants exist in literature, but the standard and simplest cases are one by Sheth and Tormen (1999)

$$f_{\rm ST} = A\sqrt{\frac{2a}{\pi}}\left[1+\left(\frac{\sigma^2}{a\delta_c^2}\right)^p\right]\frac{\delta_c}{\sigma}\,\exp\left(-\frac{a\delta_c^2}{2\sigma^2}\right) ~, \qquad A=0.3222~,\quad a=0.707~,\quad p=0.3~, \tag{1.27}$$

and one by Jenkins et al. (2001)

$$f_{\rm Virgo} = 0.315\,\exp\left(-\left|\ln\sigma^{-1}+0.61\right|^{3.8}\right) ~. \tag{1.28}$$

They all have a limited range of validity.

1.2.4 Astrophysical Applications

halo merger trees, semi-analytic models for galaxy formation, void distributions, and so on.

1.3 From Probability Functional to Correlation Function

Assuming that the density probability functional is known, we derive the one-point statistics and its spatial correlation function. Those for the cumulative distributions are also derived. In this case, we focus on one population of halos or samples, rather than multiple populations considered in the mass function approach.

1.3.1 Density Probability Functional

With a probability functional of some field, one can trivially obtain a PDF at a given position, which is then related to a generating function by a Fourier transformation. The generating function indeed generates N-point correlation functions. They contain the same information, but in different format.

One-Point Statistics

Given a probability functional $P[\delta(\mathbf{x})]$ defined in all space $\mathbf{x}$, a one-point pdf (i.e., the probability to have δ_1 at a given point $\mathbf{x}_1$) is

$$P(\delta_1) = \int d\delta(\mathbf{x})\;P[\delta(\mathbf{x})]\;\delta^D[\delta(\mathbf{x}_1)-\delta_1] = \int d\delta(\mathbf{x})\;P[\delta(\mathbf{x})]\int\frac{dJ_1}{2\pi}\;e^{iJ_1(\delta(\mathbf{x}_1)-\delta_1)} \equiv \int\frac{dJ_1}{2\pi}\;e^{-iJ_1\delta_1}Z(J_1)~, \tag{1.29}$$

where the pdf is the Fourier counterpart of the generating function

$$Z(J_1) := \int d\delta(\mathbf{x})\;P[\delta(\mathbf{x})]\;e^{iJ_1\delta(\mathbf{x}_1)} = \left\langle e^{iJ_1\delta(\mathbf{x}_1)}\right\rangle = \sum_{n=0}^\infty\frac{(iJ_1)^n}{n!}\left\langle\delta^n(\mathbf{x}_1)\right\rangle \equiv e^\omega~, \tag{1.30}$$

and the cumulant generating function is

$$\omega(J_1) := \ln Z = \sum_{n=0}^\infty\frac{(iJ_1)^n}{n!}\left\langle\delta^n(\mathbf{x}_1)\right\rangle_c~. \tag{1.31}$$

Note the difference in the ensemble average in Z and ω, and we used the cumulant expansion theorem

$$\left\langle e^{iX}\right\rangle = \sum_{n=0}^\infty\frac{i^n}{n!}\left\langle X^n\right\rangle = \exp\left[\left\langle e^{iX}\right\rangle_c\right] = \exp\left[\sum_{n=1}^\infty\frac{i^n}{n!}\left\langle X^n\right\rangle_c\right]~, \tag{1.32}$$

$$\left\langle e^{i\delta}\right\rangle = e^{-\sigma^2/2} = 1-\frac{1}{2}\sigma^2+\frac{1}{2!}\left(\frac{\sigma^4}{4}\right)-\frac{1}{3!}\left(\frac{\sigma^6}{8}\right)\cdots = 1-\frac{1}{2}\left\langle\delta^2\right\rangle+\frac{1}{4!}\left\langle\delta^4\right\rangle-\frac{1}{6!}\left\langle\delta^6\right\rangle+\cdots~, \tag{1.33}$$

where we used the standard formula for the Gaussian distribution

$$\omega_G = -\frac{1}{2}J^2\sigma^2~, \qquad P_G(\delta_1) = \int\frac{dJ_1}{2\pi}\;e^{-iJ_1\delta_1-\frac{1}{2}J_1^2\sigma^2} = \frac{1}{\sqrt{2\pi}\,\sigma}\;e^{-\delta_1^2/2\sigma^2}~. \tag{1.34}$$

To incorporate a smoothed density field $\delta_R(\mathbf{x}_1)$, we just need to replace

$$iJ_1\delta(\mathbf{x}_1) \quad \rightarrow \quad iJ_1\delta_R(\mathbf{x}_1) = iJ_1 \int d^3\mathbf{y}_1 \; \delta(\mathbf{y}_1) W_R(\mathbf{x}_1 - \mathbf{y}_1) \; . \tag{1.35}$$

N-Point Statistics

Similarly, the one-point calculations can be extended to a two-point pdf of δ_1 at $\mathbf{x}_1$ and δ_2 at $\mathbf{x}_2$

$$P(\delta_1, \delta_2) = \int d\delta(\mathbf{x}) \; P[\delta(\mathbf{x})] \; \delta^D[\delta(\mathbf{x}_1) - \delta_1] \; \delta^D[\delta(\mathbf{x}_2) - \delta_2] = \int \frac{dJ_1}{2\pi} \; e^{-iJ_1\delta_1} \int \frac{dJ_2}{2\pi} \; e^{-iJ_2\delta_1} Z(J_1, J_2) \; , \tag{1.36}$$

where the two-point generating function is

$$Z = \left\langle e^{iJ_1\delta(\mathbf{x}_1) + iJ_2\delta(\mathbf{x}_2)} \right\rangle \; . \tag{1.37}$$

By the same token, we generalize to ∞-point pdf and derive

$$P[\delta(\mathbf{x})] = \prod_{i=1}^{\infty} \int \frac{dJ_i}{2\pi} \; e^{iJ_i\delta_i} Z \; , \qquad Z = \int d\delta(\mathbf{x}) \; P[\delta(\mathbf{x})] \; e^{\sum_i^{\infty} iJ_i\delta(\mathbf{x}_i)} = \left\langle e^{\int d^3\mathbf{x} \; iJ(\mathbf{x})\delta(\mathbf{x})} \right\rangle = e^{\omega} \; , \tag{1.38}$$

$$\omega = \left\langle e^{\int d^3\mathbf{x} \; iJ(\mathbf{x})\delta(\mathbf{x})} \right\rangle_c \; , \tag{1.39}$$

and from the generating function we "generate" cumulants

$$\langle \delta(\mathbf{x}_1) \cdots \delta(\mathbf{x}_n) \rangle_c =: \xi_c^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \frac{1}{i^n} \frac{\delta^n \ln Z}{\delta J(\mathbf{x}_1) \cdots \delta J(\mathbf{x}_n)} \bigg|_{\mathbf{J}=0} \; . \tag{1.40}$$

Further Generalization

In fact, this can be further generalized by considering not only the density field $\delta(\mathbf{x})$, but also other fields $A_\mu(\mathbf{x})$ with $\mu = 1, \cdots, N$ such as the velocity field $\eta_i(\mathbf{x}) = \partial_i\delta(\mathbf{x})$, the tidal gravitational field $\zeta_{ij}(\mathbf{x}) = \partial_{ij}\delta(\mathbf{x})$, and so on. For example, the peak model PDF requires $N = 10$ (1 for δ, 3 for η_i, 6 for ζ_{ij}) all at the same spatial point (such that $N \times M$-number of fields for M different spatial points). Another example is $N = \infty$ for the density field. In general, the generating function can be re-constructed for N-number of A_μ by using the cumulants as (here, N is the total number including different quantities and spatial points)

$$M^{(n)}_{\mu_1, \cdots, \mu_n} \;\; := \;\; \langle A_{\mu_1} \cdots A_{\mu_n} \rangle_c = \frac{1}{i^n} \frac{\delta^n \ln Z}{\delta J_{\mu_1} \cdots \delta J_{\mu_n}} \bigg|_{\mathbf{J}=0} \; , \tag{1.41}$$

$$\ln Z[\mathbf{J}] \;\; = \;\; \sum_{n=1}^{\infty} \frac{i^n}{n!} \sum_{\mu_1=1}^{N} \cdots \sum_{\mu_n=1}^{N} M^{(n)}_{\mu_1, \cdots, \mu_n} J_{\mu_1} \cdots J_{\mu_n} \; , \qquad Z = e^{-\mathbf{J}\cdot\mathbf{M}^T\cdot\mathbf{J}} \exp\left[\sum_{n=3}^{\infty} \frac{i^n}{n!} \sum_{\mu_1, \cdots, \mu_n}^{N} M^{(n)}_{\mu_1, \cdots, \mu_n} J_{\mu_1} \cdots J_{\mu_n}\right] \; ,$$

where we isolated the Gaussian part in the generating function. Finally, using the trick $J_\mu \rightarrow i\partial/\partial A_\mu$ we can express the one-point pdf with N-A_μ in terms of Gaussian PDF as

$$\begin{aligned} P(\mathbf{A}) \;\; &= \;\; \prod_{i=1}^{N} \left[\int_{-\infty}^{\infty} \frac{dJ_i}{(2\pi)}\right] \; e^{-i\mathbf{J}\cdot\mathbf{A}} \; Z(\mathbf{J}) = \exp\left[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \sum_{\mu_1, \cdots, \mu_n}^{N} M^{(n)}_{\mu_1, \cdots, \mu_n} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}}\right] \prod_{i=1}^{N} \left[\int_{-\infty}^{\infty} \frac{dJ_i}{(2\pi)}\right] \; e^{-\mathbf{J}\cdot\mathbf{M}^T\cdot\mathbf{J} - i\mathbf{J}\cdot\mathbf{A}} \\ &= \;\; \exp\left[\sum_{n=3}^{\infty} \frac{(-1)^n}{n!} \sum_{\mu_1, \cdots, \mu_n}^{N} M^{(n)}_{\mu_1, \cdots, \mu_n} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}}\right] P_G(\mathbf{A}) \; . \end{aligned} \tag{1.42}$$

Another useful generalization for a one-point PDF is that two probability distribution functions and their generating functions are related to each other as

$$P_a(\delta) = \exp\left[\sum_{n=1}^{\infty} \frac{\kappa_n^a - \kappa_n^b}{n!} \left(-\frac{d}{d\delta}\right)^n\right] P_b(\delta) \; , \qquad Z_a(J) = \exp\left[\sum_{n=1}^{\infty} \frac{\kappa_n^a - \kappa_n^b}{n!} \, (iJ)^n\right] Z_b(J) \; , \tag{1.43}$$

where the cumulant is $\kappa_n = \langle \delta^n \rangle_c$.

Two-Point Correlation

Now we consider a probability at a given position or two positions, in which the density fluctuation is above some threshold. The cumulative N-point probability distributions are from Eq. (1.42)

$$P_1(>\nu) = \frac{1}{\sqrt{2\pi}}\int_\nu^\infty dy\, \exp\left[\sum_{N=3}^\infty (-1)^N \frac{\langle\delta^N\rangle_c}{\sigma^N N!}\frac{d^N}{dy^N}\right] e^{-y^2/2} , \tag{1.44}$$

$$P_2(>\nu,r) = \frac{1}{2\pi}\int_\nu^\infty dy_1 \int_\nu^\infty dy_2\, \exp\left[\sum_{N=2}^\infty\sum_{m=0}^N (-1)^N \frac{w_s^{(N,m)}(r)}{m!(N-m)!}\frac{\partial^N}{\partial y_1^m \partial y_2^{N-m}}\right] e^{-\frac{1}{2}(y_1^2+y_2^2)} ,$$

where the N-point matter correlation functions are

$$w_s^{(N,m)}(r) := \begin{cases} w_s^{(2,m)} = \xi_s^{(2,m)}(r)/\sigma_{0s}^2 & (m=1) \\ w_s^{(2,m)} = 0 & (m = 0 \text{ or } 2) \\ w_s^{(N,m)} = \xi_s^{(N,m)}(r)/\sigma_{0s}^N & (N>2) \end{cases} , \qquad \xi_s^{(N,m)}(r) := \Big\langle \underbrace{\delta_s(\boldsymbol{x}_1)\cdots\delta_s(\boldsymbol{x}_1)}_{m \text{ times}}\underbrace{\delta_s(\boldsymbol{x}_2)\cdots\delta_s(\boldsymbol{x}_2)}_{N-m \text{ times}}\Big\rangle_c . \tag{1.45}$$

Again, keeping only the linear order term in Eq. (1.44), i.e., $\exp[\cdots] = 1+\cdots$, we have

$$P_1(>\nu) \approx \frac{1}{2}\,\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) + \sum_{N=3}^\infty \frac{1}{\sqrt{2\pi}}\frac{w_s^{(N,0)}}{N!}H_{N-1}(\nu)\, e^{-\nu^2/2} , \tag{1.46}$$

$$P_2(>\nu,r) \approx \left[\frac{1}{2}\,\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^2 + \sqrt{\frac{1}{2\pi}}\,\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\sum_{N=3}^\infty \frac{w_s^{(N,0)}}{N!}H_{N-1}(\nu)e^{-\nu^2/2}$$
$$+\frac{1}{2\pi}\sum_{N=2}^\infty\sum_{m=1}^{N-1}\frac{w_s^{(N,m)}}{m!(N-m)!}H_{m-1}(\nu)H_{N-m-1}(\nu)e^{-\nu^2} ,$$

where $H_m(x)$ is the Hermite polynomials. Note that for P_2, there exists a non-vanishing term with $N = 2$, otherwise it would simply describe two independent PDF.

1.3.2 Mildly Non-Gaussian Halo Mass Function

Given the non-Gaussian PDF, the non-Gaussian mass function can be derived as

$$\frac{dn}{dM} = -2\,\frac{\bar\rho_m}{M}\frac{dP(>\nu_R)}{dM} \tag{1.47}$$
$$= 2\,\frac{\bar\rho_m}{M}\frac{d\nu}{dM}\frac{1}{\sqrt{2\pi}}\exp\left[\sum_{n=3}^\infty \frac{S_n\sigma^n}{n!\sigma^2}\left(-\frac{d}{d\nu}\right)^n\right]e^{-\nu^2/2} - 2\,\frac{\bar\rho_m}{M}\int_\nu^\infty dx\,\frac{1}{\sqrt{2\pi}}\frac{d}{dM}\left\{\exp\left[\sum_{n=3}^\infty\frac{S_n\sigma^n}{n!\sigma^2}\left(-\frac{d}{dx}\right)^n\right]e^{-x^2/2}\right\} ,$$

where the reduced cumulant is $S_n = \kappa_n/\kappa_2^{n-1}$, i.e., $\kappa_n = (S_n\sigma^n/\sigma^2)\sigma^n$. Note the reduced cumulant follows the convention in literature, but it is dimensionful. One needs to solve the above equation for the non-Gaussian mass function. Various non-Gaussian mass functions differ in truncation of the expansion above (see below). For example, Matarrese et al. (2000) truncated at some cumulant order. LoVerde et al. (2008) truncated by using the Edgeworth expansion. LoVerde and Smith (2011) truncated by using the log of Edgeworth.

Only keeping the linear order term $\exp[x] = 1 + x + \cdots$ and using Eq. (1.42), the one-point non-Gaussian PDF is by (often known as Gram-Chalier A Series)

$$P_a(\delta) = \exp\left[\sum_{n=3}^\infty \frac{\kappa_n}{n!}\left(-\frac{d}{d\delta}\right)^n\right]\frac{1}{\sqrt{2\pi}\sigma}e^{-\delta^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma}e^{-\delta^2/2\sigma^2}\left[1+\frac{S_3\sigma}{6}H_3+\frac{S_4\sigma^2}{24}H_4+\cdots\right] . \tag{1.48}$$

However, since it only accounts for the linear order term in the exponential, this series is not positive-definite (hence P_a is not a proper PDF) and often diverges.

Following Lucchin & Matarrese (1988) and manipulating the variables,

$$\nu := \frac{\delta}{\sigma} , \qquad J := \frac{z-i\nu}{\sigma} , \tag{1.49}$$

we derive

$$-iJ\delta+\omega(J)=\left(-\frac{1}{2}J^2\sigma^2-iJ\delta\right)+\left(\omega+\frac{1}{2}J^2\sigma^2\right)=-\frac{1}{2}z^2-\frac{1}{2}\nu^2+\frac{1}{\sigma^2}\sum_{n=3}^{\infty}\frac{S_n}{n!}\sigma^n(\nu+iz)^n\,, \tag{1.50}$$

and by using the contour integration, we finally derive the one-point PDF, known as the Edgeworth expansion,

$$\begin{aligned}P(\delta) &= e^{-\frac{1}{2}\nu^2}\int_{-\infty}^{\infty}\frac{dz}{2\pi\sigma}\exp\left[-\frac{1}{2}z^2+\frac{1}{\sigma^2}\sum_{n=3}^{\infty}\frac{S_n}{n!}\sigma^n(\nu+iz)^n\right] \qquad (1.51)\\ &= \frac{e^{-\frac{1}{2}\nu^2}}{\sqrt{2\pi}\sigma}\left[1+\frac{S_3\sigma}{6}H_3(\nu)+\sigma^2\left(\frac{S_4}{24}H_4(\nu)+\frac{S_3^2}{72}H_6(\nu)\right)+\sigma^3\left(\frac{S_5}{120}H_5(\nu)+\frac{S_3S_4}{144}H_7(\nu)+\frac{S_3^3}{1296}H_9(\nu)\right)+\cdots\right]\,,\end{aligned}$$

where the probabilists' Hermite polynomial is

$$H_L(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dz\,e^{-\frac{1}{2}z^2}(x+iz)^L\,, \tag{1.52}$$

but one can also derive from Eq. (1.42) by going to higher order terms in the exponential $\exp[x]=1+x+x^2/2+\cdots$.

1.3.3 Peak-Background Split and Statistics of Thresholded Region

Here we exclusively focus on the statistics of the density distribution above a given threshold.

One-Point Statistics

When the underlying smoothed density field (R_s) obeys the Gaussian statistics, the probability P_1 of exceeding the threshold ν and the effect of adding a long-wavelength (background) perturbation δ_l of characteristic wavelength $R_l\gg R_s$ to the small scale density field (peak) δ_s are

$$P_1(>\nu)=\frac{1}{\sqrt{2\pi}}\int_\nu^\infty dx\,e^{-x^2/2}=\frac{1}{2}\,\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\,,\qquad P_1(>\nu,\delta_l)=P_1\left(>\nu-\frac{\delta_l}{\sigma_s}\right)\,. \tag{1.53}$$

We define the peak-background split *cumulative* bias factors c_N as the fractional change of P_1 with δ_l via

$$c_N:=\frac{1}{P_1(>\nu)}\frac{d^N\,P_1(>\nu,\delta_l)}{d\,\delta_l^N}=\left(-\frac{1}{\sigma_s}\right)^N\frac{1}{P_1(>\nu)}\frac{d^N[P_1(>\nu)]}{d\nu^N}=\sqrt{\frac{2}{\pi}}\left[\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-1}\frac{e^{-\nu^2/2}}{\sigma_s^N}H_{N-1}(\nu)\,, \tag{1.54}$$

and in the high peak limit $\nu\to\infty$

$$H_N\to\nu^N\,,\qquad c_N\approx\nu H_{N-1}(\nu)/\sigma_s^N\approx\nu^N/\sigma_s^N\,. \tag{1.55}$$

Here, H_N is the statistician's Hermite polynomial defined by

$$H_N(x):=(-1)^Ne^{x^2/2}\frac{d^N}{dx^N}\left(e^{-x^2/2}\right)\,,\qquad H_N^{\rm phys}(x):=2^{N/2}H_N\left(\sqrt{2}x\right)\,, \tag{1.56}$$

$$H_0=1\,,\qquad H_1=x\,,\qquad H_2=x^2-1\,,\qquad H_3=x^3-3x\,,\qquad H_4=x^4-6x^2+3\,. \tag{1.57}$$

For differential mass function, or halos in a mass bin (vs. cumulative), we derive

$$\bar n_h(M):=-2\frac{\bar\rho_m}{M}\frac{d}{dM}P_1\left(>\nu\right)=2\frac{\bar\rho}{M^2}\frac{\nu e^{-\nu^2/2}}{\sqrt{2\pi}}\left|\frac{d\ln\sigma_s}{d\ln M}\right|\,,\qquad P_1(>\nu)=\frac{1}{2\bar\rho_m}\int_M^\infty dM'M'\bar n_h(M')\,. \tag{1.58}$$

where the factor of 2 is introduced to account for the fact that regions with $\delta<\delta_c$ may be embedded in regions with $\delta>\delta_c$ on scale $>R_s$ (clouds-in-clouds). Using the definition of the cumulative bias, we have

$$c_N=\left[\int_M^\infty dM'M'\bar n_h(M')\right]^{-1}\int_M^\infty dM'M'\left[\frac{d^N}{d\delta_l^N}\,\bar n_h(M')\right]\,,\qquad\left[\frac{d^N}{d\delta_l^N}\,\bar n_h(M')\right]=:b_N(M')\bar n_h(M')\,, \tag{1.59}$$

where the PBS bias is

$$b_N(M)=\frac{1}{\nu e^{-\nu^2/2}}\left(-\frac{1}{\sigma_s}\right)^N\frac{d^N}{d\nu^N}\left(\nu e^{-\nu^2/2}\right)=\frac{1}{\nu}\frac{H_{N+1}(\nu)}{\sigma_M^N}\,. \tag{1.60}$$

It is only in the high-peak limit ($\nu\gg1$) that the mass-weighted cumulative bias c_N and the bias $b_N(M)$ asymptotic to the same values.

Two-Point Statistics

For a Gaussian, we have (see Eq. 1.44)

$$\xi_{>\nu}(r) = \frac{P_2(r)}{P_1^2} - 1 = \frac{2}{\pi}\left[\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2}\int_\nu^\infty dy_1\int_\nu^\infty dy_2\,\exp\left[\frac{\xi_s(r)}{\sigma_s^2}\frac{\partial^2}{\partial y_1\partial y_2}\right]e^{-\frac{1}{2}\left(y_1^2+y_2^2\right)} - 1\,, \tag{1.61}$$

and the double integration is

$$\int_\nu^\infty dy_1\int_\nu^\infty dy_2\sum_{N=0}^\infty\frac{1}{N!}\left[\frac{\xi_s(r)}{\sigma_s^2}\right]^N\left(\frac{\partial^2}{\partial y_1\partial y_2}\right)^N e^{-\frac{1}{2}\left(y_1^2+y_2^2\right)} = \sum_{N=0}^\infty\frac{1}{N!}\left[\frac{\xi_s(r)}{\sigma_s^2}\right]^N\left[\int_\nu^\infty dy\left(\frac{\partial}{\partial y}\right)^N e^{-y^2/2}\right]^2\,, \tag{1.62}$$

where the $N=0$ term is simply the erfc function for the Gaussian piece and the integration is

$$\int_\nu^\infty dy\left(\frac{\partial}{\partial y}\right)^N e^{-y^2/2} = -\left(\frac{\partial}{\partial y}\right)^{N-1}e^{-y^2/2} = (-1)^N e^{-y^2/2}H_{N-1}(y)\,. \tag{1.63}$$

Therefore, the correlation function is

$$\xi_{>\nu}(r) = \frac{2}{\pi}\left[\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2}\sum_{N=1}^\infty\frac{\left[\xi_s(r)\right]^N}{N!\sigma_s^{2N}}\left[H_{N-1}(\nu)\right]^2 e^{-\nu^2} = \sum_{N=1}^\infty\frac{c_N^2}{N!}\left[\xi_s(r)\right]^N = c_1^2\xi_s(r) + \frac{1}{2}\,c_2^2\xi_s^2(r) + \cdots\,. \tag{1.64}$$

Since the local bias expansion can be written as

$$\delta_{>\nu}(\boldsymbol{x}) = \sum_{N=1}^\infty\frac{\tilde{c}_N}{N!}\left[\delta_s(\boldsymbol{x})\right]^N = \tilde{c}_1\delta_s + \frac{1}{2}\,\tilde{c}_2\delta_s^2 + \cdots\,, \tag{1.65}$$

we see that the coefficient $\tilde{c}_N$ is different from the c_N appearing in the correlation function: the coefficient c_N^2 includes not only $\tilde{c}_N^2$, but also terms such as $\tilde{c}_N\tilde{c}_{N+2m}\sigma_s^{2m}$ for all positive integers $m \geq N/2$. This clearly shows that the bias parameters c_N from the peak-background split are to be seen as "renormalized" bias parameters, which take all the higher order moments into account.

Non-Gaussian Two-Point Statistics

Similarly, the correlation function in the non-Gaussian case is obtained by using Eq. (1.46) and keeping the linear order only

$$\xi_{>\nu}(r) = \frac{2}{\pi}\left[\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2}\sum_{N=2}^\infty\sum_{m=1}^{N-1}\frac{w_s^{(N,m)}}{m!(N-m)!}H_{m-1}(\nu)H_{N-m-1}(\nu)e^{-\nu^2} = \sum_{N=2}^\infty\sum_{m=1}^{N-1}\frac{c_m c_{N-m}}{m!(N-m)!}\,\xi_s^{(N,m)}(r)\,. \tag{1.66}$$

Now we compute the power spectrum. For simplicity and without loss of generality, we will assume that a single non-Gaussian N-point function ($N \geq 3$) dominates. We then have

$$\xi_{>\nu}(r) = c_1^2\xi_s(r) + \sum_{m=1}^{N-1}\frac{c_m c_{N-m}}{m!(N-m)!}\,\xi_s^{(N,m)}(r)\,, \qquad P_{>\nu}(k) = c_1^2 P_s(k) + \sum_{m=1}^{N-1}\frac{c_m c_{N-m}}{m!(N-m)!}\tilde{\xi}_s^{(N,m)}(k)\,, \tag{1.67}$$

where for $m=1$ (and $m=N-1$) we have

$$\tilde{\xi}_s^{(N,1)}(k) = \mathcal{M}_s(k)\prod_{i=1}^{N-2}\left(\int\frac{d^3k_i}{(2\pi)^3}\mathcal{M}_s(k_i)\right)\mathcal{M}_s(q)\,\xi_\Phi^{(N)}(\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_{N-2},\mathbf{q};X)\,, \tag{1.68}$$

$$\tilde{\xi}_s^{(N,2)}(k) = \prod_{i=1}^{N-2}\left(\int\frac{d^3k_i}{(2\pi)^3}\mathcal{M}_s(k_i)\right)\mathcal{M}_s(|\mathbf{k}-\mathbf{k}_1|)\,\mathcal{M}_s(q)\,\xi_\Phi^{(N)}(\mathbf{k}-\mathbf{k}_1,\mathbf{k}_1,\ldots,\mathbf{k}_{N-2},\mathbf{q};X)\,, \tag{1.69}$$

where $\mathbf{q} = -\mathbf{k}_1 - \cdots - \mathbf{k}_{N-2} - \mathbf{k}$, and X is a set of variables characterizing the primordial N-point function such as $f_{\rm NL}$, $g_{\rm nl}$ depending on the details of the model of non-Gaussianity. On large scales, $\tilde{\xi}_s^{(N,m)}$ terms ($m = 2\cdots N-2$) all add white-noise contributions to the power spectrum of thresholded regions, and only the terms with $m=1,\ N-1$ contribute to the scale-dependent bias. Therefore, the power spectrum is

$$P_{>\nu}(k) \simeq c_1^2 P_s(k) + 2\frac{c_1 c_{N-1}}{(N-1)!}\tilde{\xi}_s^{(N,1)}(k) = \left[c_1^2 + 2\frac{4}{(N-1)!}c_1 c_{N-1}\sigma_s^2\mathcal{M}_s^{-1}(k)\mathcal{F}_s^{(N)}(k,X)\right]P_s(k)\,, \tag{1.70}$$

where the shape factor is

$$\mathcal{F}_s^{(N)}(k,X) \equiv \frac{\mathcal{M}_s^{-1}(k)}{4\sigma_s^2 P_\phi(k)}\tilde{\xi}_s^{(N,1)}(k) = \frac{1}{4\sigma_s^2\, P_\phi(k)}\left\{\prod_{i=1}^{N-2}\int\frac{d^3k_i}{(2\pi)^3}\mathcal{M}_s(k_i)\right\}\mathcal{M}_s(q)\,\xi_\Phi^{(N)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_{N-2},\boldsymbol{q},k\hat{\boldsymbol{z}};X)\,. \quad (1.71)$$

Noting that $P_{>\nu}=(c_1^2+2c_1\Delta c_1)P_s$ to leading order in the non-Gaussian corrections, we can identify the scale-dependent correction to the linear bias, and for the $f_{\rm NL}$-case we have

$$\Delta c_1(k) = \frac{4c_{N-1}}{(N-1)!}\sigma_s^2\frac{\mathcal{F}_s^{(N)}(k)}{\mathcal{M}_s(k)}\,, \qquad P_{>\nu}(k) = \left[c_1^2+4f_{\rm NL}c_1c_2\sigma_s^2\mathcal{M}_s^{-1}(k)\right]P_s(k) \overset{\nu\gtrsim 1}{=} b_1^2\left[1+4f_{\rm NL}\frac{\delta_c}{\mathcal{M}_s(k)}\right]P_s(k)\,. \quad (1.72)$$

Note that in general the scale-dependent correction is c_{N-1}, not the usual (b_1-1).

1.4 Beyond the Spherical Model: Peak Models

1.4.1 Basic Idea of Peak Models

Going beyond the spherical collapse model, we consider the density peak positions for the sites for halo formation. Compared to the simplest version of the spherical collapse models, not only the density threshold, but also its derivative is considered to describe the halo formation process. In the framework of this peak model, we derive the statistics of such peaks including the number density, the shape parameters (deviation from sphericity), and the correlation. For the Gaussian probability distribution, analytic calculations are possible, and much of the work was done in Bardeen, Bond, Kaiser, and Szalay (1986).

1.4.2 Multivariate Gaussian Joint Probability Distribution

The notational convention is

$$F(\mathbf{x},t)=\delta_m(\mathbf{x},t)\,, \qquad \eta_i(\mathbf{r})=\nabla_iF(\mathbf{r})\,, \qquad \zeta_{ij}(\mathbf{r})=\nabla_i\nabla_jF(\mathbf{r})\,, \qquad N_{pk}(\nu)=\langle N_{pk}(\mathbf{r},\nu)\rangle\,. \quad (1.73)$$

The joint Gaussian probability distribution of F, η_i, ζ_{ij} at a given one-point $\mathbf{r}$ is described by its correlations

$$\langle FF\rangle=\sigma_0^2\,,\quad \langle\eta_i\eta_j\rangle=\frac{\sigma_1^2}{3}\,\delta_{ij}\,,\quad \langle F\zeta_{ij}\rangle=-\frac{\sigma_1^2}{3}\,\delta_{ij}\,,\quad \langle\zeta_{ij}\zeta_{kl}\rangle=\frac{\sigma_2^2}{15}\,(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,,\quad \langle F\eta_i\rangle=\langle\eta_i\zeta_{jk}\rangle=0\,, \quad (1.74)$$

where $\sigma_j^2=\int d\ln k\;\Delta_k^2\;k^{2j}$ and $[\sigma_j^2]=L^{2j}$ dimensionful. Note that if one consider a PDF for density F only, one just needs σ_0^2 (it is one-point, not spatial two-point correlation). Because of the symmetry of ζ_{ij}, only six components are independent: $A=1-6$, referring to the $ij=11,22,33,23,13,12$ components. The covariance matrix $\mathbf{M}$ has dimension 10.

$$\mathbf{M}=\begin{pmatrix}
\sigma_0^2 & 0 & 0 & 0 & -\sigma_1^2/3 & -\sigma_1^2/3 & -\sigma_1^2/3 & 0 & 0 & 0\\
0 & \sigma_1^2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \sigma_1^2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \sigma_1^2/3 & 0 & 0 & 0 & 0 & 0 & 0\\
-\sigma_1^2/3 & 0 & 0 & 0 & \sigma_2^2/5 & \sigma_2^2/15 & \sigma_2^2/15 & 0 & 0 & 0\\
-\sigma_1^2/3 & 0 & 0 & 0 & \sigma_2^2/15 & \sigma_2^2/5 & \sigma_2^2/15 & 0 & 0 & 0\\
-\sigma_1^2/3 & 0 & 0 & 0 & \sigma_2^2/15 & \sigma_2^2/15 & \sigma_2^2/5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2^2/15 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2^2/15 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2^2/15
\end{pmatrix}\,. \quad (1.75)$$

To diagonalize the remaining four dimensions, we introduce a new set of variables $\{\zeta_A,\ A=1,2,3\}\to\{x,y,z\}$, where

$$\sigma_2x=-\nabla^2F=-(\zeta_1+\zeta_2+\zeta_3)\,, \qquad \sigma_2y=-(\zeta_1-\zeta_3)/2\,, \qquad \sigma_2z=-(\zeta_1-2\,\zeta_2+\zeta_3)/2\,, \quad (1.76)$$

$$\zeta_1=-\frac{\sigma_2}{3}(x+3y+z)\,, \qquad \zeta_2=-\frac{\sigma_2}{3}(x-2z)\,, \qquad \zeta_3=-\frac{\sigma_2}{3}(x-3y+z)\,. \quad (1.77)$$

Further manipulation gives

$$\left(\frac{F}{\sigma_0},\frac{\zeta_1}{\sigma_2},\frac{\zeta_2}{\sigma_2},\frac{\zeta_3}{\sigma_2}\right)\to\begin{pmatrix}1 & -\gamma/3 & -\gamma/3 & -\gamma/3\\ -\gamma/3 & 1/5 & 1/15 & 1/15\\ -\gamma/3 & 1/15 & 1/5 & 1/15\\ -\gamma/3 & 1/15 & 1/15 & 1/5\end{pmatrix}\,, \qquad (\nu,x,y,z)\to\begin{pmatrix}1 & \gamma & 0 & 0\\ \gamma & 1 & 0 & 0\\ 0 & 0 & 1/15 & 0\\ 0 & 0 & 0 & 1/15\end{pmatrix}\,, \quad (1.78)$$

where γ is dimensionless and

$$\left\langle\nu^2\right\rangle = 1~, \quad \left\langle x^2\right\rangle = 1~, \quad \langle x\nu\rangle = \gamma~, \quad \left\langle y^2\right\rangle = \frac{1}{15}~, \quad \left\langle z^2\right\rangle = \frac{1}{5}~, \quad \gamma \equiv \frac{\left\langle k^2\right\rangle}{\left\langle k^4\right\rangle^{1/2}} = \frac{\sigma_1^2}{\sigma_2\sigma_0}~. \tag{1.79}$$

Note $\left\langle k^2\right\rangle = \sigma_1^2/\sigma_0^2 = -3\xi''(0)/\xi(0)$. So, if $P_m(k) \sim \delta^D(k-k_0)$, then $\gamma = 1$, and if $\Delta_k^2 \simeq$constant, then $\gamma \ll 1$.

For multivariate Gaussians $(y_1, \cdots, y_n)$, the joint Gaussian probability is

$$P(y_1,\cdots,y_n)\,dy_1\cdots dy_n = \frac{e^{-Q}}{\left[(2\pi)^n \det \mathbf{M}\right]^{1/2}}\,dy_1\cdots dy_n~, \tag{1.80}$$

where $Q = \frac{1}{2}\left(\mathbf{y}-\langle\mathbf{y}\rangle\right)^T \mathbf{M}^{-1}\left(\mathbf{y}-\langle\mathbf{y}\rangle\right)$ and the covariance matrix $\mathbf{M} = \left\langle\left(\mathbf{y}-\langle\mathbf{y}\rangle\right)\left(\mathbf{y}-\langle\mathbf{y}\rangle\right)^T\right\rangle$. Therefore, for the Gaussian variables ν, $\boldsymbol{\eta}$, ζ_A ($A = 1-6$), the joint probability is

$$P(F,\eta_i,\zeta_A)\,dF\,d^3\boldsymbol{\eta}\,d^6\zeta = \frac{e^{-Q}}{\left[(2\pi)^{10}\det\mathbf{M}\right]^{1/2}}\,dF\,d^3\boldsymbol{\eta}\,d^6\zeta~, \tag{1.81}$$

where $\mathbf{M}$ is the covariance matrix and

$$2Q = \nu^2 + \frac{(x-x_*)^2}{(1-\gamma^2)} + 15y^2 + 5z^2 + \frac{3\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}}{\sigma_1^2} + \sum_{A=4}^{6}\frac{15\,\zeta_A^2}{\sigma_2^2}~, \qquad\qquad x_* \equiv \gamma\nu~. \tag{1.82}$$

Now we choose the principal axes of ζ_{ij} and let the eigenvalues λ_i, where $i = 1,2,3$:

$$\boldsymbol{\zeta} = -\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}~, \tag{1.83}$$

where $\zeta_A = -\lambda_A$ for $A = 1,2,3$ (note $x,y,z \neq 0$) and $\zeta_A = 0$ for $A = 4,5,6$ (no off-diagonal terms). The volume element in the six-dimensional space of symmetric real matrices is

$$\begin{aligned} d^6\zeta_A &= \prod_{A=1}^{6} d\zeta_A = |(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)|d\lambda_1\,d\lambda_2\,d\lambda_3\,\frac{d\Omega_{S^3}}{6}\\ &= 2\,\sigma_2^3\,|y\,(y^2-z^2)|\,\frac{2}{3}\,\sigma_2^3\,dx\,dy\,dz\,\frac{d\Omega_{S^3}}{6} = \frac{2}{9}\,\sigma_2^6\,|y\,(y^2-z^2)|\,dx\,dy\,dz\,d\Omega_{S^3}~, \end{aligned} \tag{1.84}$$

$$d\lambda_1\,d\lambda_2\,d\lambda_3 = \sigma_2^3|J_R^{-1}|\,dx\,dy\,dz = \frac{2}{3}\,\sigma_2^3\,dx\,dy\,dz~, \tag{1.85}$$

$$y\,(y^2-z^2) = -\frac{1}{2\sigma_2^3}\,(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)~. \tag{1.86}$$

Therefore, the joint probability is

$$P(F,\eta_i,\zeta_A)\,dF\,d^3\boldsymbol{\eta}\,d^6\zeta = \frac{e^{-Q}}{\left[(2\pi)^{10}\det\mathbf{M}\right]^{1/2}}\,(\sigma_0 d\nu)\,d^3\boldsymbol{\eta}\,\left(\frac{2}{9}\,\sigma_2^6\,|y\,(y^2-z^2)|\,dx\,dy\,dz\,d\Omega_{S^3}\right)~, \tag{1.87}$$

and since Q is independent of the Euler angles, integrating over the angles gives $2\pi^2/3!$ and the joint probability is

$$\begin{aligned} P(\nu,\boldsymbol{\eta},x,y,z)\,d\nu\,d^3\boldsymbol{\eta}\,dx\,dy\,dz &= \frac{e^{-Q}}{\left[(2\pi)^{10}\det\mathbf{M}\right]^{1/2}}\,\sigma_0\,\frac{2}{9}\,\sigma_2^6\,|y\,(y^2-z^2)|\,\frac{2\pi^2}{3!}\,d\nu\,d^3\boldsymbol{\eta}\,dx\,dy\,dz\\ &= \frac{(15)^{5/2}}{32\pi^3}\,\frac{\sigma_0^3}{\sigma_1^3(1-\gamma^2)^{1/2}}\,|2y\,(y^2-z^2)|\,e^{-Q}d\nu\,dx\,dy\,dz\,\frac{d^3\boldsymbol{\eta}}{\sigma_0^3}~, \end{aligned} \tag{1.88}$$

where the prefactor is N and the determinant is

$$\det\mathbf{M} = \left(\frac{\sigma_1^2}{3}\right)^3\left(\frac{\sigma_2^2}{15}\right)^3\frac{1-\gamma}{3^3 15^2}\,\sigma_0^2\sigma_2^6~. \tag{1.89}$$

1.4.3 Gaussian Peaks Models

Given $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and peaks are at extremum $\boldsymbol{\eta}(\mathbf{r}_p) = 0$, the density fluctuation and its derivative near peaks are

$$F(\mathbf{r}) \simeq F(\mathbf{r}_p) + \frac{1}{2}\sum_{ij}\zeta_{ij}(\mathbf{r}_p)(\mathbf{r}-\mathbf{r}_p)_i(\mathbf{r}-\mathbf{r}_p)_j \,, \qquad \eta_i(\mathbf{r}) \simeq \sum_j \zeta_{ij}(\mathbf{r}_p)(\mathbf{r}-\mathbf{r}_p)_j \,. \tag{1.90}$$

Therefore, the full density field for the maxima of height between ν_0 and $\nu_0 + d\nu$ (differential number density of peaks) is

$$\begin{aligned} N_{pk}(\mathbf{r},\nu_0)\,d\nu &= \sum_i \delta^D(\mathbf{r}-\mathbf{r}_{p,i}) = \delta^D[(\zeta^{-1})\boldsymbol{\eta}]\,\theta(\lambda_1)\,\theta(\lambda_2)\,\theta(\lambda_3)\,\delta^D(\nu-\nu_0)\,d\nu \qquad (1.91)\\ &= |\det\zeta|\,\delta^D(\boldsymbol{\eta})\,\theta(\lambda_1)\,\theta(\lambda_2)\,\theta(\lambda_3)\delta^D(\nu-\nu_0)\,d\nu\,. \end{aligned}$$

Similarly, the phase-space distribution function is $f_{pk}(\mathbf{r},\mathbf{v}_0) = \delta^D(\mathbf{v}-\mathbf{v}_0)\,n_{pk}(\mathbf{r})$. Unfortunately, the probability function of $n_{pk} = N_{pk}(>\nu)$ is analytically intractable. However, the average $N_{pk} = \langle N_{pk}(\mathbf{r})\rangle$ can be obtained by integrating Eq. (1.88) as

$$\begin{aligned} N_{pk}(\nu,x,y,z)\,d\nu\,dx\,dy\,dz &= \langle N_{pk}\rangle\,d\nu\,dx\,dy\,dz = \int d^3\boldsymbol{\eta}\,N_{pk}\,P(\nu,\boldsymbol{\eta},x,y,z)\,d\nu\,dx\,dy\,dz \\ &= \frac{5^{5/2}3^{1/2}}{(2\pi)^3}\left(\frac{\sigma_2}{\sigma_1}\right)^3\frac{1}{(1-\gamma^2)^{1/2}}e^{-\tilde{Q}}F(x,y,z)\,\chi\,d\nu\,dx\,dy\,dz\,, \qquad (1.92)\end{aligned}$$

where

$$F(x,y,z) = \frac{27}{2}\frac{\lambda_1\lambda_2\lambda_3(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}{\sigma_2^6} = (x-2z)\left[(x+z)^2-(3y)^2\right]\,y(y^2-z^2)\,, \tag{1.93}$$

$$\tilde{Q} = \frac{\nu^2}{2} + \frac{(x-x_*)^2}{2(1-\gamma^2)} + \frac{5}{2}\left(3y^2+z^2\right)\,, \tag{1.94}$$

and $\chi = 1$ if the constraints in the x, y, z domain are satisfied ($\chi = 0$ otherwise). Further integration of Eq. (1.92) yields

$$N_{pk}(\nu,x)\,d\nu\,dx = \frac{e^{-\nu^2/2}}{(2\pi)^2R_*^3}\,f(x)\,\frac{\exp\left[-(x-x_*)^2/2(1-\gamma^2)\right]}{[2\pi(1-\gamma^2)]^{1/2}}\,d\nu\,dx\,, \tag{1.95}$$

with

$$\begin{aligned} f(x) &= \frac{3^25^{5/2}}{\sqrt{2\pi}}\left[\int_0^{x/4}dy\,e^{-(15/2)y^2}\int_{-y}^{y}dz\,F(x,y,z)\,e^{-(5/2)z^2} + \int_{x/4}^{x/2}dy\,e^{-(15/2)y^2}\int_{3y-x}^{y}dz\,F(x,y,z)\,e^{-(5/2)z^2}\right] \qquad (1.96)\\ &= (x^3-3x)\left\{\mathrm{erf}\left[\left(\frac{5}{2}\right)^{1/2}x\right]+\mathrm{erf}\left[\left(\frac{5}{2}\right)^{1/2}\frac{x}{2}\right]\right\}\Big/2 + \left(\frac{2}{5\pi}\right)^{1/2}\left[\left(\frac{31x^2}{4}+\frac{8}{5}\right)e^{-5x^2/8}+\left(\frac{x^2}{2}-\frac{8}{5}\right)e^{-5x^2/2}\right]. \end{aligned}$$

The asymptotic limits of this function is

$$f(x) \to \frac{3^55^{5/2}}{7\,2^{11}\sqrt{2\pi}}\,x^8\left(1-\frac{5x^2}{8}\right) \quad \text{for } x\to 0\,, \qquad f(x)\to x^3-3x \quad \text{for } x\to\infty\,. \tag{1.97}$$

Final integration to get the average number density $N_{pk}(\nu)\,d\nu$ needs to be done numerically over x as

$$N_{pk}(\nu)\,d\nu = \frac{1}{(2\pi)^2R_*^3}\,e^{-\nu^2/2}\int_0^\infty dx\,f(x)\,\frac{\exp\left[-(x-x_*)^2/2(1-\gamma^2)\right]}{[2\pi(1-\gamma^2)]^{1/2}}\,, \tag{1.98}$$

where $R_* \equiv \sqrt{3}\,(\sigma_1/\sigma_2)$ and $x_* = \gamma\nu$. The integral is denoted as $G(\gamma,\ x_*)$ and its fitting formula is given. The cumulative number density of peaks higher than height ν is

$$n_{pk}(\nu) = \int_\nu^\infty d\nu'\,N_{pk}(\nu')\,,\quad n_{pk}(-\infty) = \frac{29-6\sqrt{6}}{5^{3/2}2(2\pi)^2R_*^3} = 0.016R_*^{-3}\,, \tag{1.99}$$

and the high peak limit $\nu\to\infty$ is

$$N_{pk}\,d\nu \to \frac{[\langle k^2\rangle/3]^{3/2}}{(2\pi)^2}\,(\nu^3-3\nu)\,e^{-\nu^2/2}\,d\nu\,,\quad n(\nu)\to\frac{[\langle k^2\rangle/3]^{3/2}}{(2\pi)^2}\,(\nu^2-1)\,e^{-\nu^2/2}\,,\quad \langle k^2\rangle \equiv \frac{\sigma_1^2}{\sigma_0^2} = 3(\gamma/R_*)^2\,. \tag{1.100}$$

For a Gaussian filtering, the power spectrum is $P(k,R_f)\sim k^n\,e^{-(kR_f)^2}$, and other quantities are

$$\frac{\sigma_1^2(R_f)}{\sigma_0^2(R_f)} = \frac{n+3}{2R_f^2}\,,\quad \frac{\sigma_2^2(R_f)}{\sigma_0^2(R_f)} = \frac{(n+5)(n+3)}{4R_f^4}\,,\quad \gamma^2 = \frac{n+3}{n+5}\,,\quad R_* = \left(\frac{6}{n+5}\right)^{1/2}R_f\,,\quad N_{pk}d\nu \propto 1/R_*^3 \propto 1/R_f^3\,. \tag{1.101}$$

1.4.4 Conditional Probability for Ellipticity and Prolateness

The conditional probability for $x = -\nabla^2 F/\sigma_2$ is independent of e and p and is

$$P(x|\nu)\,dx = \frac{N_{pk}(\nu,x)\,dx}{N_{pk}(\nu)} = \frac{\exp[-(x-x_*)^2/2(1-\gamma^2)]}{[2\pi(1-\gamma^2)]^{1/2}}\frac{f(x)\,dx}{G(\gamma,x_*)}\,. \tag{1.102}$$

For high peaks (large x_*), it is more likely that the peaks are located at large x. The conditional probability for y, z given ν, x is simply

$$P(y,z|\nu,x)\,dy\,dz = \frac{N_{pk}(\nu,x,y,z)\,d\nu\,dx\,dy\,dz}{N_{pk}(\nu,x)\,d\nu\,dx} = \frac{3^2 5^{5/2}}{\sqrt{2\pi}}\,\frac{F(x,y,z)\,\chi}{f(x)}\,\exp\left[-\frac{5}{2}\,(3y^2+z^2)\right]\,, \tag{1.103}$$

and independent of ν. We characterize the asymmetry as ellipticity and prolateness

$$e = \frac{\lambda_1-\lambda_3}{2\sum_i\lambda_i} = \frac{y}{x}\,, \qquad\qquad p = \frac{\lambda_1-2\lambda_2+\lambda_3}{2\sum_i\lambda_i} = \frac{z}{x}\,. \tag{1.104}$$

Thus, e $(0\le e\le 1/2)$ is a measure of the ellipticity of the distribution in the 1-3 plane, and p determines the degrees of oblateness $(0\le p\le e)$ or prolateness $(0\ge p\ge -e)$ of the triaxial ellipsoid. Oblate spheroids (football) have $p=e$, and prolate spheroids (disk) have $p=-e$. The characteristic function $\chi=1$ if $0\le e\le 1/4$ and $-e\le p\le e$ or $1/4\le e\le 1/2$ and $-(1-3e)\le p\le e$ (zero, otherwise). (e,p) is confined within $(0,0)$, $(1/4,-1/4)$, $(1/2,1/2)$.

Therefore, the conditional probability for ellipticity and prolateness is

$$P_{ep}(e,p|\nu,x)\,de\,dp = P_{ep}(e,p|x)\,de\,dp = \frac{3^2 5^{5/2}}{\sqrt{2\pi}}\frac{x^8}{f(x)}\,e^{-(5/2)x^2(3e^2+p^2)}\,W(e,p)\,de\,dp\,, \tag{1.105}$$

where

$$W(e,p) = \frac{F(x,y=ex,z=px)\,\chi}{x^8} = e\,(e^2-p^2)(1-2p)[(1+p)^2-9\,e^2]\,\chi\,. \tag{1.106}$$

Notice that the most likely value of p quickly goes to zero. High ν peaks are neither oblate nor prolate, but they are definitely triaxially asymmetric, since $\lambda_2\simeq(\lambda_1+\lambda_3)/2$. Indeed, in the large x limit, e, p are small, and we can approximate the PDF by a Gaussian:

$$P_{ep}(e,p) \simeq P_{ep}(e_m,p_m)\,\exp\left[-\frac{(e-e_m)^2}{2\sigma_e^2}-\frac{(p-p_m)^2}{2\sigma_p^2}\right]\,, \tag{1.107}$$

where

$$e_m = \frac{1}{\sqrt{5}x[1+6/(5x^2)]^{1/2}}\,,\quad p_m = \frac{6}{5x^4[1+6/(5x^2)]^2}\,,\quad \sigma_e = \frac{e_m}{\sqrt{6}}\,,\quad \sigma_p = \frac{e_m}{\sqrt{3}}\,. \tag{1.108}$$

For high peaks, x is large and thus $e_m\simeq p_m\simeq 0$ (more spherically symmetric).