2 Newtonian Perturbation Theory and Galaxy Bias

In this chapter we study the Eulerian and the Lagrangian perturbation theories in Newtonian dynamics and their connection to modeling the galaxy (or halo) distribution.

2.1 Standard Perturbation Theory

In Newtonian dynamics, fully nonlinear equation presureless fluid can be written down:

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{v}=-\frac{1}{a}\nabla\cdot(\mathbf{v}\delta)\ ,\qquad \nabla\cdot\dot{\mathbf{v}}+H\nabla\cdot\mathbf{v}+\frac{3H^2}{2}a\Omega_m\delta=-\frac{1}{a}\nabla\cdot\left[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\right]\ ,\qquad \nabla^2\phi=4\pi G\rho\ . \tag{2.1}$$

The Euler equation can be split into one for divergence and one for vorticity. The vorticity vector $\nabla\times\mathbf{v}$ decays at the linear order. At nonlinear level, if no anisotropic pressure and no initial vorticity, the vorticity vanishes at all orders. However, in reality, the anisotropic pressure arises from shell crossing, generating vorticity on small scales, even in the absence of the initial vorticity. This modifies the SPT equation, such that there exist additional source terms for two kernels.

2.1.1 Basic Formalism

We consider multi-component fluids in the presence of isotropic pressure. In case of n-fluids with the mass densities ϱ_i, the pressures p_i, the velocities $\mathbf{v}_i$ ($i=1,2,\ldots n$), and the gravitational potential Φ, we have

$$\dot{\varrho}_i+\nabla\cdot(\varrho_i\mathbf{v}_i)=0\ ,\qquad \dot{\mathbf{v}}_i+\mathbf{v}_i\cdot\nabla\mathbf{v}_i=-\frac{1}{\varrho_i}\nabla p_i-\nabla\Phi\ ,\qquad \nabla^2\Phi=4\pi G\sum_{j=1}^{n}\varrho_j\ . \tag{2.2}$$

Assuming the presence of spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

$$\varrho_i=\bar{\varrho}_i+\delta\varrho_i,\qquad p_i=\bar{p}_i+\delta p_i,\qquad \mathbf{v}_i=H\mathbf{r}+\mathbf{u}_i,\qquad \Phi=\bar{\Phi}+\delta\Phi, \tag{2.3}$$

where $H\equiv\dot{a}/a$, and $a(t)$ is a cosmic scale factor. We move to the comoving coordinate $\mathbf{x}$ where

$$\mathbf{r}\equiv a(t)\mathbf{x},\qquad \nabla=\nabla_{\mathbf{r}}=\frac{1}{a}\nabla_{\mathbf{x}},\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}\Big|_{\mathbf{r}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}+\left(\frac{\partial}{\partial t}\Big|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}}=\frac{\partial}{\partial t}\Big|_{\mathbf{x}}-H\mathbf{x}\cdot\nabla_{\mathbf{x}}. \tag{2.4}$$

In the following we neglect the subindex $\mathbf{x}$. To the background order we have

$$\dot{\varrho}_i+3H\varrho_i=0,\qquad \frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\sum_j\varrho_j,\qquad H^2=\frac{8\pi G}{3}\sum_j\varrho_j+\frac{2E}{a^2}, \tag{2.5}$$

where E is an integration constant which can be interpreted as the specific total energy in Newton's gravity; in Einstein's gravity we have $2E=-Kc^2$ where K can be normalized to be the sign of spatial curvature. To the perturbed order we have

$$\dot{\delta}_i+\frac{1}{a}\nabla\cdot\mathbf{u}_i=-\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i)\ ,\qquad \dot{\mathbf{u}}_i+H\mathbf{u}_i+\frac{1}{a}\mathbf{u}_i\cdot\nabla\mathbf{u}_i=-\frac{1}{a\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i}-\frac{1}{a}\nabla\delta\Phi\ ,\qquad \frac{1}{a^2}\nabla^2\delta\Phi=4\pi G\sum_j\bar{\varrho}_j\delta_j\ . \tag{2.6}$$

By introducing the expansion θ_i and the rotation $\vec{\omega}_i$ of each component as

$$\theta_i\equiv-\frac{1}{a}\nabla\cdot\mathbf{u}_i,\qquad \vec{\omega}_i\equiv\frac{1}{a}\nabla\times\mathbf{u}_i, \tag{2.7}$$

we derive

$$\dot{\theta}_i+2H\theta_i-4\pi G\sum_j\bar{\varrho}_j\delta_j=\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i)+\frac{1}{a^2\bar{\varrho}_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right), \tag{2.8}$$

$$\dot{\vec{\omega}}_i+2H\vec{\omega}_i=-\frac{1}{a^2}\nabla\times(\mathbf{u}_i\cdot\nabla\mathbf{u}_i)+\frac{1}{a^2\bar{\varrho}_i}\frac{(\nabla\delta_i)\times\nabla\delta p_i}{(1+\delta_i)^2}. \tag{2.9}$$

By introducing decomposition of perturbed velocity into the potential- and transverse parts as

$$\mathbf{u}_i \equiv -\nabla U_i + \mathbf{u}_i^{(v)}, \qquad \nabla\cdot\mathbf{u}_i^{(v)} \equiv 0; \qquad \theta_i = \frac{\Delta}{a}U_i, \qquad \vec{\omega}_i = \frac{1}{a}\nabla\times\mathbf{u}_i^{(v)}, \tag{2.10}$$

instead of Eq. (2.9) we have

$$\dot{\mathbf{u}}_i^{(v)} + H\mathbf{u}_i^{(v)} = -\frac{1}{a}\left[\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar\varrho_i}\frac{\nabla\delta p_i}{1+\delta_i} - \nabla\Delta^{-1}\nabla\cdot\left(\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar\varrho_i}\frac{\nabla\delta p_i}{1+\delta_i}\right)\right]. \tag{2.11}$$

Combining equations above, we can derive

$$\ddot\delta_i + 2H\dot\delta_i - 4\pi G\sum_j \bar\varrho_j\delta_j = -\frac{1}{a^2}\left[a\nabla\cdot(\delta_i\mathbf{u}_i)\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar\varrho_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right). \tag{2.12}$$

These equations are valid to fully nonlinear order. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.

2.1.2 Recurrence Relation and Third Order Solution

For a single presureless medium, the governing equation simplifies as

$$\dot\delta - \theta = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}) \ , \qquad \dot\theta + 2H\theta - 4\pi G\bar\rho_m\delta = \frac{1}{a^2}\nabla\cdot\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] \ . \tag{2.13}$$

From the linear order calculations, we derived

$$\theta = Hf\delta \ , \qquad \mathbf{u}_k = ia\frac{\mathbf{k}}{k^2}\theta_k \ . \tag{2.14}$$

By assuming the separability of the time and the spatial dependences, the standard perturbation theory (SPT) takes a perturbative approach to the nonlinear solution:

$$\delta_N(t,\mathbf{k}) \equiv \sum_{n=1}^{\infty} D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat\delta(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})F_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n) \equiv \sum_{n=1}^{\infty}D^n(t)\delta^{(n)}(\mathbf{k}) \ , \tag{2.15}$$

$$\frac{\theta_N(t,\mathbf{k})}{Hf_1} \equiv \sum_{n=1}^{\infty} D^n(t)\left[\prod_i^n\int\frac{d^3\mathbf{q}_i}{(2\pi)^3}\,\hat\delta(\mathbf{q}_i)\right](2\pi)^3\delta^D(\mathbf{k}-\mathbf{q}_{12\cdots n})G_n^{(s)}(\mathbf{q}_1,\cdots,\mathbf{q}_n) \equiv \sum_{n=1}^{\infty}D^n\theta^{(n)}(\mathbf{k}) \ , \tag{2.16}$$

where $\mathbf{q}_{12\cdots n}\equiv\mathbf{q}_1+\cdots+\mathbf{q}_n$, $\delta^{(n)}(\mathbf{k})$ and $\theta^{(n)}(\mathbf{k})$ are time-independent n-th order perturbations, $F_n^{(s)}$ and $G_n^{(s)}$ are the SPT kernels symmetrized over its arguments. The (dimensionless) Newtonian linear-order growth factor $D(t)$ is normalized to unity at some early epoch t_0 when the nonlinearities are ignored, and the initial linear density perturbation is set up in terms of which the perturbative expansion is given:

$$\delta_N(t_0,\mathbf{k}) \equiv \delta_1^{(1)}(t_0,\mathbf{k}) \equiv \hat\delta(\mathbf{k}) \ , \qquad D(t) \equiv \frac{D_1(t)}{D_1(t_0)} \ , \qquad \ddot D + 2H\dot D - 4\pi G\bar\rho_m D = 0 \ . \tag{2.17}$$

With these decompositions in the Fourier space, the LHS of the Newtonian dynamical equations become

$$\dot\delta_N - \theta_N = Hf_1\sum_{n=1}^{\infty}D^n\left(n\delta^{(n)} - \theta^{(n)}\right) \ , \qquad \dot\theta_N + 2H\theta_N - 4\pi G\bar\rho_m\delta_N = H^2f_1^2\sum\frac{D^n}{2}\left[(1+2n)\theta^{(n)} - 3\delta^{(n)}\right] \ , \tag{2.18}$$

where we adopted the usual assumption $\Omega_m = f_1 = 1$ in SPT and utilized the relation between the growth factor and the growth rate $\dot D = HDf_1$. The RHS of the Newtonian dynamical equations are the convolution in the Fourier space:

$$\left[-\frac{1}{a}\nabla\cdot(\delta_N\mathbf{v}_N)\right](\mathbf{k}) = \int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\alpha_{12}\theta_N(\boldsymbol{Q}_1,t)\delta_N(\boldsymbol{Q}_2,t) \equiv Hf_1\sum_{n=1}^{\infty}D^nA_n(\mathbf{k}) \ , \tag{2.19}$$

$$\left\{\frac{1}{a^2}\nabla\cdot\left[(\mathbf{v}_N\cdot\nabla)\mathbf{v}_N\right]\right\}(\mathbf{k}) = \int\frac{d^3\boldsymbol{Q}_1}{(2\pi)^3}\int\frac{d^3\boldsymbol{Q}_2}{(2\pi)^3}(2\pi)^3\delta^D(\mathbf{k}-\boldsymbol{Q}_{12})\beta_{12}\theta_N(\boldsymbol{Q}_1,t)\theta_N(\boldsymbol{Q}_2,t) \equiv H^2f_1^2\sum_{n=1}^{\infty}D^nB_n(\mathbf{k}) \ , \tag{2.20}$$

where the vertex functions are defined as

$$\alpha_{12} \equiv \alpha(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \equiv 1 + \frac{\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2}{Q_1^2} , \qquad \beta_{12} \equiv \beta(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \equiv \frac{|\boldsymbol{Q}_1 + \boldsymbol{Q}_2|^2 \boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2}{2Q_1^2 Q_2^2} , \tag{2.21}$$

and the n-th order perturbation kernels $A_n(\mathbf{k})$ and $B_n(\mathbf{k})$ are

$$A_n(\mathbf{k}) = \left[\prod_i^n \int \frac{d^3\mathbf{q}_i}{(2\pi)^3} \, \hat\delta(\mathbf{q}_i)\right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\cdots n}) \sum_{i=1}^{n-1} \alpha_{12} G_i(\mathbf{q}_1, \cdots, \mathbf{q}_i) F_{n-i}(\mathbf{q}_{i+1}, \cdots, \mathbf{q}_n) , \tag{2.22}$$

$$B_n(\mathbf{k}) = \left[\prod_i^n \int \frac{d^3\mathbf{q}_i}{(2\pi)^3} \, \hat\delta(\mathbf{q}_i)\right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\cdots n}) \sum_{i=1}^{n-1} \beta_{12} G_i(\mathbf{q}_1, \cdots, \mathbf{q}_i) G_{n-i}(\mathbf{q}_{i+1}, \cdots, \mathbf{q}_n) , \tag{2.23}$$

with $\boldsymbol{Q}_1 = \mathbf{q}_{1\cdots i}$ and $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \mathbf{k}$.

Therefore, the two Newtonian dynamical equations become algebraic equations with the time-dependence removed:

$$n\delta^{(n)} - \theta^{(n)} = A_n , \qquad (1+2n)\theta^{(n)} - 3\delta^{(n)} = 2B_n , \tag{2.24}$$

and the well-known recurrence formulas for the solutions are

$$\delta^{(n)} = \frac{(1+2n)A_n + 2B_n}{(2n+3)(n-1)} , \qquad \theta^{(n)} = \frac{3A_n + 2nB_n}{(2n+3)(n-1)} , \tag{2.25}$$

and similarly so for the SPT kernels

$$F_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n+3)(n-1)} \left[(1+2n)\alpha_{12} F_{n-i} + 2\beta_{12} \, G_{n-i}\right] , \qquad G_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n+3)(n-1)} \left[3\alpha_{12} F_{n-i} + 2n\beta_{12} G_{n-i}\right] . \tag{2.26}$$

Up to the third order in perturbations, with $F_1 = G_1 = 1$ these SPT kernels are explicitly

$$F_2 = \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2} + \frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) , \qquad G_2 = \frac{3}{7} + \frac{4}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2} + \frac{\mathbf{q}_1\cdot\mathbf{q}_2}{2q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) , \tag{2.27}$$

$$F_3 = \frac{2k^2}{54}\left[\frac{\mathbf{q}_1\cdot\mathbf{q}_{23}}{q_1^2 q_{23}^2} G_2(\mathbf{q}_2,\mathbf{q}_3) + \text{cycl.}\right] + \frac{7}{54}\mathbf{k}\cdot\left[\frac{\mathbf{q}_{12}}{q_{12}^2} G_2(\mathbf{q}_1,\mathbf{q}_2) + \text{cycl.}\right] + \frac{7}{54}\mathbf{k}\cdot\left[\frac{\mathbf{q}_1}{q_1^2} F_2(\mathbf{q}_2,\mathbf{q}_3) + \text{cycl.}\right] , \tag{2.28}$$

$$G_3 = \frac{k^2}{9}\left[\frac{\mathbf{q}_1\cdot\mathbf{q}_{23}}{q_1^2 q_{23}^2} G_2(\mathbf{q}_2,\mathbf{q}_3) + \text{cycl.}\right] + \frac{1}{18}\mathbf{k}\cdot\left[\frac{\mathbf{q}_{12}}{q_{12}^2} G_2(\mathbf{q}_1,\mathbf{q}_2) + \text{cycl.}\right] + \frac{1}{18}\mathbf{k}\cdot\left[\frac{\mathbf{q}_1}{q_1^2} F_2(\mathbf{q}_2,\mathbf{q}_3) + \text{cycl.}\right] . \tag{2.29}$$

Using the recurrence relations, the SPT kernels $F_n \sim G_n \propto k^2$ for $n > 1$ in the limit $k \to 0$, with the individual momentum $\mathbf{q}_i$ held finite. This originates from the momentum conservation of the nonlinear evolution.

• Compute the one-loop power spectrum

2.1.3 Asymptotic Behavior

The leading-order terms for the one-loop power spectrum are

$$P_{22} = 2\int \frac{d^3q}{(2\pi)^3} \, P_L(q) P_L(|\mathbf{k}-\mathbf{q}|) \left[F_2(\mathbf{q}, \mathbf{k}-\mathbf{q})\right]^2 \qquad \lim_{q\to 0} F_2(q, k-q) = \lim_{q\to k} F_2(q, k-q) , \tag{2.30}$$

$$= \int d\ln q \, \frac{q^3 P_L(q)}{2\pi^2} \int_{-1}^{1} d\mu \, P_L(|\mathbf{k}-\mathbf{q}|) \left[\frac{3r + 7\mu - 10r\mu^2}{14\, r(1 + r^2 - 2r\mu)}\right]^2 ,$$

$$2P_{13} = 6P_L(\mathbf{k}) \int \frac{d^3\mathbf{q}}{(2\pi)^3} \, P_L(\mathbf{q}) F_3(\mathbf{q}, -\mathbf{q}, \mathbf{k}) \tag{2.31}$$

$$= 6P_L(\mathbf{k}) \int d\ln q \, \frac{q^3 P_L(\mathbf{q})}{2\pi^2} \frac{1}{24}\left[\frac{6k^6 - 79k^4q^2 + 50k^2q^4 - 21q^6}{63k^2q^4} + \frac{(q^2-k^2)^3(7q^2+2k^2)}{42k^3q^5} \ln\left|\frac{k+q}{k-q}\right|\right] ,$$

where the solid angle integration of F_3 is performed over $\hat{\mathbf{q}}$. The asymptotic behaviors are

$$F_2 \to \begin{cases} \frac{k\mu}{2q} \to \infty & \text{if} \quad k \to \infty \ (q \to 0) \\ \frac{(3-5\mu^2)k^2}{7q^2} \to 0 & \text{if} \quad k \to 0 \ (q \to \infty) \end{cases} , \qquad \hat{F}_3 \to \begin{cases} -\frac{61k^2}{1890q^2} \to \infty & \text{if} \quad k \to \infty \ (q \to 0) \\ -\frac{k^2}{18q^2} \to 0 & \text{if} \quad k \to 0 \ (q \to \infty) \end{cases} , \tag{2.32}$$

where the angle-averaged $\hat{F}_3$ is the square bracket above with $1/24$. The one-loop terms asymptotically approach, and the leading correction after cancellation of two terms is

$$\lim_{k\to\infty} P_{22} = \lim_{k\to\infty} -2P_{13} = \frac{k^2 P(k)}{6\pi^2}\int_0^\infty dq\ P(q)\ , \tag{2.33}$$

$$\lim_{k\to\infty} (P_{22}+2P_{13}) \propto 0 + P(k)\int_0^\infty dq\ q^2 P(q) \propto P(k)\times\infty\ , \tag{2.34}$$

where the learding term $k^2P(k)$ is cancelled. For a scale-free power spectrum $P_L \propto k^n$, P_{22} diverges with $n \geq 1/2$ at UV and $n \leq -1$ at IR, while P_{13} diverges with $n \geq -1$ at UV and $n \leq -1$ at IR. The sum diverges with $n > -3$ at UV and $n \leq -3$ at IR.

Peebles Argument

Given the matter number density (or galaxies), the nonlinear correction can only give rise to k^2 in the large scale limit. The galaxy number density can be written as

$$n_g(\mathbf{x}) = \sum_i^N \delta^D(\mathbf{x}-\mathbf{x}_i)\ , \qquad n_g(\mathbf{k}) = \sum_i^N e^{-i\mathbf{k}\cdot\mathbf{x}_i}\ . \tag{2.35}$$

Compared to the position at the initial time, the galaxy number density can evolve due to nonlinear gravity as

$$\begin{aligned}\Delta n_g(\mathbf{k}) &= \int d^3\mathbf{x}\,[n_g(\mathbf{x}) - n_g(\mathbf{x};t_i)]\,e^{-i\mathbf{k}\cdot\mathbf{x}} = \sum_i^N e^{-i\mathbf{k}\cdot\mathbf{x}_i} - e^{-i\mathbf{k}\cdot\mathbf{x}_i^{\rm ini}} = \sum_i^N e^{-i\mathbf{k}\cdot\mathbf{x}_i^{\rm ini}}\left[e^{-i\mathbf{k}\cdot\Delta\mathbf{x}_i} - 1\right] \\ &= \sum_i^N e^{-i\mathbf{k}\cdot\mathbf{x}_i^{\rm ini}}\left[\sum_j^N (1-1) - i\mathbf{k}\cdot\Delta\mathbf{x}_j - (\mathbf{k}\cdot\Delta\mathbf{x}_j)^2 + \cdots\right] \propto k^2 n_g^{\rm ini}(\mathbf{k})\ .\end{aligned} \tag{2.36}$$

The first term vanishes due to mass conservation, and the second term vanishes due to momentum conservation. Therefore, the correction should fall as k^2, compared to the initial.

2.1.4 Unified Treatment of the Standard Perturbation Theory

The master equation for SPT can be rephrased as

$$\begin{aligned}\left[\delta_{ab}\frac{\partial}{\partial\eta} + \Omega_{ab}(\eta)\right]\Phi_b(\mathbf{k};t) &= \Lambda_{ab}\Phi_b(\mathbf{k};t) \qquad \eta := \ln D(t)\ , \\ &= \int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_2}{(2\pi)^3}\ \delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\gamma_{abc}(\mathbf{k}_1,\mathbf{k}_2)\,\Phi_b(\mathbf{k}_1;t)\,\Phi_c(\mathbf{k}_2;t)\ ,\end{aligned} \tag{2.37}$$

where $\Phi_a = (\delta_m, -\theta/f) \rightarrow (\delta_m, \delta_m)$ in the linear regime, $\theta = \nabla\cdot\mathbf{v}/\mathcal{H}$, the time-dependent matrix and the vertex funtion are

$$\Omega_{ab}(\eta) = \begin{pmatrix} 0 & -1 \\ -\dfrac{3}{2f^2}\Omega_{\rm m}(\eta) & \dfrac{3}{2f^2}\Omega_{\rm m}(\eta) - 1 \end{pmatrix}\ , \qquad \gamma_{abc}(\mathbf{k}_1,\mathbf{k}_2) = \begin{cases} \frac{1}{2}\left\{1+\frac{\mathbf{k}_2\cdot\mathbf{k}_1}{|\mathbf{k}_2|^2}\right\} & ;\quad (a,b,c) = (1,1,2) \\ \frac{1}{2}\left\{1+\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{|\mathbf{k}_1|^2}\right\} & ;\quad (a,b,c) = (1,2,1) \\ \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)|\mathbf{k}_1+\mathbf{k}_2|^2}{2|\mathbf{k}_1|^2|\mathbf{k}_2|^2} & ;\quad (a,b,c) = (2,2,2) \\ 0 & ;\quad \text{otherwise} \end{cases}\ . \tag{2.38}$$

The formal solution can be obtained as

$$\Phi_a(\mathbf{k};\eta) = g_{ab}(\eta,\eta_0)\,u_b\,\delta_0(\mathbf{k}) + \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta,\eta')\int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_2}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\gamma_{bcd}(\mathbf{k}_1,\mathbf{k}_2)\Phi_c(\mathbf{k}_1;\eta')\Phi_d(\mathbf{k}_2;\eta')\ . \tag{2.39}$$

From the master equation (2.38), the power spectrum is the integral of bispectrum, and so on. This hierarchy arises from the nonlinearity, and it must be truncated to solve it self-consistently.

With the definition of the nonlinear propogator, the linear propagator satisfies

$$\delta^D(\mathbf{k}-\mathbf{k}')G_{ab}(|\mathbf{k}|,\eta,\eta') := \left\langle\frac{\delta\Phi_a(\mathbf{k};\eta)}{\delta\Phi_b(\mathbf{k}';\eta')}\right\rangle\ , \qquad 0 = \left[\delta_{ab}\frac{\partial}{\partial\eta} + \Omega_{ab}(\eta)\right]g_{bc}(\eta,\eta') = \Lambda_{ab}g_{bc}\ , \qquad g_{ab}(\eta,\eta) = \delta_{ab}\ , \tag{2.40}$$

and the power spectrum can be expressed as

$$P_{ab}(k;\eta)=G_{ac}(k|\eta,\eta_0)G_{bd}(k|\eta,\eta_0)u_cu_dP_{\rm lin}(k;\eta_0)+P_{ab}^{(\rm MC)}(k;\eta,\eta_0)~,\qquad G_{ab}(k|\eta,\eta')=g_{ab}(\eta,\eta')+G_{ab}^{(\rm MC)}(k;\eta,\eta')~,\tag{2.41}$$

where $u_a=(1,1)$ represents the growing mode solution and the mode-coupling terms are defined by the above equation. See Eq. (2.182).

2.2 Galaxy Bias Primer

Galaxies form in an over-dense region of the matter density fluctuation. The matter density becomes more linear and smooth at high redshift, so it is easier to deal with the matter density evolution at higher redshifts. However, galaxies form in a peak, which becomes increasingly rare at higher redshifts. Consequently, galaxies of the same mass or luminosity are more biased at higher redshifts and more difficult to model.

At the linear order in perturbations, the matter density fluctuations evolve as $\dot\delta_m=\theta$. Assuming the linear bias relation $\delta_g=b\,\delta_m$ and no velocity bias (i.e., the motion of galaxies is the same as the matter distribution, in response to the gravity), we derive the simple evolution equation for galaxy bias:

$$\dot\delta_m=\theta\equiv\dot\delta_g=\dot b\delta_m+b\dot\delta_m~,\qquad d\ln(b-1)+d\ln D=0~,\qquad b(z)-1=\frac{b_0-1}{D}~,\tag{2.42}$$

where the growth factor is normalized today. Biased objects ($b_0>1$) are even more biased ($b\gg1$) at higher redshifts, while unbiased objects ($b_0=1$) remain always unbiased ($b=1$). However, smaller objects ($b_0<1$) are less and less biased at higher redshifts, yielding $b\leq0$ beyond the redshift z, where $b_0=1-D$. Certainly, this model is a simple approximation.

At higher redshift, the halo power spectrum goes as $b^2(z)D^2(z)$, compared to the power spectrum at $z=0$. Even for $b_0>1$, the product

$$b(z)D(z)=D(z)+b_0-1~,\tag{2.43}$$

decreases with redshift. While true that biased objects with $b_0>1$ are even more highly biased, the increase in the bias factor is not enough to make up for the decrease in the growth factor D.

2.2.1 Renormalizing the Bias Parameter

Beyond the linear-order in perturbations, the galaxy bias model (and the perturbation theory to some degree) has several issues, which demands a certain type of renormalization (McDonald, 2006; McDonald and Roy, 2009). Consider a simple galaxy bias model:

$$n_g=n_0+n_0'\,\delta+\frac{1}{2}n_0''\,\delta^2+\frac{1}{6}n_0'''\,\delta^3+\epsilon+\mathcal{O}(4)~,\tag{2.44}$$

where ϵ is a shot noise and n_0 is some function. We will work only up to the third order in perturbations. The mean number density is then

$$\langle n_g\rangle=n_0+\frac{1}{2}n_0''\sigma^2+\mathcal{O}(4)~,\qquad \sigma^2:=\left\langle\delta^2\right\rangle~,\tag{2.45}$$

and in the absence of a UV cutoff, the variance σ^2 is infinite, or at least potentially large, which signals the breakdown of the perturbation theory. This is not necessarily problem, as we work with the fluctuations around the mean:

$$\delta_g:=\frac{n_g-\langle n_g\rangle}{\langle n_g\rangle}=:c_1\delta+\frac{1}{2}c_2\left(\delta^2-\sigma^2\right)+\frac{1}{6}c_3\delta^3+\epsilon+\mathcal{O}(4)~,\tag{2.46}$$

where the bias parameters c_i are the usual and ϵ has been re-scaled. In Fourier space, the galaxy fluctuation field is

$$\delta_g(\mathbf{k})=c_1\delta_\mathbf{k}+\frac{1}{2}c_2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\delta_\mathbf{q}\delta_{\mathbf{k}-\mathbf{q}}+\frac{1}{6}c_3\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3}\,\delta_{\mathbf{q}_1}\delta_{\mathbf{q}_2}\delta_{\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2}+\epsilon_\mathbf{k}+\mathcal{O}(4)~,\tag{2.47}$$

while the matter density fluctuation is

$$\delta_\mathbf{k}=\delta_1(\mathbf{k})+\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\delta_1(\mathbf{q})\delta_1(\mathbf{k}-\mathbf{q})F_2(\mathbf{q},\mathbf{k}-\mathbf{q})+\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3}\,\delta_1(\mathbf{q}_1)\delta_1(\mathbf{q}_2)\delta_1(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)F_3(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)+\mathcal{O}(4)~,\tag{2.48}$$

where the subscripts indicate the perturbation order.

Combining these two equations, we derive the galaxy power spectrum as

$$P_g(k) = N_0 + \left[c_1^2 + c_1c_3\sigma^2 + \frac{68}{21}c_1c_2\sigma^2\right]P(k)$$
$$+\frac{1}{2}c_2^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,P(q)P(|\mathbf{k}-\mathbf{q}|) + 2\,c_1c_2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,P(q)P(|\mathbf{k}-\mathbf{q}|)F_2(\mathbf{q},\mathbf{k}-\mathbf{q}) + \mathcal{O}(6)\,, \tag{2.49}$$

where $P(k)$ is the linear-order matter power spectrum. With quite ill-defined σ^2 present, this equation suggests the redefinition of the linear bias parameter:

$$b_1^2 := c_1^2 + c_1c_3\sigma^2 + \frac{68}{21}c_1c_2\sigma^2\,. \tag{2.50}$$

The point is that b_1 should now be treated as the free parameter of the model, with c_1 and c_3 eliminated from the equation by substitution. In addition, there exists another potentially divergent UV-sensitive term:

$$\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,P(q)P(|\mathbf{k}-\mathbf{q}|)\,, \tag{2.51}$$

which diverges at high q if the asymptotic logarithmic slope of the power spectrum is $n_{\rm eff}(q\to\infty)\geq -1.5$ (this is not the case for ΛCDM). Since the initial power spectrum at high k is quickly completely erased and the theory is unlikely to be valid at such high k, the perturbation theory results should be devoid of high-k sensitivity. Another issue with this term is that it is non-zero in the limit $k\to 0$, and it appears as a correction to the scale-independent shot-noise. Therefore, we now perform our final renormalization (up to this order), absorbing this potentially divergent term, evaluated at $k=0$, into the shot-noise term, which becomes

$$N := N_0 + \frac{1}{2}c_2^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}P^2(q)\,. \tag{2.52}$$

After this piece is absorbed into the shot-noise, the remaining integral

$$\int\frac{d^3\mathbf{q}}{(2\pi)^3}P(q)\Big[P(|\mathbf{k}-\mathbf{q}|)-P(q)\Big] \simeq \int\frac{d^3\mathbf{q}}{(2\pi)^3}P(q)\left(-\mu k\frac{dP(q)}{dq}\right) + \mathcal{O}(k^2)\,, \qquad \mu := \frac{\mathbf{k}\cdot\mathbf{q}}{kq}\,, \tag{2.53}$$

is clearly convergent for any reasonable power spectrum.

The second bias parameter c_2 is not renormalized at this order, but for notational compactness we define

$$b_2 = c_2\,, \tag{2.54}$$

and the final result for the power spectrum is:

$$P_g(k) = N + b_1^2P(k) + \frac{1}{2}b_2^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,P(q)\Big[P(|\mathbf{k}-\mathbf{q}|)-P(q)\Big] + 2\,b_1b_2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,P(q)\,P(|\mathbf{k}-\mathbf{q}|)\,F_2(\mathbf{q},\mathbf{k}-\mathbf{q})\,. \tag{2.55}$$

Note since the last term with c_1c_2 is already $\mathcal{O}(4)$, c_1 in that term can be freely replaced by b_1, completely removing c_3 at this order as an independent parameter at this order in perturbation. The only correction to the linear-order galaxy power spectrum (which are free b_1 and N parameters) is now b_2.

The calculations become more involved, as we include more observable statistics such as the galaxy bispectrum, the galaxy-matter cross power spectrum, and so on.

2.2.2 Summary of Spherical Collapse Model

Full extension of Kaiser (1984) to N-point without approximation (thresholded sample), they have

$$1+\xi_\nu^{(N)} = \sum_{m=0}^{\infty}\frac{w_{12}^{m_{12}}}{m_{12}!}\frac{w_{13}^{m_{13}}}{m_{13}!}\cdots A_{m_1}A_{m_2}\cdots A_{m_N}\,, \qquad A_0 = 1\,, \qquad A_n = \frac{2xH_{n-1}(x)2^{-n/2}}{\sqrt{\pi}xe^{x^2}{\rm erfc}(x)}\,, \tag{2.56}$$

where $x=\nu/\sqrt{2}$, $w(r)=\xi(r)/\sigma^2$, and

$$m = \sum_{k,l} m_{kl}\,, \qquad m_{kl} = 0 \quad \text{if } k\geq l\,. \tag{2.57}$$

In the limit $\lim_{\nu\to\infty} A_n = \nu^n$ (thresholded sample becomes a peak sample), we have

$$1+\xi_\nu^{(N)} = \sum_{m=0}^{\infty} \frac{\nu^2 w_{12}^{m_{12}}}{m_{12}!}\frac{\nu^2 w_{13}^{m_{13}}}{m_{13}!}\cdots = \exp\left[\nu^2(w_{12}+w_{13}+\cdots)\right] , \tag{2.58}$$

corresponding to the Politzer and Wise (1984) equation. For the two-point correlations, we have

$$\xi_\nu = \sum_{m=1}^{\infty} \frac{w^m}{m!} A_m^2 \quad \to \quad \nu^2 w . \tag{2.59}$$

The thresholded correlation vanishes, whenever the matter correlation vanishes.

2.2.3 Renormalized Galaxy Bias

Assuming the galaxy number density is a function of smoothed density field, one can perform a naive expansion:

$$n_g(\mathbf{x}) = F_g[\delta(\mathbf{x});\mathbf{x}] , \qquad \delta_g(\mathbf{x}) = c_0 + c_1\delta(\mathbf{x}) + \frac{c_2}{2}\delta^2(\mathbf{x}) + \ldots , \tag{2.60}$$

where δ is a smoothed matter density with some filter R, c_n are the bias parameters. Additional dependence on $\mathbf{x}$ indicates that there exists some stochasticity on small scales (this scatter is equivalent to the dependence of n_g on the small-scale fluctuations δ_s in the given region), but for Gaussian case this stochasticity does not matter on large scales. When two-point correlation function is computed, the correlation function depends on the zero-lag correlators (e.g., σ^2 and hence the dependence on the smoothing scale R), signaling the break-down of the local expansion.

This is to be renormalized, absorbing the zero-lag terms into the bare bias parameters. It is observed that when δ_g and δ_m are plotted, the bare bias parameters determined from the scatter plot change as a function of smoothing radius R. But on large scales, the correlation function should be independent of R. Physically, the renormalized bias parameters quantify the response of the mean abundance of tracers to a change in the background matter density $\bar\rho$ of the Universe

$$b_N = \frac{\bar\rho^N}{\bar n_g}\frac{\partial^N \bar n_g}{\partial\bar\rho^N} , \qquad \xi_g(r) = \sum_{N=1}^{\infty}\frac{b_N^2}{N!}\left[\xi(r)\right]^N , \tag{2.61}$$

where the latter holds for a Gaussian density. Note that renormalization removes the zero-lag matter correlators from the expression for the tracer correlation function at all orders, and that the same bias parameters b_N describe both the tracer auto- and the cross-correlation with matter.

Basics

The PBS argument can be summarized as follows: if the description of the clustering solely through their dependence on δ is sufficient, the expected abundance of tracers in a region with smoothed overdensity $\delta = D$ is sufficiently well approximated by the mean abundance $\langle n_g\rangle$ in a fictitious Universe with modified background density $\bar\rho' = \bar\rho(1+D)$. The advantage of this approach is that we only need $\langle n_g\rangle$, not F_g:

$$\langle n_g\rangle|_D = \langle F_g[0]\rangle \sum_{n=0}^{\infty}\frac{c_n}{n!}\langle(\delta+D)^n\rangle , \qquad b_N := \frac{1}{\langle n_g\rangle}\frac{\partial^N\langle n_g[D]\rangle}{\partial D^N}\Big|_{D=0} = \frac{\bar\rho^N}{\langle n_g\rangle}\frac{\partial^N\langle n_g\rangle}{\partial\bar\rho^N} . \tag{2.62}$$

Starting with the assumption of n_g above and assuming that the small-scale fluctuations are not correlated, we have

$$n_g(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}F_g^{(n)}[\delta=0;\mathbf{x}]\,[\delta(\mathbf{x})]^n , \qquad \langle F_g^{(n)}[0;\mathbf{x}]\,[\delta(\mathbf{x})]^n\rangle = \langle F_g^{(n)}[0;\mathbf{x}]\rangle\,\langle[\delta(\mathbf{x})]^n\rangle , \tag{2.63}$$

and then the mean is

$$\langle n_g(\mathbf{x})\rangle = \sum_n \frac{1}{n!}\left\langle F_g^{(n)}[0;\mathbf{x}]\right\rangle\langle\delta^n\rangle = \langle F_g[0]\rangle\left(1+\frac{c_2}{2}\sigma_L^2+\frac{c_3}{6}\langle\delta_L^3\rangle+\ldots\right) , \qquad c_n := \frac{1}{\langle F_g[0]\rangle}\left\langle F_g^{(n)}[0]\right\rangle . \tag{2.64}$$

The renormalized bias parameter can be derived by using the PBS argument in Eq. (2.62)

$$b_N = \frac{1}{\mathcal{N}}\sum_{n=N}^{\infty}\frac{c_n}{n!}\frac{n!}{(n-N)!}\langle\delta^{n-N}\rangle , \qquad \mathcal{N} = \sum_{n=0}^{\infty}\frac{c_n}{n!}\langle\delta_L^n\rangle . \tag{2.65}$$

Using the binomial expansion

$$\langle\delta^n(1)\delta(2)\rangle\sum_{N=0}^{n}\binom{n}{N}\langle\delta^{n-N}\rangle\langle\delta^N(1)\delta(2)\rangle_c\,,\qquad\qquad \langle\delta_1^n\delta_2^m\rangle=\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\langle\delta_1^k\rangle\binom{m}{l}\langle\delta_2^l\rangle\langle\delta_1^{n-k}\delta_2^{m-l}\rangle_{\rm nzl}\,, \tag{2.66}$$

the two-point correlation function is derived as

$$\xi_g(r) = \frac{\sum_{n,m=0}^{\infty}\frac{c_nc_m}{n!m!}\langle\delta^n(1)\delta^m(2)\rangle}{\sum_{n,m=0}^{\infty}\frac{c_nc_m}{n!m!}\langle\delta^n\rangle\langle\delta^m\rangle}-1=\cdots=\sum_{N,M=1}^{\infty}\frac{b_N}{N!}\frac{b_M}{M!}\langle\delta^N(1)\delta^M(2)\rangle_{\rm nzl}\,,\qquad \mathcal{N}:=\sum_{n=0}^{\infty}\frac{c_n}{n!}\langle\delta^n\rangle\,, \tag{2.67}$$

$$\xi_{gm}(r) = \frac{\sum_{n=1}^{\infty}\frac{c_n}{n!}\langle\delta^n(1)\delta(2)\rangle}{\sum_{n=0}^{\infty}\frac{c_n}{n!}\langle\delta^n\rangle}=\frac{1}{\mathcal{N}}\sum_{n=1}^{\infty}\frac{c_n}{n!}\sum_{N=0}^{n}\binom{n}{N}\langle\delta^{n-N}\rangle\langle\delta^N(1)\delta(2)\rangle_c=\sum_{N=1}^{\infty}\frac{b_N}{N!}\langle\delta_L^N(1)\delta_L(2)\rangle_c\,,$$

where nzl: no zero-lag correlator.

• Example 1 — Universal mass function:

$$b_N=\frac{(-1)^N}{\langle n_g\rangle}\frac{\partial^N\langle n_g\rangle}{\partial\delta_c^N}=\frac{(-1)^N}{\sigma^N}\frac{1}{f(\nu_c)}\frac{d^Nf(\nu_c)}{d\nu_c^N}\,. \tag{2.68}$$

• Example 2 — Thresholded sample:

$$\langle n_g\rangle = P_1(\nu_c)=\frac{1}{\sqrt{2\pi}}\int_{\nu_c}^{\infty}dx e^{-x^2/2}=\frac{1}{2}\text{erfc}\left(\frac{\nu_c}{\sqrt{2}}\right)\,, \tag{2.69}$$

$$\xi_g(r) = \frac{P_2(\nu_c;r)}{[P_1(\nu_c)]^2}-1=\frac{2}{\pi}\left[\text{erfc}\left(\frac{\nu_c}{\sqrt{2}}\right)\right]^{-2}\sum_{N=1}^{\infty}\frac{\left[\xi(r)\right]^N}{N!\sigma^{2N}}\left[H_{N-1}(\nu_c)\right]^2e^{-\nu_c^2}\,, \tag{2.70}$$

which are consistent with the renormalized bias parameters.

Curvature bias

Now, consider additional dependence on the coarse-grained Laplacian of the density field, i.e. the curvature, and we follow the same procedure:

$$n_g(\mathbf{x}) = F_g[\delta(\mathbf{x});\nabla^2\delta(\mathbf{x});\mathbf{x}]\,,\qquad c_{\nabla^2\delta}\equiv\frac{1}{\langle F_g[0]\rangle}\left\langle\frac{\partial F_g}{\partial(\nabla^2\delta)}\Big|_{\delta=0,\nabla^2\delta=0}\right\rangle\,, \tag{2.71}$$

$$\xi_g(r) = c_1^2\langle\delta(1)\delta(2)\rangle+2c_1c_{\nabla^2\delta}\left\langle\delta(1)\nabla^2\delta(2)\right\rangle+\mathcal{O}(\nabla^4\xi)=c_1^2\left[\xi(r)+2R^2\nabla^2\xi(r)\right]+2c_1c_{\nabla^2\delta}\nabla^2\xi(r)+\cdots\,, \tag{2.72}$$

which is again phrased in terms of disconnected matter correlators and R-dependent bare bias parameters. We need to introduce a R-independent PBS bias parameter for $\nabla^2\delta$ as before. We would like a transformation where the Laplacian of the density perturbation shifts by a constant:

$$\nabla^2\delta_\alpha(\mathbf{x})=\nabla^2\delta(\mathbf{x})+\frac{\alpha}{l^2}\quad\longrightarrow\quad\delta_\alpha(\mathbf{x})=\delta(\mathbf{x})+\frac{\alpha}{6l^2}\left(\mathbf{x}^2+\vec{A}\cdot\mathbf{x}+C\right)\quad\longrightarrow\quad\delta_\alpha(\mathbf{x})=\delta(\mathbf{x})+\frac{\alpha}{6l^2}\mathbf{x}^2\,, \tag{2.73}$$

where α is a dimensionless small parameter, we have added a constant length scale l (which will disappear in b_N), and we used symmetry argument to remove A and C. We can now defined a (renormalized) PBS bias parameter through

$$b_{\nabla^2\delta}=\frac{l^2}{\langle n_g\rangle}\frac{\partial\langle n_g[0;\alpha]\rangle}{\partial\alpha}\Big|_{\alpha=0}\,. \tag{2.74}$$

non-Gaussianity

In a traditional way for computing the primordial non-Gaussianity, we have

$$\xi_g(r)=b_1^2\xi_L(r)+b_1b_2\left\langle\delta(1)\delta^2(2)\right\rangle+\mathcal{O}(\delta^4)\,,\qquad\qquad\left\langle\delta(1)\delta^2(2)\right\rangle=4f_{\rm NL}\sigma^2\xi_{\phi\delta}(r)\,, \tag{2.75}$$

where the appearance of σ^2 indicates that the description of the tracer density as a function of the matter density δ alone is insufficient even on large scales in the non-Gaussian case. Instead, we need to include a dependence of the tracer density on the amplitude of small-scale fluctuations. This dependence is present regardless of the nature of the initial conditions; however, only in the non-Gaussian case are there large-scale modulations of the small-scale fluctuations, due to mode coupling, whereas in the Gaussian case we were able to neglect the small-scale fluctuations in the large-scale description. For simplicity, we will parametrize the dependence through the variance of the density field on a single scale $R_* < R < r$.

We first define the small-scale density field as the local fluctuations around the coarse-grained field δ:

$$\delta_s(\mathbf{x}) \equiv \delta_*(\mathbf{x}) - \delta(\mathbf{x}) = \int d^3\mathbf{y}[W_*(\mathbf{x}-\mathbf{y}) - W_R(\mathbf{x}-\mathbf{y})]\delta(\mathbf{y}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{W}_s(k)\delta(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \,, \qquad \tilde{W}_s(k) = \tilde{W}_*(k) - \tilde{W}_R(k) \,. \tag{2.76}$$

We quantify the dependence of the tracer abundance on the amplitude of small-scale fluctuations through

$$y_*(\mathbf{x}) \equiv \frac{1}{2}\left(\frac{\delta_s^2(\mathbf{x})}{\sigma_s^2} - 1\right) \,, \qquad \sigma_s^2 := \langle\delta_s^2\rangle = \int \frac{d^3k}{(2\pi)^3}|\tilde{W}_s(k)|^2 P(k) \,, \tag{2.77}$$

In the Gaussian case, $\xi_s(r) \to 0$ for $r \gg R_L$, so that the small-scale density field and y_* have no large-scale correlations. We now generalize to explicitly include the dependence on y_*[1]

$$n_g(\mathbf{x}) = F_g[\delta(\mathbf{x}), y_*(\mathbf{x}); \mathbf{x}] \,, \qquad \langle n_g\rangle = \langle F_g[0]\rangle \sum_{n,m} \frac{c_{nm}}{n!m!}\langle\delta^n y_*^m\rangle \,, \qquad c_{nm} := \frac{1}{\langle F_g[0]\rangle}\left\langle \frac{\partial^{n+m}F_g}{\partial\delta^n\partial y_*^m}\Big|_{\delta=0,y_*=0}\right\rangle \,. \tag{2.78}$$

Then the two-point correlation function is

$$\xi_g(r) = \frac{1}{\mathcal{N}^2}\sum_{n,m,n',m'=0}^{\infty}\frac{c_{nm}c_{n'm'}}{n!m!n'!m'!}\left\langle\delta^n(1)y_*^m(1)\delta^{n'}(2)y_*^{m'}(2)\right\rangle - 1 \,, \qquad \mathcal{N} := \sum_{n,m=0}^{\infty}\frac{c_{nm}}{n!m!}\langle\delta^n y_*^m\rangle \,, \tag{2.79}$$

and to the lowest order

$$\xi_g(r) = \frac{1}{\mathcal{N}^2}\Big[\left(c_{10}^2 + c_{10}c_{30}\sigma^2\right)\xi(r) + \frac{c_{20}^2}{2}\xi(r)^2 + \left(2c_{10}c_{01} + c_{01}c_{30}\sigma^2 + 2c_{10}c_{20}\sigma^2\right)2f_{\rm NL}\xi_{\phi\delta}(r) + 2c_{11}c_{20}2f_{\rm NL}\xi_{\phi\delta}(r)\xi(r)\Big] \,. \tag{2.80}$$

We would like to introduce a physically motivated bias parameter which quantifies the response of the tracer number density to a change in the amplitude of small-scale fluctuations, without making reference to any coarse-graining on the scale R. The simplest way to parametrize such a dependence is to rescale all perturbations by a factor of $1+\epsilon$ from their fiducial value, where ϵ is an infinitesimal parameter. Note that this means that the scaled cumulants $\langle\delta_*^n\rangle_c/\sigma_*^n$ are invariant, whereas the primordial non-Gaussianity parameter $f_{\rm NL} \sim B_\Phi/P_\Phi^2$, if non-zero, scales as $(1+\epsilon)^{-1}$ under this transformation. Specifically, under this rescaling δ and y_* transform as

$$\delta(\mathbf{x}) \to (1+\epsilon)\delta(\mathbf{x}) \,, \qquad y_*(\mathbf{x}) \to y_*(\mathbf{x}) + \left(\epsilon + \frac{\epsilon^2}{2}\right)\frac{\delta_s^2(\mathbf{x})}{\sigma_s^2} \,, \tag{2.81}$$

where the parameter σ_s^2 in the definition of y_* is just a constant normalization, and does not change under the ϵ-transformation. This is in analogy to keeping $\bar\rho$ fixed in the D-transformation. We can then define a set of bivariate PBS bias parameters b_{NM} as

$$b_{NM} \equiv \frac{1}{\langle n_g\rangle_{D=0,\epsilon=0}}\frac{\partial^{N+M}\langle n_g\rangle_{D,\epsilon}}{\partial D^N\partial\epsilon^M}\Big|_{D=0,\epsilon=0} \,. \tag{2.82}$$

2.2.4 Gravitational Tidal Tensor Bias

The traceless tidal tensor and its simplest scalar are

$$s_{ij} = \partial_i\partial_j\Phi - \frac{1}{3}\delta_{ij}^{\rm K}\delta \,, \qquad s^2 = s_{ij}s_{ij} \,, \qquad s^2(\mathbf{k}) = \int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\delta_\mathbf{q}\delta_{\mathbf{k}-\mathbf{q}}S_2(\mathbf{q},\mathbf{k}-\mathbf{q}) \,, \qquad S_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{\mathbf{k}_1^2\mathbf{k}_2^2} - \frac{1}{3} \,, \tag{2.83}$$

[1] Although this approach here is formally similar to the bivariate local expansion in δ and ϕ adopted in Giannantonio and Porciani (2010), there is somewhat of a conceptual difference. The effect of non-Gaussianity, and the fact that it derives from a potential ϕ, only enter through the expressions for the correlators between δ and y_* here. The nature of non-Gaussianity thus decouples from the description of the tracers (which only know about the matter density field) in this approach.

where $\Phi \propto \alpha_\chi$ is a normalized potential. Interestingly, the gravitational instability generates the tidal contribution. By using the SPT, we have

$$\delta^{(2)}(\mathbf{k}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\,\delta_\mathbf{q}\delta_{\mathbf{k}-\mathbf{q}}F_2(\mathbf{q},\mathbf{k}-\mathbf{q})\,, \qquad F_2 = \frac{17}{21} + \frac{1}{2}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\left[\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2 - \frac{1}{3}\right]\,, \tag{2.84}$$

where we have re-arranged the usual F_2, representing the growth, the shift, and the anisotropy. In configuration space, we derive

$$\delta^{(2)}(\mathbf{x}) = \frac{17}{21}\left[\delta^{(1)}\right]^2 - \mathbf{\Psi}\cdot\nabla\delta^{(1)}(\mathbf{x}) + \frac{2}{7}s^2\,. \tag{2.85}$$

Note that

$$v_\chi = -\frac{\mathcal{H}f}{k}\delta\,, \qquad \mathbf{v} = -\frac{v_{\chi,i}}{k} = \mathcal{H}f\mathbf{\Psi}\,, \qquad \mathbf{\Psi} = \frac{ik_i}{k^2}\,\delta\,, \tag{2.86}$$

where $\mathbf{\Psi}$ is the usual Lagrangian displacement vector.

Up to the second order in perturbation, the nonlinear matter fluctuation is

$$\delta(\mathbf{x}) = \delta(\mathbf{q}) + \mathbf{\Psi}\cdot\nabla\delta(\mathbf{q}) + \mathcal{O}(3)\,, \tag{2.87}$$

and using the conservation the halo bias in Eulerian space is related to one in Lagrangian space as

$$1+\delta_h(\mathbf{x}) = (1+\delta(\mathbf{x}))(1+\delta_h(\mathbf{q}))\,, \qquad \delta_h(\mathbf{q}) = b_1^L\delta(\mathbf{q}) + \frac{1}{2}b_2^L\delta^2(\mathbf{q}) + \mathcal{O}(3)\,. \tag{2.88}$$

By removing the displacement field $\mathbf{\Psi}$ in favor of s^2 and $\delta^{(2)}$, we derive the halo bias in Eulerian space:

$$\delta_h(\mathbf{x}) = \left(1+b_1^L\right)\left(\delta^{(1)}(\mathbf{x}) + \delta^{(2)}(\mathbf{x})\right) + \left(\frac{4}{21}b_1^L + \frac{1}{2}b_2^L\right)\delta^2(\mathbf{x}) - \frac{2}{7}b_1^L s^2 + \mathcal{O}(3)\,. \tag{2.89}$$

This shows that despite the fact that the halo bias depends only on the matter density in Lagrangian space the nonlinear evolution generates a bias factor in proportion to the tidal tensor in Eulerian space.

2.2.5 Halo Exclusion

Here we consider the effect of a finite size of halos on the power spectrum. By definition halos are some objects with a finite size, such that at a separation below the size, the correlation function ξ_h^d becomes -1, and this is called the halo exclusion, where we used the super-script d to emphasize halos are discrete objects with $\xi_h^d \equiv -1$ at some separation, as opposed to continuous fields with super-script c. The power spectrum of halos is then

$$P_h(k) = \int d^3x\, e^{-ikx}\,\xi_h^d(x) = 4\pi\int_0^\infty dr\, r^2\,\xi_h^d(r)j_0(kr) = 4\pi\left[\int_0^R dr + \int_R^\infty dr\right] r^2\xi_h^d(r)j_0(kr)\,, \tag{2.90}$$

where we simply split the integration range around the size of halos (R: virial radius) and assumed for simplicity all halos are of the same size.

The first term is simple, as $\xi_h^d \equiv -1$ over $r \leq R$:

$$4\pi\int_0^R dr\, r^2 j_0(kr) =: V\times W_R(k)\,, \qquad V := \frac{4\pi}{3}R^3\,, \tag{2.91}$$

where we defined the window function $W_R(k)$ and V is the exclusion volume. The second term can also be arranged in a more illuminating form as

$$4\pi\int_R^\infty dr\, r^2\xi_h^d(r)j_0(kr) = 4\pi\left[\int_R^\infty dr\,\left[\xi_{h,\mathrm{NL}} - \xi_{h,\mathrm{lin}}\right](r) + \int_0^\infty dr\,\xi_{h,\mathrm{lin}}^c(r) - \int_0^R dr\,\xi_{h,\mathrm{lin}}^c(r)\right] r^2 j_0(kr)\,, \tag{2.92}$$

where we removed d, because $xi_h \neq -1$ at $r \geq R$. Adding two terms, the halo power spectrum is then obtained as

$$P_h^d(k) = -VW_R(k) + 4\pi\int_R^\infty dr\,\left(\xi_{h,\mathrm{NL}} - \xi_{h,\mathrm{lin}}\right) r^2 j_0(kr) + P_{h,\mathrm{lin}}^c(k) + \frac{1}{n_h} - 4\pi\int_0^R dr\, r^2\xi_{h,\mathrm{lin}}^c(r)j_0(kr)\,. \tag{2.93}$$

In the limit size is zero ($R \to 0$), the halo power spectrum is the usual non-linear power spectrum plus a Poisson shot-noise:

$$P_h^d(k) = 4\pi \int_0^\infty dr\, \xi_{h,\mathrm{NL}}^c(r) r^2 j_0(kr) + \frac{1}{n_h} . \tag{2.94}$$

In the limit $k \to 0$ the halo power spectrum gives

$$\lim_{k\to 0} P_h^d(k) = -V + \int_R^\infty d^3r\, (\xi_{h,\mathrm{NL}} - \xi_{h,\mathrm{lin}}) + 0 + \frac{1}{n_h} - \int_0^R d^3r\, \xi_{h,\mathrm{lin}}^c(r) , \tag{2.95}$$

which should be the shot-noise in reality. Compared to the Poisson shot-noise, the exclusion terms (first and last) contribute negative power to the shot-noise, while the nonlinear effect contributes positive power.

2.2.6 Miscellaneous

Stochasticity

The stochasticity can be defined as

$$r(R) = \frac{\langle \delta_\mathrm{m}(R)\delta_\mathrm{X}(R)\rangle}{\sigma_\mathrm{m}(R)\sigma_\mathrm{X}(R)} , \qquad r(k) = \frac{P_\mathrm{mX}(k)}{\sqrt{P_\mathrm{m}(k)P_\mathrm{X}(k)}} , \tag{2.96}$$

and $r(k) = 1$ to the linear order, while $r(R)$ can be scale-dependent to the linear order, if $b_1(k)$ is scale-dependent. To the lowest order contribution to the stochasticity, we have

$$1 - r(k) = \frac{1}{4[b_1(k)]^2 P_\mathrm{L}(k)} \int \frac{d^3k'}{(2\pi)^3} \left[b_2(\boldsymbol{k}', \boldsymbol{k} - \boldsymbol{k}')\right]^2 P_\mathrm{L}(k') P_\mathrm{L}(|\boldsymbol{k} - \boldsymbol{k}'|) \;\to\; \frac{(b_2^\mathrm{L})^2}{4[b_1(k)]^2 P_\mathrm{L}(k)} \int \frac{d^3k'}{(2\pi)^3} \left[P_\mathrm{L}(k')\right]^2 , \tag{2.97}$$

where the latter is the large-scale limit ($b_2 \to b_2^L$) with constant Lagrangian bias. Nonzero b_2^L can generate stochasticity.

Deterministic (nonlocal) Bias Arguments

The peak-background split method is necessary because the halo approach is based on a statistical nature of extended Press-Schechter mass function. In such an approach, the local mass function is obtained by averaging over small-scale fluctuations, while large-scale fluctuations are considered as background modulation field, which leads spatial fluctuations of number density of halos. Comparing the fluctuations of the halo number density field and those of mass, the halo bias is analytically derived. However, the biasing can be seen as a deterministic process at a most fundamental level, in which any statistical information is not required. One can think of getting a halo catalog in numerical simulations to understand the situation. Just one realization of the initial condition deterministically gives subsequent nonlinear evolutions and formation sites of halos. When only leading growing modes are considered in a perturbation theory, any structure in the universe is deterministically related to the linear density field. The biasing relation should not require statistical information of the field. Any statistical quantities, such as the short-mode power spectrum in the method of peak-background split, are not expected to appear at the most fundamental level.

2.3 Lagrangian Perturbation Theory

2.3.1 Basic Idea: Zel'dovich Approximation

Zel'dovich provides a model for the Universe, which is the linear order approximation of the Lagrangian perturbation theory. The Lagrangian Perturbation Theory attempts to provide description of the matter and the galaxy distributions today by modeling those at the very early (initial) time and tracing their motion until today. The matter density is more linear and smooth at early times. So the critical quantity in this approach is the displacement field $\boldsymbol{\Psi}$ that relates the initial (Lagrangian) position $\boldsymbol{q}$ to the final (Eulerian) position $\boldsymbol{x}$:

$$\boldsymbol{x}(\boldsymbol{q}, t) = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q}, t) . \tag{2.98}$$

Rather than modeling the density and the velocity in the Standard Perturbation Theory, the Lagrangian Perturbation Theory models the evolution of the displacement field. Therefore, when completely expanded at each order, they both agree, but in general the LPT expressions correspond to the SPT expression with non-trivial ressumation of different perturbation orders.

With the mass conservation, the matter density today is related to its Lagrangian quantities as

$$(1 + \delta_m) d^3x = (1 + \delta_q) d^3q , \qquad \delta(\boldsymbol{x}) = \frac{1 + \delta_q}{J} - 1 = \int d^3q\, (1 + \delta_q)\, \delta^D\Big[\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\Psi}(\boldsymbol{q})\Big] - 1 , \tag{2.99}$$

where the last equation can be verified by integrating over d^3x, and the Jacobian matrix is

$$J_{ij} := \frac{\partial x^i}{\partial q^j} = \delta_{ij} + \Psi_{i,j} \,, \qquad (J^{-1})_{ij} \simeq \delta_{ij} - \Psi_{i,j} + \cdots \,, \tag{2.100}$$

and its determinant is

$$\begin{aligned} J &= \det J_{ij} = \exp\left[\mathrm{Tr}\left(\ln \delta_{ij} + \Psi_{i,j}\right)\right] = \exp\left[\mathrm{Tr}\left(\sum_{n=1} \frac{(-1)^{n-1}(\Psi_{i,j})^n}{n}\right)\right] \\ &= 1 + \mathrm{Tr}\left(\Psi_{i,j}\right) + \frac{1}{2}\left[\mathrm{Tr}^2\left(\Psi_{i,j}\right) - \mathrm{Tr}\left(\Psi_{i,j}\right)^2\right] + \cdots = 1 + \nabla\cdot\mathbf{\Psi} + \left(\partial_x\Psi_x\,\partial_y\Psi_y + \cdots - \partial_x\Psi_y\,\partial_y\Psi_x - \cdots\right)\cdots \,. \end{aligned} \tag{2.101}$$

For sufficiently early t_i, the initial density field for matter is often assumed to be zero:

$$\delta_q = 0 \quad \text{at } t = t_i \,, \tag{2.102}$$

hence the nonlinear matter density today is

$$1 + \delta_m = \frac{d^3q}{d^3x} = \frac{1}{J} \,. \tag{2.103}$$

Similarly, with the number conservation of galaxies, we derive

$$(1+\delta_g)d^3x = \left(1+\delta_g^i\right)d^3q \,, \qquad 1+\delta_g = \left(1+\delta_g^i\right)(1+\delta_m) \,, \tag{2.104}$$

where the (Eulerian) galaxy bias today and the Lagrangian bias are modeled as

$$\delta_g(\mathbf{x}) = \sum b_n^E[\delta_m(x)]^n \,, \qquad \delta_g^i(\mathbf{q}) = \sum b_n^L[\delta_L(q)]^n \,, \tag{2.105}$$

in terms of the nonlinear matter distribution δ_m today and the linear matter density δ_L (linearly) extrapolated to the late time.

Summary

At the linear order (Zel'dovich approximation), we will obtain

$$\mathbf{\Psi}^{(1)} = i\frac{\boldsymbol{k}}{k^2}\delta_m^{(1)}(t,\mathbf{k}) \,, \qquad \boldsymbol{v} = \mathcal{H} f \mathbf{\Psi}^{(1)} \,. \tag{2.106}$$

In general, the displacement field in Fourier space is generally represented as

$$\tilde{\mathbf{\Psi}}^{(n)}(\boldsymbol{p}) = \frac{iD^n}{n!}\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}(2\pi)^3\delta^D\left(\sum_{j=1}^n \boldsymbol{p}_j - \boldsymbol{p}\right)\boldsymbol{L}^{(n)}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)\delta_0(\boldsymbol{p}_1)\cdots\delta_0(\boldsymbol{p}_n) \,, \tag{2.107}$$

and we have

$$\boldsymbol{L}^{(1)}(\boldsymbol{p}_1) = \frac{\boldsymbol{k}}{k^2} \,, \qquad \boldsymbol{L}^{(2)}(\boldsymbol{p}_1,\boldsymbol{p}_2) = \frac{3}{7}\frac{\boldsymbol{k}}{k^2}\left[1 - \left(\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1p_2}\right)^2\right] \,. \tag{2.108}$$

Equation of motion

Pressureless particles in an expanding universe are subject to the equation of motion and the Poisson equation as

$$\ddot{\mathbf{r}} = -\nabla_r\phi \,, \qquad \nabla_{\mathbf{x}}^2\Phi = 4\pi G\bar\rho_m a^2(1+\delta) \,, \tag{2.109}$$

where the usual relations are

$$\mathbf{r} =: a\,\mathbf{x} \,, \qquad \nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}} \,, \qquad \Phi = \bar\phi + \phi \,, \qquad \mathbf{x} = \mathbf{q} + \mathbf{\Psi} \,. \tag{2.110}$$

Using Jean's swindle, we remove the constant motion in the Poisson equation:

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\bar\rho_m \,, \qquad \nabla_{\mathbf{x}}\cdot\left(\frac{\ddot a}{a}\mathbf{x}\right) = -4\pi G\bar\rho_m = -\frac{1}{a^2}\nabla_{\mathbf{x}}^2\bar\phi \,, \tag{2.111}$$

to obtain

$$\nabla_{\mathbf{x}}^2\phi = 4\pi G\bar{\rho}_m a^2\delta = \frac{3}{2}\mathcal{H}^2\Omega_m(z)\left[\frac{1}{J}-1\right] . \tag{2.112}$$

The equation of motion in terms of comoving coordinates is then

$$\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla_{\mathbf{x}}\phi , \qquad \ddot{\mathbf{\Psi}} + 2H\dot{\mathbf{\Psi}} = -\frac{1}{a^2}\nabla_{\mathbf{x}}\phi\left(\mathbf{q}+\mathbf{\Psi}\right) \quad\rightarrow\quad J\nabla_{\mathbf{x}}\cdot\ddot{\mathbf{\Psi}} + 2JH\nabla_{\mathbf{x}}\cdot\dot{\mathbf{\Psi}} = \frac{3}{2}H^2\Omega_m(J-1) ,$$
$$\mathbf{x}'' + \mathcal{H}\mathbf{x}' = -\nabla_{\mathbf{x}}\phi , \qquad \mathbf{\Psi}'' + \mathcal{H}\mathbf{\Psi}' = -\nabla_{\mathbf{x}}\phi\left(\mathbf{q}+\mathbf{\Psi}\right) \quad\rightarrow\quad J\nabla_{\mathbf{x}}\cdot\mathbf{\Psi}'' + J\mathcal{H}\nabla_{\mathbf{x}}\cdot\mathbf{\Psi}' = \frac{3}{2}\mathcal{H}^2\Omega_m(J-1) , \tag{2.113}$$

where prime is the derivative with respect to conformal time.

Assuming that the time dependence is $\mathbf{\Psi}^{(n)} \propto D^n$, we derive[2]

$$\dot{D} = HDf , \qquad \dot{\mathbf{\Psi}}^{(n)} = nHf\mathbf{\Psi}^{(n)} , \qquad \ddot{\mathbf{\Psi}}^{(n)} = H^2f^2n\left[\frac{\dot{H}}{H^2f}+\frac{\dot{f}}{Hf^2}+n\right]\mathbf{\Psi}^{(n)} , \tag{2.115}$$

and by plugging in the equation of motion we obtain

$$\nabla_{\mathbf{x}}\cdot\ddot{\mathbf{\Psi}} + 2H\nabla_{\mathbf{x}}\cdot\dot{\mathbf{\Psi}} = \sum_n \nabla_{\mathbf{x}}\cdot\mathbf{\Psi}^{(n)}H^2f^2n\left[\frac{\dot{H}}{H^2f}+\frac{\dot{f}}{Hf^2}+n+\frac{2}{f}\right] = \sum_n \nabla_{\mathbf{x}}\cdot\mathbf{\Psi}^{(n)}H^2f^2n\left[n-1+\frac{3}{2}\frac{\Omega_m}{f^2}\right] . \tag{2.116}$$

For a ΛCDM universe, we derive a non-trivial identity

$$\frac{\dot{H}}{H^2f}+\frac{\dot{f}}{Hf^2}+\frac{2}{f} = \frac{3}{2}\frac{\Omega_m}{f^2}-1 , \tag{2.117}$$

but as in the SPT, as long as $\Omega_m(z)/f^2 \simeq 1$ the time-dependence of the displacement field can be separated, and the equation of motion is

$$\nabla_{\mathbf{x}}\cdot\ddot{\mathbf{\Psi}} + 2H\nabla_{\mathbf{x}}\cdot\dot{\mathbf{\Psi}} = \frac{3}{2}H^2\Omega_m\left(1-\frac{1}{J}\right) \quad\rightarrow\quad \nabla_{\mathbf{x}}\cdot\mathbf{\Psi}^{(n)}n\left[n-1+\frac{3}{2}\frac{\Omega_m}{f^2}\right] = \frac{3}{2}\frac{\Omega_m}{f^2}\left(1-\frac{1}{J}\right)^{(n)} , \tag{2.118}$$

The peculiar velocity field needs some care due to coordinates.

$$\mathbf{v}(\mathbf{q},t) = a\dot{\mathbf{x}} = a\dot{\mathbf{\Psi}}(\mathbf{q},t) , \qquad \mathbf{v}(\mathbf{x},t) = a\dot{\mathbf{\Psi}}(\mathbf{x}-\mathbf{\Psi},t) = a\dot{\Psi}_i - a\dot{\Psi}_{i,j}\Psi_j + \cdots , \qquad \boldsymbol{\nabla}_x\times\boldsymbol{v} = \mathbf{0} , \tag{2.119}$$

where we have

$$\nabla_{\mathbf{x}} = \frac{\partial}{\partial x^i} = \frac{\partial q^j}{\partial x^i}\frac{\partial}{\partial q^j} = \left(J^{-1}\right)_{ji}\frac{\partial}{\partial q^j} , \qquad \frac{\partial}{\partial q^i} = J_{ji}\frac{\partial}{\partial x^j} . \tag{2.120}$$

Note that $\mathbf{v}(\mathbf{q},t)$ is the velocity assigned to particles that are initially placed in a grid at $\mathbf{q}$ and then displaced to a position $\mathbf{x}$ in the Zel'dovich or 2LPT simulations.

In the original Zel'dovich paper, strange notations are used:

$$\mathbf{\Psi} := D(z)\mathbf{p}(\mathbf{q}) := -D(z)\nabla\Phi , \qquad \mathbf{p} = -\nabla\Phi = -\frac{i\mathbf{k}}{k^2}\frac{\delta_m^{(1)}(t,\mathbf{k})}{D(z)} , \qquad \Phi = -\frac{\delta_m^{(1)}(t,\mathbf{k})}{k^2D(z)} = -\frac{\delta^{(1)}(\mathbf{q})}{k^2} . \tag{2.121}$$

Initial Condition in Numerical Simulations

In numerical simulations the initial density distribution is set up at the initial redshift $z_i \simeq 50$. By using $P(k,z_i)$ one generates $\delta^{\rm Eul.}(x,z_i) = \delta^{\rm Eul.}(q,z_i) + \mathcal{O}(2)$ and use $\delta^{\rm Eul.}(q,z_i)$ to generate the displacement field at each grid. Finally, particles at each grid is then displaced by using $\mathbf{\Psi}$.

[2] Sometimes, it is assumed

$$\mathbf{\Psi}^{(n)} \propto D_n , \qquad D_1 = D , \qquad D_2 = \frac{3}{7}D^2 , \qquad f_2 = \frac{d\ln D_2}{d\ln a} = 2f_1 . \tag{2.114}$$

2.3.2 Resummation in LPT: One-Loop Power Spectrum

Following Matsubara (2008), the polyspectra of the displacement field are defined as

$$\left\langle\tilde\Psi_{i_1}(\boldsymbol{p}_1)\cdots\tilde\Psi_{i_N}(\boldsymbol{p}_N)\right\rangle_{\rm c}=(2\pi)^3\delta^D(\boldsymbol{p}_1+\cdots+\boldsymbol{p}_N)(-i)^{N-2}C_{i_1\cdots i_N}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N)\,, \tag{2.122}$$

where $i=x,y,z$, representing a vector (including scalar) quantity and the Fourier transformation is

$$\tilde\Psi_i(\boldsymbol{p})=\int d^3q\,e^{-i\boldsymbol{p}\cdot\boldsymbol{q}}\Psi_i(\boldsymbol{q})\,. \tag{2.123}$$

The relation $\boldsymbol{p}_1+\cdots+\boldsymbol{p}_N=\boldsymbol{0}$ is always satisfied because of the translational invariance. The factors $(-i)^{N-2}$ in the RHS are there to ensure that the polyspectra $C_{i_1\cdots}$ are real numbers:

$$\tilde\Psi_i(-\mathbf{p})=\tilde\Psi_i^*(\mathbf{p})\;\to\;C_{i_1\cdots i_N}(-\mathbf{p}_1,\cdots,-\mathbf{p}_N)=(-1)^NC^*_{i_1\cdots i_N}(\mathbf{p}_1,\cdots,\mathbf{p}_N)\,, \tag{2.124}$$

Note that this relation holds in general. For the displacement field, we have another condition:

$$\left\langle\tilde\Psi_{i_1}(\boldsymbol{p}_1)\cdots\tilde\Psi_{i_N}(\boldsymbol{p}_N)\right\rangle^*_{\rm c}=(-1)^N\left\langle\tilde\Psi_{i_1}(\boldsymbol{p}_1)\cdots\tilde\Psi_{i_N}(\boldsymbol{p}_N)\right\rangle_{\rm c}\,, \tag{2.125}$$

which together guarantees that the polyspectra are real. For $N=2$, $C_{ij}(\boldsymbol{p})=C_{ij}(\boldsymbol{p},-\boldsymbol{p})$, for simplicity. By using Eq. (2.99), the power spectrum is obtained as

$$\begin{aligned}P(\boldsymbol{k})&=\int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left\{\left\langle e^{-i\boldsymbol{k}\cdot[\boldsymbol{\Psi}(\boldsymbol{q}_1)-\boldsymbol{\Psi}(\boldsymbol{q}_2)]}\right\rangle-1\right\}\\&=\exp\left[-2\sum_{n=1}^\infty\frac{k_{i_1}\cdots k_{i_{2n}}}{(2n)!}A^{(2n)}_{i_1\cdots i_{2n}}\right]\int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left\{\exp\left[\sum_{N=2}^\infty\frac{k_{i_1}\cdots k_{i_N}}{N!}B^{(N)}_{i_1\cdots i_N}(\boldsymbol{q})\right]-1\right\},\end{aligned} \tag{2.126}$$

where we have

$$\left\langle[\boldsymbol{k}\cdot(\boldsymbol{\Psi}_{\boldsymbol{q}_1}-\boldsymbol{\Psi}_{\boldsymbol{q}_2})]^N\right\rangle_{\rm c}=[1+(-1)^N]\left\langle[\boldsymbol{k}\cdot\Psi(\boldsymbol{0})]^N\right\rangle_{\rm c}+\sum_{j=1}^{N-1}(-1)^{N-j}\binom{N}{j}\left\langle(\boldsymbol{k}\cdot\Psi_{\boldsymbol{q}_1})^j(\boldsymbol{k}\cdot\Psi_{\boldsymbol{q}_2})^{N-j}\right\rangle_{\rm c}\,, \tag{2.127}$$

$$A^{(2n)}_{i_1\cdots i_{2n}}=\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_{2n}}{(2\pi)^3}(2\pi)^3\delta^D(\boldsymbol{p}_1+\cdots+\boldsymbol{p}_{2n})\,C_{i_1\cdots i_{2n}}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{2n})\,, \tag{2.128}$$

$$B^{(N)}_{i_1\cdots i_N}(\boldsymbol{q})=\sum_{j=1}^{N-1}(-1)^{j-1}\binom{N}{j}\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_N}{(2\pi)^3}(2\pi)^3\delta^D(\boldsymbol{p}_1+\cdots+\boldsymbol{p}_N)\,e^{i(\boldsymbol{p}_1+\cdots+\boldsymbol{p}_j)\cdot\boldsymbol{q}}\,C_{i_1\cdots i_N}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N)\,.$$

The calculation is done at the one-loop power spectrum: all we need are

$$\begin{aligned}&A^{(2)}_{ij}=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,C_{ij}(\mathbf{p})\,,\qquad B^{(2)}_{ij}=2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,e^{i\mathbf{p}\cdot\mathbf{q}}C_{ij}(\mathbf{p})\,,\\&B^{(3)}_{ijk}=3\int\frac{d^3\mathbf{p}_1}{(2\pi)^3}\int\frac{d^3\mathbf{p}_2}{(2\pi)^3}\left[e^{i\mathbf{p}_1\cdot\mathbf{q}}-e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{q}}\right]C_{ijk}(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2)\,,\\&C_{ij}(\mathbf{p},-\mathbf{p})=C^{(11)}_{ij}+C^{(22)}_{ij}+C^{(13)}_{ij}+C^{(31)}_{ij}\,,\qquad C_{ijk}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=C^{(112)}_{ijk}+C^{(121)}_{ijk}+C^{(211)}_{ijk}\,,\end{aligned} \tag{2.129}$$

and hence the power spectrum is

$$\begin{aligned}P(\mathbf{k})&=\exp\left[-k_ik_jA^{(2)}_{ij}\right]\int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left[\frac{k_ik_j}{2}B^{(2)}_{ij}+\frac{k_ik_jk_k}{6}B^{(3)}_{ijk}+\frac{k_ik_jk_kk_l}{8}B^{(2)}_{ij}B^{(2)}_{kl}\right]\\&=\exp\left[-k_ik_jA^{(2)}_{ij}\right]\left[k_ik_jC_{ij}(\mathbf{k},-\mathbf{k})+k_ik_jk_k\int\frac{d^3\mathbf{p}}{(2\pi)^3}C_{ijk}(\mathbf{k},-\mathbf{p},\mathbf{p}-\mathbf{k})+\frac{k_ik_jk_kk_l}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}C_{ij}(\mathbf{p})C_{ij}(\mathbf{k},-\mathbf{p})\right].\end{aligned} \tag{2.130}$$

With this, the resulting one-loop power spectrum is

$$P(k)=\exp\left[-\frac{k^2}{6\pi^2}\int dp\,P_{\rm L}(p)\right]\left[P_{\rm L}(k)+P^{\text{1-loop}}_{\rm SPT}(k)+\frac{k^2}{6\pi^2}P_{\rm L}(k)\int dp\,P_{\rm L}(p)\right]\,, \tag{2.131}$$

where we defined

$$k_{\rm NL}^{-2}:=\frac{1}{6\pi^2}\int dp\,P_{\rm L}(p)\,. \tag{2.132}$$

When expanded, it is identical to the SPT at one-loop, but due to the resummation, it differs from SPT. The LPT formula breaks down at $k\simeq k_{\rm NL}$ due to the exponential damping.

2.3.3 Zel'dovich Power Spectrum

The Zel'dovich approximation is basically the linear order LPT, in which the displacement field is computed only at the linear order in perturbations. However, these quantities in the power spectrum are not expanded, but kept in the exponential, such that it is a nonlinear analytic solution under the assumption that the displacement field is only linear.

Defining the 1-D rms displacement, we have

$$\sigma^2_{\Psi,1\mathrm{D}} = \frac{1}{3}\left\langle|\mathbf{\Psi}|^2\right\rangle = \frac{1}{3}\int\frac{d^3k}{(2\pi)^3}\,\frac{P_m(k)}{k^2} = \frac{1}{6\pi^2}\int_0^\infty dk\,P(k) = I_0(0)\,, \tag{2.133}$$

where we defined

$$I_l(q) := \frac{1}{3}\int_0^\infty\frac{dp}{2\pi^2}\,P_m(p)j_l(pq)\,. \tag{2.134}$$

To the linear order, we just need

$$A^{(2)}_{ij} = \langle\mathbf{\Psi}_i(0)\mathbf{\Psi}_j(0)\rangle\,,\qquad\qquad k_ik_jA^{(2)}_{ij} = k^2\sigma^2_\Psi\,, \tag{2.135}$$

to derive the power spectrum as

$$P(\boldsymbol{k}) = \exp\left[-k_ik_jA^{(2)}_{ij}\right]\int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left\{\exp\left[\frac{k_ik_j}{2}B^{(2)}_{ij}(\boldsymbol{q})\right]-1\right\}\,, \tag{2.136}$$

where we omitted 1-D in the notation. Assuming that $\mathbf{q}/\!/\hat{\mathbf{z}}$ and $\mathbf{k}$ in x-z plane ($\mu_k = \hat{\mathbf{q}}\cdot\hat{\mathbf{k}}$, $\mu_p = \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$), the term in the exponential can be written as

$$k_ik_jp_ip_j = \left(kp\sqrt{1-\mu_k^2}\sqrt{1-\mu_p^2}\cos\phi_p + kp\mu_k\mu_p\right)^2\,, \tag{2.137}$$

$$\begin{aligned}\frac{k_ik_j}{2}\,B^{(2)}_{ij}(\mathbf{q}) &= \int_0^\infty\frac{p^2dp}{2\pi^2}\int\frac{d\mu_p}{2}\int_0^{2\pi}\frac{d\phi_p}{2\pi}\,e^{ipq\mu_p}\frac{k_ik_jp_ip_j}{2p^4}\,P_m(p) \\ &= \int_0^\infty\frac{p^2dp}{2\pi^2}\,\frac{P_m(p)}{p^4}\int\frac{d\mu_p}{2}\,e^{ipq\mu_p}k^2p^2\left[\frac{1}{2}(1-\mu_k^2)(1-\mu_p^2)+\mu_k^2\mu_p^2\right] \\ &= k^2(1-\mu_k^2)[I_0(q)+I_2(q)] + k^2\mu_k^2[I_0(q)-2I_2(q)] = -\frac{1}{2}k^2(1-\mu_k^2)\sigma^2_\perp(q) - \frac{1}{2}k^2\mu_k^2\sigma^2_\parallel(q) + k^2\sigma^2_\Psi\,,\end{aligned} \tag{2.138}$$

where the $\cos\phi_p$ term averages out, the $\cos^2\phi_p$ term yields 1/2, $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ is the Lagrangian space position in configuration space, and

$$\sigma^2_\parallel = \left\langle\left[\Psi_\parallel(\mathbf{q}_1)-\Psi_\parallel(\mathbf{q}_2)\right]^2\right\rangle = 2\sigma^2_\Psi - 2I_0(q) + 4I_2(q)\,, \tag{2.139}$$

$$\sigma^2_\perp = \left\langle\left[\Psi_\perp(\mathbf{q}_1)-\Psi_\perp(\mathbf{q}_2)\right]^2\right\rangle = 2\sigma^2_\Psi - 2I_0(q) - 2I_2(q)\,,\qquad \lim_{q\to0}\sigma^2_\perp(q) = \lim_{q\to0}\sigma^2_\parallel(q) = 0\,. \tag{2.140}$$

2.3.4 Galaxy Bias in the Lagrangian Frame

The local Lagrangian (deterministic) bias factor is introduced as

$$\langle F\rangle = 1\,,\qquad \rho^{\mathrm{L}}_{\mathrm{obj}}(\boldsymbol{q}) = \bar\rho_{\mathrm{obj}}\,F[\delta_R(\boldsymbol{q})]\;\to\;1+\delta_L = F[\delta_R(\boldsymbol{q})] = \int\frac{d\lambda}{2\pi}\,e^{i\delta_R\lambda}\,\tilde F(\lambda)\,, \tag{2.141}$$

and then the power spectrum is then

$$\begin{aligned}P(\mathbf{k}) &= \int d^3q\,e^{-i\mathbf{k}\cdot\mathbf{q}}\left[\left\langle e^{-ik_z[\Psi_z(\mathbf{q}_2)-\Psi_z(\mathbf{q}_1)]}\left(1+\delta^L_{\mathbf{q}_1}\right)\left(1+\delta^L_{\mathbf{q}_2}\right)\right\rangle-1\right] \\ &= \int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left[\int\frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}\tilde F(\lambda_1)\tilde F(\lambda_2)\left\langle e^{i[\lambda_1\delta_R(\boldsymbol{q}_1)+\lambda_2\delta_R(\boldsymbol{q}_2)]-i\boldsymbol{k}\cdot[\mathbf{\Psi}(\boldsymbol{q}_1)-\mathbf{\Psi}(\boldsymbol{q}_2)]}\right\rangle-1\right]\,.\end{aligned} \tag{2.142}$$

Using the cumulant theorem, we have

$$\left\langle e^{i[\lambda_1\delta_R(\boldsymbol{q}_1)+\lambda_2\delta_R(\boldsymbol{q}_2)]-i\boldsymbol{k}\cdot[\mathbf{\Psi}(\boldsymbol{q}_1)-\mathbf{\Psi}(\boldsymbol{q}_2)]}\right\rangle = \exp\left[\sum_{n_1+n_2+m_1+m_2\geq1}\frac{i^{n_1+n_2+m_1+m_2}}{n_1!n_2!m_1!m_2!}{\lambda_1}^{n_1}{\lambda_2}^{n_2}B^{n_1n_2}_{m_1m_2}(\boldsymbol{k},\boldsymbol{q})\right]\,, \tag{2.143}$$

where the multinomial theorem is used and using the translational invariance and the parity symmetry $B^{n_1n_2}_{m_1m_2} = B^{n_2n_1}_{m_2m_1}$, we have

$$\begin{aligned} B^{n_1n_2}_{m_1m_2}(\boldsymbol{k},\boldsymbol{q}) &= (-1)^{m_1}\left\langle[\delta_R(\boldsymbol{q}_1)]^{n_1}[\delta_R(\boldsymbol{q}_2)]^{n_2}(\boldsymbol{k}\cdot\boldsymbol{\Psi}_{\boldsymbol{q}_1})^{m_1}(\boldsymbol{k}\cdot\boldsymbol{\Psi}_{\boldsymbol{q}_2})^{m_2}\right\rangle_{\rm c} \\ &= (-1)^{m_1+m_2}B^{n_2n_1}_{m_2m_1}(\boldsymbol{k},-\boldsymbol{q}) = (-1)^{m_1+m_2}B^{n_1n_2}_{m_1m_2}(\boldsymbol{k},-\boldsymbol{q})\,. \end{aligned} \tag{2.144}$$

For a Gaussian (initial) random field, we have a few cases, where we can solve

$$B^{n_1n_2}_{00}(\boldsymbol{k},\boldsymbol{q}) = \begin{cases} \xi_R(|\boldsymbol{q}|)\,, & n_1=n_2=1, \\ \sigma_R^2\,, & (n_1=2,n_2=0) \text{ or } (n_1=0,n_2=2), \\ 0\,, & \text{otherwise,} \end{cases} \tag{2.145}$$

and for $n_1 = n_2 = 0$ (matter part) we have

$$A_{2m}(\boldsymbol{k}) \equiv \left\langle[\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{0})]^{2m}\right\rangle_{\rm c}\,, \qquad B_{m_1m_2}(\boldsymbol{k},\boldsymbol{q}) \equiv (-1)^{m_1}\left\langle[\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q}_1)]^{m_1}[\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q}_2)]^{m_2}\right\rangle_{\rm c}\,, \tag{2.146}$$

$$B^{00}_{m_1m_2}(\boldsymbol{k},\boldsymbol{q}) = \begin{cases} A_{2m}(\boldsymbol{k})\,, & (m_1=2m,m_2=0) \text{ or } (m_1=0,m_2=2m), \\ B_{m_1m_2}(\boldsymbol{k},\boldsymbol{q})\,, & m_1\ge1 \text{ and } m_2\ge1, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.147}$$

Therefore, the resulting power spectrum is in full generality

$$\begin{aligned} P_{\rm obj}(\boldsymbol{k}) &= \exp\left[2\sum_{m=1}^\infty\frac{(-1)^m}{(2m)!}A_{2m}(\boldsymbol{k})\right]\int d^3q\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\exp\left[\sum_{m_1,m_2\ge1}^\infty\frac{i^{m_1+m_2}}{m_1!m_2!}B_{m_1m_2}(\boldsymbol{k},\boldsymbol{q})\right] \\ &\times\int_{-\infty}^\infty\frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}\,\tilde F(\lambda_1)\tilde F(\lambda_2)\,e^{-\frac12\left[(\lambda_1\sigma_R)^2+(\lambda_2\sigma_R)^2\right]}\exp\left[-\lambda_1\lambda_2\xi_R(|\boldsymbol{q}|)+\sum_{\substack{n_1+n_2\ge1\\ m_1+m_2\ge1}}^\infty\frac{i^{n_1+n_2+m_1+m_2}}{n_1!n_1!m_1!m_2!}{\lambda_1}^{n_1}{\lambda_2}^{n_2}B^{n_1n_2}_{m_1m_2}(\boldsymbol{k},\boldsymbol{q})\right] \\ &-(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k})\,. \end{aligned} \tag{2.148}$$

With one-loop corrections, the real-space power spectrum can be obtained by expanding the above big square bracket as

$$\begin{aligned} P_{\rm obj}(k) &= \exp\left[-\left(k/k_{\rm NL}\right)^2\right]\left\{\left(1+\langle F'\rangle\right)^2P_{\rm L}(k)+\frac{9}{98}Q_1(k)+\frac37Q_2(k)+\frac12Q_3(k)+\langle F'\rangle\left[\frac67Q_5(k)+2Q_7(k)\right]\right. \\ &+\langle F''\rangle\left[\frac37Q_8(k)+Q_9(k)\right]+\langle F'\rangle^2\left[Q_9(k)+Q_{11}(k)\right]+2\langle F'\rangle\langle F''\rangle Q_{12}(k)+\frac12\langle F''\rangle^2Q_{13}(k) \\ &\left.+\frac67\left(1+\langle F'\rangle\right)^2\left[R_1(k)+R_2(k)\right]-\frac{8}{21}\left(1+\langle F'\rangle\right)R_1(k)\right\}\,, \end{aligned} \tag{2.149}$$

where the Lagrangian linear bias factors are

$$b^L_n = \left\langle\frac{\partial^n\delta_L}{\partial\delta_R^n}\right\rangle = \int_{-\infty}^\infty\frac{d\lambda}{2\pi}\tilde F(\lambda)e^{-\lambda^2\sigma_R^2/2}(i\lambda)^n = \frac{1}{\sqrt{2\pi}\,\sigma_R}\int_{-\infty}^\infty d\delta e^{-\delta^2/2\sigma_R{}^2}\frac{d^nF}{d\delta^n}\equiv\left\langle F^{(n)}\right\rangle\,. \tag{2.150}$$

At the linear order, we have the Eulerian bias $b_1 = 1 + \langle F'\rangle$, derived without assuming spherical collapse.
• compute the scale-dependent coefficients

2.3.5 Perturbation Theory and Nonlocal Bias

Considering the translational invariance (given density fields, coordinates can be freely chosen, i.e., origin and so on), the galaxy fluctuation field can be written in terms of non-local bias (deterministic) functions as

$$\delta_{\rm X}(\boldsymbol{x}) = \sum_{n=1}^\infty\frac{1}{n!}\int d^3x_1\cdots d^3x_n\,b_n(\boldsymbol{x}-\boldsymbol{x}_1,\ldots,\boldsymbol{x}-\boldsymbol{x}_n)\delta_{\rm m}(\boldsymbol{x}_1)\cdots\delta_{\rm m}(\boldsymbol{x}_n)\,, \tag{2.151}$$

and its Fourier transform is

$$\delta_{\rm X}(\boldsymbol{k})=\sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k}_{1\cdots n}-\boldsymbol{k})\times b_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\,\delta_{\rm m}(\boldsymbol{k}_1)\cdots\delta_{\rm m}(\boldsymbol{k}_n)\,,\tag{2.152}$$

where $\boldsymbol{k}_{1\cdots n}\equiv\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n$, the bias functional is (and the renormalized bias functional)

$$b_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)=(2\pi)^{3n}\int\frac{d^3k'}{(2\pi)^3}\left.\frac{\delta^n\delta_{\rm X}(\boldsymbol{k}')}{\delta\delta_m(\boldsymbol{k}_1)\cdots\delta\delta_m(\boldsymbol{k}_n)}\right|_{\delta_{\rm L}=0}\,,\tag{2.153}$$

$$c_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)=(2\pi)^{3n}\int\frac{d^3k'}{(2\pi)^3}\left\langle\frac{\delta^n\delta_{\rm X}(\boldsymbol{k}')}{\delta\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta\delta_m(\boldsymbol{k}_n)}\right\rangle\,,\tag{2.154}$$

and should be rotationally invariant, i.e., $b_1(k)$, $b_2(k_1,k_2,k_{12})$ and so on. Furthermore, the mass density contrast $\delta_{\rm m}(\boldsymbol{x})$ is also a nonlocal and nonlinear functional of a linear density field $\delta_{\rm L}(\boldsymbol{x})$:

$$\delta_{\rm m}(\boldsymbol{k})=\sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k}_{1\cdots n}-\boldsymbol{k})F_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta_{\rm L}(\boldsymbol{k}_n)\,,\tag{2.155}$$

where the SPT kernels are the usual

$$F_1(\boldsymbol{k})=1\,,\qquad F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{10}{7}+\left(\frac{k_1}{k_2}+\frac{k_1}{k_2}\right)\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}+\frac{4}{7}\left(\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\right)^2\,.\tag{2.156}$$

Combining the above, the galaxy fluctuation field is

$$\delta_{\rm X}(\boldsymbol{k})=\sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k}_{1\cdots n}-\boldsymbol{k})K_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta_{\rm L}(\boldsymbol{k}_n)\,,\tag{2.157}$$

where we have $K_1(\boldsymbol{k})=b_1(\boldsymbol{k})$, $K_2(\boldsymbol{k}_1,\boldsymbol{k}_2)=b_1(\boldsymbol{k})F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)+b_2(\boldsymbol{k}_1,\boldsymbol{k}_2)$, and

$$K_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=b_1(\boldsymbol{k})F_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)+[b_2(\boldsymbol{k}_1,\boldsymbol{k}_{23})F_2(\boldsymbol{k}_2,\boldsymbol{k}_3)+\text{cyc.}]+b_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,.\tag{2.158}$$

Similarly, we express the galaxy fluctuation field in the Lagrangian space at initial time as in Eq. (2.151), and the galaxy fluctuation is moved via Lagrangian picture: $\boldsymbol{x}=\boldsymbol{q}+\boldsymbol{\Psi}$ and $(1+\delta_X)d^3x=(1+\delta_{\rm X}^L)d^3q$ as

$$\begin{aligned}\delta_{\rm X}(\boldsymbol{k})&=\int d^3q e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}\left[1+\delta^{\rm L}_{\rm X}(\boldsymbol{q})\right]e^{-i\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q})}-(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k})=\sum_{n+m\geq1}^{\infty}\frac{(-i)^m}{n!m!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}\frac{d^3k'_1}{(2\pi)^3}\cdots\frac{d^3k'_m}{(2\pi)^3}\\&\times(2\pi)^3\delta^3_{\rm D}(\boldsymbol{k}_{1\cdots n}+\boldsymbol{k}'_{1\cdots m}-\boldsymbol{k})b^{\rm L}_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta_{\rm L}(\boldsymbol{k}_n)[\boldsymbol{k}\cdot\tilde{\boldsymbol{\Psi}}(\boldsymbol{k}'_1)]\cdots[\boldsymbol{k}\cdot\tilde{\boldsymbol{\Psi}}(\boldsymbol{k}'_m)]\,.\end{aligned}\tag{2.159}$$

Combining the above, we derive the relation (and the relation between the Eulerian and the Lagrangian bias functionals)

$$\begin{aligned}&K_1(\boldsymbol{k})=\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k})+b^{\rm L}_1(\boldsymbol{k})\,,\\&K_2(\boldsymbol{k}_1,\boldsymbol{k}_2)=\boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_1,\boldsymbol{k}_2)+[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)][\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)]+b^{\rm L}_1(\boldsymbol{k}_1)[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)]+b^{\rm L}_1(\boldsymbol{k}_2)[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)]+b^{\rm L}_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\,,\\&K_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\boldsymbol{k}\cdot\boldsymbol{L}_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)+\{[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)][\boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_2,\boldsymbol{k}_3)]+\text{cyc.}\}+[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)][\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)][\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_3)]\\&+\left\{b^{\rm L}_1(\boldsymbol{k}_1)[\boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_2,\boldsymbol{k}_3)]+\text{cyc.}\right\}+\left\{b^{\rm L}_1(\boldsymbol{k}_1)[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)][\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_3)]+\text{cyc.}\right\}+\left\{b^{\rm L}_2(\boldsymbol{k}_1,\boldsymbol{k}_2)[\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_3)]+\text{cyc.}\right\}+b^{\rm L}_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,.\end{aligned}\tag{2.160}$$

2.3.6 Examples of Bias Models

• Local Lagrangian bias: with $\langle F_X(\delta_R)\rangle=0$

$$\delta^{\rm L}_{\rm X}=F_{\rm X}(\delta_R)\,,\qquad b^{\rm L}_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)=F^{(n)}_{\rm X}(0)\,,\tag{2.161}$$

$$c^{\rm L}_n(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)=\left\langle F^{(n)}_{\rm X}(\delta_R)\right\rangle=(-1)^n\int_{-\infty}^{\infty}d\delta_R\,\mathcal{P}^{(n)}(\delta_R)\,F_{\rm X}(\delta_R)\,,\tag{2.162}$$

where we assume the smoothing kernel is unity $W(kR)=1$, valid in the large scale limit. The thresholded sample is

$$F_{\rm X}(\delta_R)=A\Theta(\delta_R-\delta_{\rm t})-1\,,\qquad A=\langle\Theta(\delta_R-\delta_{\rm t})\rangle^{-1}=\left[\int_{\delta_{\rm t}}^{\infty}d\delta_R\,\mathcal{P}(\delta_{\mathcal{R}})\right]^{-1}\,,\tag{2.163}$$

$$\left\langle F^{(n)}_{\rm X}(\delta_R)\right\rangle=(-1)^nA\int_{\delta_{\rm t}}^{\infty}d\delta_R\,\mathcal{P}^{(n)}(\delta_R)\,.\tag{2.164}$$

Halo model bias factors are spherically symmetric, and hence they are local in both Lagrangian and Eulerian.

• Spherical collapse model: The number density of halos of mass M, identified at redshift z, in a region of Lagrangian radius R_0 in which the linear overdensity extrapolated to the present time is δ_0, is given by

$$n(M,z|\delta_0,R_0)dM=\frac{\bar\rho}{M}f_{\rm MF}(\nu')\,d\ln\nu'\,,\qquad \nu'=\frac{\delta_{\rm c}(z)-\delta_0}{[\sigma^2(M)-\sigma_0^2]^{1/2}}\,,\qquad \sigma_0=\sigma(M_0)\,,\qquad M_0=\frac{4\pi}{3}\bar\rho R_0^3\,. \tag{2.165}$$

The halo of mass M is collapsed at z, while M_0 is assumed uncollapsed at $z=0$, and thus we always have $\delta_{\rm c}(z)>\delta_0$. The conditional number density represents the biasing for the Lagrangian number density of halos. The smoothed density contrast δ_0 of mass modulates the number of halos, and the number density becomes unconditional at $R_0\to\infty$ ($\delta_0\to0$, $\sigma_0\to0$). The density contrast of halos in Lagrangian space is given by

$$\frac{d\ln\nu'}{d\ln\nu}=\frac{\sigma^2(M)}{[\sigma^2(M)-\sigma_0^2]}\,,\qquad \delta_{\rm h}^{\rm L}=\frac{n(M,z|\delta_0,R_0)}{n(M,z)}-1=\frac{\sigma^2(M)}{\sigma^2(M)-\sigma_0^2}\frac{f_{\rm MF}(\nu')}{f_{\rm MF}(\nu)}-1\ \to\ F_X(\delta_{R_0})\,,\qquad \delta_R\to D(z)\delta_0\,. \tag{2.166}$$

To evaluate the bias functions, the derivatives $F_{\rm X}^{(n)}$ need to be derived, and we consider a limit of the peak-background split $\sigma^2(M)\gg\sigma_0^2$ for simplicity. In this limit, we have

$$F_{\rm X}^{(n)}(\delta_R)\simeq\frac{1}{D^n(z)}\left(\frac{\partial}{\partial\delta_0}\right)^n\delta_{\rm h}^{\rm L}=\left(\frac{-1}{D(z)\sigma(M)}\right)^n\frac{f_{\rm MF}^{(n)}(\nu')}{f_{\rm MF}(\nu)}\,, \tag{2.167}$$

$$F_{\rm X}^{(n)}(0)\simeq\left\langle F_{\rm X}^{(n)}(\delta_R)\right\rangle=b_n^{\rm L}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)=c_n^{\rm L}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\,. \tag{2.168}$$

Using the PS, we have

$$b_n^{\rm L}(M)=F_X^{(n)}(0)=\frac{\nu^{n-1}H_{n+1}(\nu)}{{\delta_{\rm c}}^n}\,,\qquad H_n(\nu)=e^{\nu^2/2}\left(-\frac{d}{d\nu}\right)^n e^{-\nu^2/2}\,,\qquad H_{n+1}(\nu)=\nu H_n(\nu)-nH_{n-1}(\nu)\,. \tag{2.169}$$

• Halo model example: Assuming a universal mass function (i.e., only depends on ν), we have

$$n(\boldsymbol{x},M)=-\frac{2\bar\rho_0}{M}\frac{\partial}{\partial M}\Theta[\delta_M(\boldsymbol{x})-\delta_{\rm c}]\,, \tag{2.170}$$

$$\langle\Theta(\delta_M(\boldsymbol{x})-\delta_{\rm c})\rangle=P(M,\delta_{\rm c})=\frac{1}{2}F(\nu)\,,\qquad F(\nu)=\int_\nu^\infty\frac{f_{\rm MF}(\nu)}{\nu}\,d\nu\,, \tag{2.171}$$

where for the original PS Θ is a step function, but it can be modified for other MF. Also, note that for the PS, there is no statistical nature at a given point: it is either collapsed or not. Defining $\Theta^{(n)}(x):=d^n\Theta(x)/dx^n$ and using

$$\frac{\delta^n n(\boldsymbol{x},M)}{\delta\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta\delta_{\rm L}(\boldsymbol{k}_n)}=-\frac{e^{i(\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n)\cdot\boldsymbol{x}}}{(2\pi)^{3n}}\frac{2\bar\rho_0}{M}\frac{\partial}{\partial M}\left[\Theta^{(n)}(\delta_M-\delta_{\rm c})W(k_1R)\cdots W(k_nR)\right]\,, \tag{2.172}$$

$$\frac{\delta\delta_M(\boldsymbol{x})}{\delta\delta_L(\boldsymbol{k}_1)}=\frac{e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}}{(2\pi)^3}W(k_1R)\,,\qquad \frac{\delta n(\boldsymbol{x},M)}{\delta\delta_L(\boldsymbol{k}_1)}=-\frac{2\bar\rho_0}{M}\frac{\partial}{\partial M}\left[\Theta^{(1)}(\delta_M-\delta_{\rm c})\,\frac{e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}}{(2\pi)^3}W(k_1R)\right]\,, \tag{2.173}$$

and noting that $\delta_{\rm h}^{\rm L}(\boldsymbol{x})=n(\boldsymbol{x},M)/n(M)-1$ we have two equivalent expressions

$$\begin{aligned}
c_n^{\rm L}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n) &= \frac{(-1)^n\dfrac{\partial}{\partial M}\left[\dfrac{\partial^n P(M,\delta_{\rm c})}{\partial{\delta_{\rm c}}^n}W(k_1R)\cdots W(k_nR)\right]}{\dfrac{\partial P(M,\delta_{\rm c})}{\partial M}} \\
&= \frac{A_n(M)}{{\delta_{\rm c}}^n}W(k_1R)\cdots W(k_nR)+\frac{A_{n-1}(M)\,{\sigma_M}^n}{{\delta_{\rm c}}^n}\frac{d}{d\ln\sigma_M}\left[\frac{W(k_1R)\cdots W(k_nR)}{{\sigma_M}^n}\right] \\
&= b_n^{\rm L}(M)W(k_1R)\cdots W(k_nR)+\frac{A_{n-1}(M)}{{\delta_{\rm c}}^n}\frac{d}{d\ln\sigma_M}\left[W(k_1R)\cdots W(k_nR)\right]\,,
\end{aligned}$$

$$A_n(M):=\sum_{j=0}^n\frac{n!}{j!}{\delta_{\rm c}}^j\,b_j^{\rm L}(M)\,,\qquad A_n=nA_{n-1}+{\delta_{\rm c}}^n b_n^{\rm L}\,,$$

where $A_0 = 1$, $A_1 = 1 + \delta_c b_1^L(M)$, $A_2 = 2 + 2\delta_c b_1^L(M) + \delta_c^2 b_2^L(M)$, and note that $c_n^L \to b_n^L$ in the limit $k \to 0$ (there are a few steps in the above calculations and c_n^L for $n \leq 2$ are explicitly computed).

• Multivariate Lagrangian bias: for multiple variables χ_α,

$$\chi_\alpha(\boldsymbol{q}) = \int d^3q' U_\alpha(\boldsymbol{q}-\boldsymbol{q}')\delta_{\rm L}(\boldsymbol{q}'), \qquad \delta_{\rm X}^{\rm L} = F_{\rm X}(\chi_1, \chi_2, \ldots) , \qquad \chi_i = \phi \to U = -\frac{Ga^2\bar\rho_m}{|\boldsymbol{q}-\boldsymbol{q}'|} , \tag{2.174}$$

where the last one is an example of additional variable.

$$b_n^{\rm L}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = \sum_{\alpha_1,\ldots,\alpha_n} \left.\frac{\partial^n F_{\rm X}}{\partial\chi_{\alpha_1}\cdots\partial\chi_{\alpha_n}}\right|_{\chi_\alpha=0} U_{\alpha_1}(\boldsymbol{k}_1)\cdots U_{\alpha_n}(\boldsymbol{k}_n) , \tag{2.175}$$

$$c_n^{\rm L}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = \sum_{\alpha_1,\ldots,\alpha_n} \left\langle\frac{\partial^n F_{\rm X}}{\partial\chi_{\alpha_1}\cdots\partial\chi_{\alpha_n}}\right\rangle U_{\alpha_1}(\boldsymbol{k}_1)\cdots U_{\alpha_n}(\boldsymbol{k}_n) . \tag{2.176}$$

The peaks formalism is a ten-dimensional multi-variate case:

$$n_{\rm pk} = \theta\left(\delta_R/\sigma_R - \nu\right)\delta_{\rm D}^3\left(\boldsymbol{\nabla}\delta_R\right)\left|\det\left(\boldsymbol{\nabla\nabla}\delta_R\right)\right|\theta\left(\lambda_3\right) , \tag{2.177}$$

$$(\chi_\alpha) = (\delta_R, \boldsymbol{\nabla}\delta_R, \boldsymbol{\nabla\nabla}\delta_R) , \qquad (U_\alpha) = [W(kR), ik_iW(kR), -k_ik_jW(kR)] , \tag{2.178}$$

and only renormalized bias functionals are defined for peaks:

$$c_1^{\rm L}(\boldsymbol{k}) = \left(A_1 + B_1k^2\right) , \qquad c_2^{\rm L}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \left[A_2 + B_2\left(k_1^2 + k_2^2\right) + C_2\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + D_2k_1^2k_2^2 + E_2\left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2\right)^2\right] , \tag{2.179}$$

where the exact coefficients are calculated by Vincent.

2.3.7 Multi-Point Propagator for Matter and Biased Tracers

The original RPT propagator was identified as one-point propagator, and it was extended to multi-point propagators. Here we consider only one-component (density, not velocity) multi-point propagator for matter and biased object:

$$\left\langle\frac{\delta^n\delta_{\rm m}(\boldsymbol{k})}{\delta\delta_{\rm L}(\boldsymbol{k}_1)\cdots\delta\delta_{\rm L}(\boldsymbol{k}_n)}\right\rangle = (2\pi)^{3-3n}\delta_{\rm D}^3(\boldsymbol{k}-\boldsymbol{k}_{1\cdots n})\Gamma_{\rm m}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) , \qquad \Gamma_{\rm m}^{(n)} = n!2^{-n}\left(\frac{D_{\rm init}}{D}\right)^n \Gamma_{1a_1\cdots a_n}^{(n)} u_{a_1}\cdots u_{a_n} , \tag{2.180}$$

where the latter shows the relation to the original multi-point propagator. Furthermore, the renormalized bias c_n is the multi-point popagators of biasing in Lagrangian space. Thus, we have in terms of the SPT kernels,

$$\Gamma_{\rm m}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = \sum_{m=0}^{\infty}\frac{1}{m!}\int\frac{d^3k_1'}{(2\pi)^3}\cdots\frac{d^3k_m'}{(2\pi)^3}F_{n+m}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{k}_1', \ldots, \boldsymbol{k}_m')\left\langle\delta_{\rm L}(\boldsymbol{k}_1')\cdots\delta_{\rm L}(\boldsymbol{k}_m')\right\rangle , \tag{2.181}$$

and the resulting power spectrum is

$$\begin{aligned} P_{\rm X}(\boldsymbol{k}) &= \sum_{n=0}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}(2\pi)^3\delta_{\rm D}^3(\boldsymbol{k}-\boldsymbol{k}_{1\cdots n})\left|\Gamma_{\rm X}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n)\right|^2 P_{\rm L}(k_1)\cdots P_{\rm L}(k_n) \\ &\to G^2(k,z)P_{\rm lin}(k) + P_{\rm mc}(k,z) , \end{aligned} \tag{2.182}$$

where we can replace F by K for the multi-point propagator for biased objects. The mode-coupling term must arise from the multi-point propagator with $n > 1$, and the one-point nonlinear propagator $G = \Gamma$. Note that as is above, the one-point (or any multi-point) nonlinear propagator can be formally expressed in terms of the SPT kernels, but it cannot be solved analytically.

2.3.8 The Recursion Relation in Lagrangian Perturbation Theory

The recursion relation in SPT is well known, but LPT has no such, because even for the irrotational fluid, the LPT displacement field has transverse component, starting at third order, which makes things complicated. However, the recursion relation for LPT kernel can be derived by comparing to SPT at each order (they should be same when the density field is literally expanded).

The density in the Eulerian space is written in terms of LPT displacement as

$$\delta(\mathbf{k},t) = \int d^3\mathbf{q}\, e^{-i\mathbf{k}\cdot\mathbf{q}} \left[e^{-i\mathbf{k}\cdot\boldsymbol{\Psi}} - 1\right] = \sum_{n=1}^{\infty} D^n(t)\delta^{(n)}(\mathbf{k}) \,, \qquad \delta^{(n)}(\mathbf{k}) = \mathrm{SPT}^{(n)} \,, \tag{2.183}$$

and this matching condition gives

$$F_1^{(s)}(\mathbf{p}_1) = \mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_1) \,, \qquad F_2^{(s)}(\mathbf{p}_1,\mathbf{p}_2) = \mathbf{k}\cdot\mathbf{S}^{(2)}_{L\oplus T}(\mathbf{p}_1,\mathbf{p}_2) + \frac{1}{2}\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_1)\,\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_2) \,,$$

$$F_3^{(s)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = \mathbf{k}\cdot\mathbf{S}^{(3)}_{L\oplus T}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) + \frac{1}{6}\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_1)\,\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_2)\,\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_3) \tag{2.184}$$

$$+\frac{1}{3}\left\{\mathbf{k}\cdot\mathbf{S}^{(1)}_{L\oplus T}(\mathbf{p}_1)\,\mathbf{k}\cdot\mathbf{S}^{(2)}_{L\oplus T}(\mathbf{p}_2,\mathbf{p}_3) + \text{two perms.}\right\} \,, \qquad \mathbf{S}^{(n)}_{L\oplus T} := \mathbf{S}^{(n)}_L + \mathbf{S}^{(n)}_T \,, \tag{2.185}$$

where $\mathbf{k} = \mathbf{p}_{1,\cdots,n}$ and $\mathbf{S}_L$ and $\mathbf{S}_T$ are LPT displacement kernels for longitudinal and transverse parts, respectively. Noting that

$$\mathbf{S}^{(1)}_T = \mathbf{S}^{(2)}_T = 0 \,, \qquad \mathbf{S}^{(n)}_L = \frac{\mathbf{p}_{1\cdots n}}{p^2_{1\cdots n}} S^{(n)}_L \,, \qquad \mathbf{k}\cdot\mathbf{S}^{(n)}_{L\oplus T}(\mathbf{p}_1,\cdots,\mathbf{p}_n) = \mathbf{k}\cdot\mathbf{S}^{(n)}_L(\mathbf{p}_1,\cdots,\mathbf{p}_n) = S^{(n)}_L \,, \tag{2.186}$$

one can derive the n-th longitudinal kernel from the lower order kernels and the n-th order SPT kernel, where $S^{(n)}_L$ is the n-th order scalar function.

The transverse kernel is a bit tricky to derive, while it is not much needed in most cases. Nevertheless, the irrotational condition gives rise to the relation:

$$0 = \nabla_\mathbf{x}\times\boldsymbol{v} \quad\rightarrow\quad \varepsilon_{ijk}\frac{d}{d\eta}\Psi_{k,j} - \varepsilon_{ijk}\Psi_{l,j}\frac{d}{d\eta}\Psi_{l,k} = \Psi_{i,n}\,\varepsilon_{njk}\left(\Psi_{l,j}\frac{d}{d\eta}\Psi_{l,k} - \frac{d}{d\eta}\Psi_{k,j}\right) \,, \tag{2.187}$$

where the time derivative is w.r.t $d\eta = dt/a^2$ and the partial derivatives are with respect to the Lagrangian coordinates. This condition can be expressed perturbatively as

$$C^{(n)}_i \equiv \sum_{p+q=n}\varepsilon_{ijk}\left(\frac{d}{d\eta}\Psi^{(n)}_{k,j} - \Psi^{(p)}_{l,j}\frac{d}{d\eta}\Psi^{(q)}_{l,k}\right) \,, \qquad \therefore\; C^{(n)}_i = -\sum_{p+q=n}\Psi^{(p)}_{i,m}C^{(q)}_m \,, \tag{2.188}$$

and the solution is $C^{(n)}_i = 0$. Therefore, the nth order solution of the transverse kernel satisfies

$$\varepsilon_{ijk}\frac{d}{d\eta}\left(\boldsymbol{\Psi}^{(n)}_T\right)_{k,j} = \sum_{p+q=n}\varepsilon_{ijk}\left(\boldsymbol{\Psi}^{(p)}_L + \boldsymbol{\Psi}^{(p)}_T\right)_{l,j}\frac{d}{d\eta}\left(\boldsymbol{\Psi}^{(q)}_L + \boldsymbol{\Psi}^{(q)}_T\right)_{l,k} \,. \tag{2.189}$$

Furthermore, by denoting that the time evolution of the nth order displacement is $\propto \eta^{-2n}$ ($\equiv D$), we can separate the time evolution of the nth order displacement from its longitudinal and transverse part: $\boldsymbol{\Psi}^{(n)}_L(\eta,\mathbf{q}) \equiv \mathbf{L}^{(n)}(\mathbf{q})\,\eta^{-2n}$, and $\boldsymbol{\Psi}^{(n)}_T(\eta,\mathbf{q}) \equiv \mathbf{T}^{(n)}(\mathbf{q})\,\eta^{-2n}$. Then, we can evaluate the temporal derivatives and obtain

$$\varepsilon_{ijk}T^{(n)}_{k,j} = \frac{1}{2}\sum_{0<p<n}\frac{n-2p}{n}\varepsilon_{ijk}\left(\mathbf{L}+\mathbf{T}\right)^{(p)}_{l,j}\left(\mathbf{L}+\mathbf{T}\right)^{(n-p)}_{l,k} \,. \tag{2.190}$$

• derive the solution

2.4 Redshift-Space Power Spectrum

The distance of objects in cosmology is estimated by measuring the redshift of such objects, and the redshift is affected not only by the cosmological expansion, but also by the peculiar motion of objects. The power spectrum we measure is therefore affected by peculiar motion, and it is called the redshift-space distortion. To separate quantities, those without redshift-space distortion are often called real-space quantities.

The real-space position $\boldsymbol{x}$ is mapped in observation as the redshift-space position $\boldsymbol{s}$ as

$$\boldsymbol{s} = \boldsymbol{x} + \frac{\hat{\boldsymbol{z}}\cdot\boldsymbol{v}}{aH}\,\hat{\boldsymbol{z}} \,, \qquad \boldsymbol{v} = a\dot{\boldsymbol{x}} \,, \tag{2.191}$$

where we assumed that the line-of-sight direction is aligned with z-axis. The redshift-space distortion can be readily incorporated into the Lagrangian picture:

$$\boldsymbol{\Psi}^{\rm s} = \boldsymbol{\Psi} + \frac{\hat{\boldsymbol{z}}\cdot\dot{\boldsymbol{\Psi}}}{H}\,\hat{\boldsymbol{z}} ~, \qquad \dot{\boldsymbol{\Psi}}^{(n)} = nHf\boldsymbol{\Psi}^{(n)} ~, \qquad \boldsymbol{\Psi}^{{\rm s}(n)} = \boldsymbol{\Psi}^{(n)} + nf(\hat{\boldsymbol{z}}\cdot\boldsymbol{\Psi}^{(n)})\,\hat{\boldsymbol{z}} ~\rightarrow~ L_i^{{\rm s}(n)} = [\delta_{ij}+nf\hat z_i\hat z_j]\,L_j^{(n)} ~, \tag{2.192}$$

and the resulting power spectrum at the one-loop level is

$$P_{\rm s}(\boldsymbol{k}) = e^{-k^2[1+f(f+2)\mu^2]\sigma_A^2}\left[(1+f\mu^2)^2P_{\rm L}(k) + P_{\rm sSPT}^{\text{1-loop}}(\boldsymbol{k}) +~ (1+f\mu^2)^2[1+f(f+2)\mu^2]k^2P_{\rm L}(k)\sigma_A^2\right] ~, \tag{2.193}$$

where $\mathcal{H}^2f^2\sigma_A^2 = \sigma_{\rm 1D}^2$ is the one-dimensional velocity dispersion.

2.4.1 Large-Scale Velocity Correlation

The velocity field is very difficult to measure in observation, but the velocity correlation is one of the basic statistics in cosmology. The velocity correlation function is

$$\Psi_{ij}(r) = \langle v_i(x)v_j(x+r)\rangle = \int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathcal{H}^2f^2P(k)\,\frac{k_ik_j}{k^4} = \Psi_\perp(r)\delta_{ij} + \left[\Psi_\parallel(r)-\Psi_\perp(r)\right]\delta_{iz}\delta_{jz} ~, \tag{2.194}$$

where z-direction is the radial direction and

$$\Psi_\perp = \Psi_{xx} = \Psi_{yy} = \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\,\frac{j_1(kr)}{kr} ~, \qquad \Psi_\parallel = \Psi_{zz} = \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\left[j_0(kr) - \frac{2j_1(kr)}{kr}\right] = \frac{d}{dr}\left[r\Psi_\perp(r)\right] ~. \tag{2.195}$$

Finally, we have

$$\sigma_{\rm 3D}^2 = \int\frac{d^3k}{(2\pi)^3}\,\frac{\mathcal{H}^2f^2P(k)}{k^2} = \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k) ~, \qquad \langle\boldsymbol{v}(x)\cdot\boldsymbol{v}(x+r)\rangle = \Psi_\parallel(r) + 2\Psi_\perp(r) ~, \tag{2.196}$$

and $\sigma_{\rm 1D}^2 = \frac{1}{3}\sigma_{\rm 3D}^2$.

Often in the redshift-space distortion literature the velocity is scaled with the conformal Hubble parameters $\tilde{\boldsymbol{v}} = \boldsymbol{v}/\mathcal{H}$ and $\tilde\theta = \theta/\mathcal{H}$, and the velocity power spectrum often mean $P_{\tilde\theta}$. The velocity vector is

$$\boldsymbol{v} = -\frac{1}{k}\,v_{,\alpha} = i\frac{\boldsymbol{k}}{k^2}a\,\dot\delta = \frac{\mathcal{H}f}{k^2}\,\delta_{,\alpha} = i\mathcal{H}f\frac{\boldsymbol{k}}{k^2}\,\delta = \mathcal{H}f\nabla\nabla^{-2}\delta ~, \tag{2.197}$$

$$\theta \equiv -\nabla\cdot\boldsymbol{v} = \mathcal{H}f\delta = -kv ~, \qquad P_\theta = \mathcal{H}^2f^2P(k) ~. \tag{2.198}$$

2.4.2 Gaussian Streaming Model

Fisher (1995): A popular streaming model (Peebles) is the convolution of the real-space correlation function with the relative velocity probability distribution:

$$\xi(r_\sigma, r_\pi) = \int dy\,\xi(r)\,P_v\left[r_\pi - y - \frac{y}{r}\,v_{12}(r)\right] ~, \qquad r^2 = y^2 + r_\sigma^2 ~, \qquad \mathcal{H}\equiv 1 ~, \tag{2.199}$$

where y/r is the cosine angle along the line-of-sight direction and r_π is the redshift-space distance. The PDF is often assumed to be an isotropic Gaussian. However, this streaming model is known to fail to recover the Kaiser formula (Hamilton) in the linear regime. In fact, the PDF is not independent: Consider a vector $\boldsymbol{\eta} = (\delta,~\delta',~v_z,~v_z')$, where primes mean quantities at $\mathbf{x}'$. By using the number conservation, we have

$$1 + \xi(r_\sigma, r_\pi) = \int d^4\boldsymbol{\eta}\,dy(1+\delta)(1+\delta')P_\eta(\boldsymbol{\eta})\,\delta^D(r_\pi - y - v_z' + v_z) ~, \tag{2.200}$$

where the PDF is a Gaussian

$$P_\eta = \frac{1}{(2\pi)^2\sqrt{\det\mathbf{C}}}\,\exp\left(-\frac{1}{2}\boldsymbol{\eta}^\dagger\mathbf{C}^{-1}\boldsymbol{\eta}\right) ~, \qquad \langle v_iv_j'\rangle = \Psi_\perp(r)\delta_{ij} + \left[\Psi_\parallel(r)-\Psi_\perp(r)\right]\hat r_i\hat r_j ~, \tag{2.201}$$

the mean relative velocity above is a number weighted as $\langle\mathbf{v}_{12}(r)\rangle = \langle(\mathbf{v}'-\mathbf{v})(1+\delta)(1+\delta')\rangle \equiv v_{12}(r)\hat{\mathbf{r}}$, and

$$\Psi_\perp(r) = \frac{\beta^2}{2\pi^2}\int dk\,P(k)\,\frac{j_1(kr)}{kr} ~, \qquad \Psi_\parallel(r) = \frac{\beta^2}{2\pi^2}\int dk\,P(k)\left[j_0(kr) - \frac{2j_1(kr)}{kr}\right] ~, \qquad \sigma_v^2 = \frac{\beta^2}{3\,2\pi^2}\int dk\,P(k) ~. \tag{2.202}$$

Using $P_\eta d^4\eta = P_{\eta'} d^4\eta'$, we change the variable η to η' as

$$\boldsymbol{\eta}' = \begin{pmatrix} \delta_+ \\ \delta_- \\ V_+ \\ V_- \end{pmatrix} \equiv \begin{pmatrix} \frac{\delta'+\delta}{\Gamma_+(r)} \\ \frac{\delta'-\delta}{\Gamma_-(r)} \\ \frac{v'+v}{\sigma_+(r)} \\ \frac{v'-v}{\sigma_-(r)} \end{pmatrix} , \qquad \Gamma^2_\pm(r) = 2\left[\xi(0)\pm\xi(r)\right] , \qquad \sigma^2_\pm(r) = 2\left[\sigma_v^2 \pm \left(\frac{y}{r}\right)^2 \Psi_\parallel(r) \pm \left(\frac{r_\sigma}{r}\right)^2\Psi_\perp(r)\right] , \tag{2.203}$$

and the covariance matrix is then

$$\mathbf{C}' = \left\langle \boldsymbol{\eta}'\boldsymbol{\eta}'^\dagger\right\rangle = \begin{pmatrix} 1 & 0 & 0 & \kappa_1 \\ 0 & 1 & -\kappa_2 & 0 \\ 0 & -\kappa_2 & 1 & 0 \\ \kappa_1 & 0 & 0 & 1 \end{pmatrix} , \qquad \mathbf{C}'^{-1} = \begin{pmatrix} \frac{1}{1-\kappa_1^2} & 0 & 0 & \frac{-\kappa_1}{1-\kappa_1^2} \\ 0 & \frac{1}{1-\kappa_2^2} & \frac{\kappa_2}{1-\kappa_2^2} & 0 \\ 0 & \frac{\kappa_2}{1-\kappa_2^2} & \frac{1}{1-\kappa_2^2} & 0 \\ \frac{-\kappa_1}{1-\kappa_1^2} & 0 & 0 & \frac{1}{1-\kappa_1^2} \end{pmatrix} , \tag{2.204}$$

where

$$\kappa_1 = \frac{y}{r}\,\frac{v_{12}(r)}{\Gamma_+\sigma_-} , \qquad \kappa_2 = \frac{y}{r}\,\frac{v_{12}(r)}{\Gamma_-\sigma_+} . \tag{2.205}$$

Therefore, the PDF in terms of new variable is

$$P_{\eta'} = \frac{1}{(2\pi)^2(1-\kappa_1^2)^{1/2}(1-\kappa_2^2)^{1/2}} \exp\left\{-\frac{1}{2}\left[V_+^2 + V_-^2 + \frac{(\sigma_+-\kappa_1V_-)^2}{1-\kappa_1^2} + \frac{(\sigma_--\kappa_2V_+)^2}{1-\kappa_2^2}\right]\right\} , \tag{2.206}$$

and the PDF becomes two independent Gaussians when $\kappa_1 \ll 1$ and $\kappa_2 \ll 1$. Now noting that $(1+\delta)(1+\delta') = 1 + \Gamma_+\delta_+ + (\Gamma_+^2\delta_+^2 - \Gamma_-^2\delta_-^2)/4$, we have

$$1+\xi(r_\sigma, r_\pi) = \int \frac{dy}{\sqrt{2\pi}\sigma_-(r)} \exp\left[-\frac{1}{2}\frac{(r_\pi-y)^2}{\sigma_-^2(r)}\right] \times \left\{1 + \frac{1}{4}(\Gamma_+^2-\Gamma_-^2) + \Gamma_+\kappa_1\frac{r_\pi-y}{\sigma_-(r)} - \frac{1}{4}\kappa_1^2\Gamma_+^2\left[1-\frac{(r_\pi-y)^2}{\sigma_-^2(r)}\right]\right\} , \tag{2.207}$$

where $d^4\eta'$ are integrated out. Finally, once the PDF is expanded to the linear order (lengthy), then it recovers the linear theory. One missing point is that this derivation is based on Gaussianity, while the Kaiser formula is just linear, independent of its Gaussianity.

2.4.3 Complete Treatment of Redshift-space Distortion

Scoccimarro (2004): Starting from the basic relation, we have the conservation relation $(1+\delta_z)d^3\mathbf{s} = (1+\delta)d^3\mathbf{r}$ and $\delta_z(\mathbf{s}) = J^{-1}\big[1+\delta(\mathbf{r})\big] - 1$ with the Jacobian $J = 1 + \nabla_z\Delta$,

$$(2\pi)^3\delta^D(\mathbf{k}) + \delta_z(\mathbf{k}) = \int d^3r\,[1+\delta(\mathbf{x})]\,e^{-i\mathbf{k}\cdot\mathbf{s}} , \qquad s \equiv r + \Delta , \qquad \Delta = V/\mathcal{H} , \tag{2.208}$$

$$\delta_z(\mathbf{k}) = \int d^3r\,(1+\delta-J)\,e^{-i\mathbf{k}\cdot\mathbf{s}} = \int d^3\mathbf{r}\left\{\delta(\mathbf{r}) - \frac{\nabla_zV(\mathbf{r})}{\mathcal{H}(z)}\right\}e^{i(k\mu\,V/H+\mathbf{k}\cdot\mathbf{r})} , \tag{2.209}$$

where two expressions are quite common and largely equivalent. Since with $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r}$, we have

$$(2\pi)^3\delta^D(\mathbf{k}) + P_z(\mathbf{k}) = \int d^3\mathbf{r}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\left\langle e^{-ik_z[\Delta(\mathbf{x}_2)-\Delta(\mathbf{x}_1)]}\,(1+\delta_{\mathbf{x}_1})\,(1+\delta_{\mathbf{x}_2})\right\rangle , \tag{2.210}$$

$$P_z(\mathbf{k}) = \int d^3\mathbf{r}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\left\langle e^{-ik_z[\Delta(\mathbf{x}_2)-\Delta(\mathbf{x}_1)]}\left(\delta - \frac{\nabla_zV}{\mathcal{H}(z)}\right)_{\mathbf{x}_1}\left(\delta - \frac{\nabla_zV}{\mathcal{H}(z)}\right)_{\mathbf{x}_2}\right\rangle ,$$

and by Fourier transforming back we have

$$1+\xi_z(\mathbf{s}) = \int\frac{dk_z}{2\pi}\int d^3\mathbf{r}\,\delta^D(s_x-x)\delta^D(s_y-y)\left\langle e^{ik_z\left(s_z-z-[\Delta_{\mathbf{x}_2}-\Delta_{\mathbf{x}_1}]\right)}\,(1+\delta_{\mathbf{x}_1})\,(1+\delta_{\mathbf{x}_2})\right\rangle , \tag{2.211}$$

$$\xi_z(\mathbf{s}) = \int\frac{dk_z}{2\pi}\int d^3\mathbf{r}\,\delta^D(s_x-x)\delta^D(s_y-y)\left\langle e^{ik_z\left(s_z-z-[\Delta_{\mathbf{x}_2}-\Delta_{\mathbf{x}_1}]\right)}\left(\delta - \frac{\nabla_zV}{\mathcal{H}(z)}\right)_{\mathbf{x}_1}\left(\delta - \frac{\nabla_zV}{\mathcal{H}(z)}\right)_{\mathbf{x}_2}\right\rangle .$$

We further define useful quantities:

$$1+\xi_z(\mathbf{s}) = \int\frac{dk_z}{2\pi}\int dz\,e^{ik_z(s_z-z)}\,[1+\xi(\mathbf{r})]\,\mathcal{M} = \int\frac{dk_z}{2\pi}\int dz\,e^{ik_z(s_z-z)}\,\mathcal{Z} = \int dz\,[1+\xi(\mathbf{r})]\,\mathcal{P} , \tag{2.212}$$

where the pairwise velocity generating function $\mathcal{M}$, the pairwise velocity probability distribution function $\mathcal{P}$,

$$\mathcal{Z} \equiv [1+\xi(\mathbf{r})]\,\mathcal{M}(k_z,\mathbf{r}) \equiv \left\langle e^{-ik_z[\Delta_{\mathbf{x}_2}-\Delta_{\mathbf{x}_1}]}\,(1+\delta_{\mathbf{x}_1})\,(1+\delta_{\mathbf{x}_2})\right\rangle , \qquad \mathcal{P}(s_z-z,\mathbf{r}) = \int \frac{dk_z}{2\pi}\, e^{ik_z(s_z-z)}\mathcal{M}(k_z,\mathbf{r}) . \tag{2.213}$$

The pairwise velocity moments are defined as

$$v_{12}(\mathbf{r}) \equiv \left(\frac{\partial \mathcal{M}}{\partial k_z}\right)_{k_z=0} , \qquad \sigma^2_{12}(\mathbf{r}) \equiv \left(\frac{\partial^2 \mathcal{M}}{\partial k_z^2}\right)_{k_z=0} . \tag{2.214}$$

In summary, we have

$$\xi_z(\mathbf{s}) = \int \frac{dk_z\, dz}{2\pi}\, e^{ik_z(s_z-z)}\,[\mathcal{Z}-1] , \qquad P_z(\mathbf{k}) = \int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\,[\mathcal{Z}-1] , \tag{2.215}$$

and for a Gaussian distribution $\mathcal{Z}$ can be exactly solved.

2.4.4 Improved Model of the Redshift-Space Distortion

The exact redshift-space power spectrum is

$$P_z(\mathbf{k}) = \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left\langle e^{-ik\mu\, f\Delta u_z}\Big\{\delta(\mathbf{r})+f\nabla_z u_z(\mathbf{r})\Big\}\Big\{\delta(\mathbf{r}')+f\nabla_z u_z(\mathbf{r}')\Big\}\right\rangle , \tag{2.216}$$

$$u_z = -\frac{v_z}{\mathcal{H}f} = -i\mu_k\frac{\delta_k}{k} , \qquad \nabla_z u_z = \mu_k^2\delta_k . \tag{2.217}$$

Note that no dynamical information for velocity and density fields, i.e., Euler equation and/or continuity equation, is invoked in deriving this equation. The above equation is in the form of

$$P_z(k,\mu) = \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\big\langle e^{j_1A_1}A_2A_3\big\rangle , \qquad j_1 = -i\,k\mu f , \tag{2.218}$$

$$A_1 = u_z(\mathbf{r}) - u_z(\mathbf{r}') , \qquad A_2(\mathbf{r}) = \delta + f\nabla_z u_z , \qquad A_3(\mathbf{r}') = \delta + f\nabla_z u_z . \tag{2.219}$$

Now consider an arbitrary vector $\mathbf{j} = (j_1, j_2, j_3)$, taking the derivative twice with respect to j_2 and j_3, and setting $j_2 = j_3 = 0$, we derive

$$\langle e^{\mathbf{j}\cdot\mathbf{A}}\rangle = \exp\left\{\langle e^{\mathbf{j}\cdot\mathbf{A}}\rangle_c\right\} \quad\rightarrow\quad \big\langle e^{j_1A_1}A_2\big\rangle = \big\langle e^{j_1A_1}A_2\big\rangle_c \exp\big\langle e^{j_1A_1}\big\rangle_c , \tag{2.220}$$

$$\langle e^{j_1A_1}A_2A_3\rangle = \exp\left\{\langle e^{j_1A_1}\rangle_c\right\}\left[\langle e^{j_1A_1}A_2A_3\rangle_c + \langle e^{j_1A_1}A_2\rangle_c\langle e^{j_1A_1}A_3\rangle_c\right] , \tag{2.221}$$

and therefore, the redshift-space power spectrum is

$$P_z(k,\mu) = \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\; \exp\left\{\langle e^{j_1A_1}\rangle_c\right\}\left[\langle e^{j_1A_1}A_2A_3\rangle_c + \langle e^{j_1A_1}A_2\rangle_c\langle e^{j_1A_1}A_3\rangle_c\right] . \tag{2.222}$$

By comparing to the phenomenolgoical models that incorporate the FoG effect, we deduce that in those models 1) the ensemble average product is zero by setting $j_1 = 0$, while keeping the exponential prefactor

$$\langle e^{j_1A_1}A_2\rangle_c\langle e^{j_1A_1}A_3\rangle_c \simeq \langle A_2\rangle_c\langle A_3\rangle_c = 0 , \tag{2.223}$$

2) the exponential prefactor becomes a Gaussian

$$A_1 = u_z(\mathbf{r}) - u_z(\mathbf{r}') , \qquad j_1 = -ik\mu f , \tag{2.224}$$

$$\exp\left\{\langle e^{j_1A_1}\rangle_c\right\} = \exp\left[\sum_n \frac{j_1^n}{n!}\right]\big\langle A_1^n\big\rangle_c \rightarrow e^{-(k\mu f\sigma_v)^2} \rightarrow \big\langle A_1^n\big\rangle_c \simeq 2\sigma_v^2 , \quad \mathrm{n}=2 , \tag{2.225}$$

i.e., the spatial correlation is ignored and other higher momements are ignored. Based on this observation, they argue that the Gaussian damping term is pretty accurate (in the sense, they get resummed), so we keep the second approximation, but make a perturbative correction to the first approximation. Up to the second order in j_1, we have

$$\begin{aligned}\langle e^{j_1A_1}A_2A_3\rangle_c + \langle e^{j_1A_1}A_2\rangle_c\langle e^{j_1A_1}A_3\rangle_c \quad\simeq\quad & \langle A_2A_3\rangle + j_1\langle A_1A_2A_3\rangle_c \\ & + j_1^2\Big\{\frac{1}{2}\,\langle A_1^2A_2A_3\rangle_c + \langle A_1A_2\rangle_c\langle A_1A_3\rangle_c\Big\} + \mathcal{O}(j_1^3),\end{aligned} \tag{2.226}$$

and the term $\langle A_1^2 A_2 A_3\rangle_c$ turns out to be higher order (not clear, doesn't matter). Therefore, we have with σ_v fitting parameter

$$P_z(k,\mu)=e^{-(k\mu f\sigma_v)^2}\left[P_{\delta\delta}(k)-2f\mu^2P_{\delta\theta}(k)+f^2\mu^4P_{\theta\theta}(k)+A(k,\mu)+B(k,\mu)\right] , \qquad (2.227)$$

where $\theta=-\nabla\cdot\mathbf{v}/(\mathcal{H}f)$, two additional corrections are

$$\begin{aligned}
A(k,\mu) &= j_1\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\langle A_1A_2A_3\rangle_c=(k\mu\, f)\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{q_z}{q^2}\left\{B_\sigma(\mathbf{q},\mathbf{k}-\mathbf{q},-\mathbf{k})-B_\sigma(\mathbf{q},\mathbf{k},-\mathbf{k}-\mathbf{q})\right\} ,\\
B(k,\mu) &= j_1^2\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\langle A_1A_2\rangle_c\langle A_1A_3\rangle_c=(k\mu f)^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,F(\mathbf{q})F(\mathbf{k}-\mathbf{q}) ,\\
F(\mathbf{q}) &= \frac{q_z}{q^2}\left[P_{\delta\theta}(q)+f\frac{q_z^2}{q^2}\,P_{\theta\theta}(q)\right] , \qquad \left\langle\theta_{\mathbf{k}_1}\left(\delta_{\mathbf{k}_2}+f\mu_2^2\theta_{\mathbf{k}_2}\right)\left(\delta_{\mathbf{k}_3}+f\mu_3^2\theta_{\mathbf{k}_3}\right)\right\rangle=(2\pi)^3\delta^D(\mathbf{k}_{123})B_\sigma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) .
\end{aligned}$$

The bispectrum B_σ is also computed to the 1-loop level, and the extension of the above with linear bias is simple.

2.4.5 Summary of Other Phenomenoligical Models of FoG

Some of the popoular models are Eq. (2.227) above, and

$$P_z(k,\mu)=D_{\rm FoG}(k\mu f\sigma_v)\;P(k,\mu) , \qquad P=\begin{cases}(1+f\mu^2)^2P_\delta(k) & \text{linear} ,\\ P_{\delta\delta}(k)+2f\mu^2P_{\delta\theta}(k)+f^2\mu^4\,P_{\theta\theta}(k) & \text{nonlinear} ,\end{cases} \qquad (2.228)$$

where $D(x)=\exp[-x^2]$ for Gaussian, $D=1/(1+x^2)$ for Lorentzian, and some variation $D=1/(1+x^2/2)^2$ (Cole et al., 1995). The nonlinear model is Scoccimarro (2004).

• LPT at the one-loop level in Eq. (2.193) is

$$P_{\rm s}(\boldsymbol{k})=e^{-k^2[1+f(f+2)\mu^2]\sigma_A^2}\left[(1+f\mu^2)^2P_{\rm L}(k)+P_{\rm sSPT}^{\text{1-loop}}(\boldsymbol{k})+\,(1+f\mu^2)^2[1+f(f+2)\mu^2]k^2P_{\rm L}(k)\sigma_A^2\right] , \qquad (2.229)$$

where $\mathcal{H}^2f^2\sigma_A^2=\sigma_{\rm 1D}^2$ is the one-dimensional velocity dispersion, but it is often taken as a free parameter.

2.5 Effective Field Theory

2.5.1 Basic Formalsim

Let $f_n(\mathbf{x},\mathbf{p})$ be the single particle phase space density defined such that $f_n(\mathbf{x},\mathbf{p})\,d^3\mathbf{x}\,d^3\mathbf{p}$ is the probability of particle n occupying an infinitesimal phase space element. For a point particle, the phase space density is

$$f_n(\mathbf{x},\mathbf{p})=\delta_D^3(\mathbf{x}-\mathbf{x}_n)\,\delta_D^3(\mathbf{p}-m\,a\,\mathbf{v}_n) \qquad (2.230)$$

(where both $\mathbf{x}$ and $\mathbf{p}$ are *co-moving*). By summing over n, we define the total phase space density f, mass density ρ, momentum density π^i, kinetic-tensor σ^{ij} as

$$\begin{aligned}
f(\mathbf{x},\mathbf{p}) &= \sum_n\delta_D^3(\mathbf{x}-\mathbf{x}_n)\,\delta_D^3(\mathbf{p}-m\,a\,\mathbf{v}_n) , \qquad \rho(\mathbf{x})=m\,a^{-3}\int d^3\mathbf{p}\,f(\mathbf{x},\mathbf{p})=\sum_n m\,a^{-3}\,\delta_D^3(\mathbf{x}-\mathbf{x}_n) ,\\
\pi^i(\mathbf{x}) &= a^{-4}\int d^3\mathbf{p}\,p^i f(\mathbf{x},\mathbf{p})=\sum_n m\,a^{-3}\,v_n^i\,\delta_D^3(\mathbf{x}-\mathbf{x}_n) , \quad \sigma^{ij}(\mathbf{x})=m^{-1}a^{-5}\int d^3\mathbf{p}\,p^ip^jf(\mathbf{x},\mathbf{p})=\sum_n m\,a^{-3}\,v_n^i\,v_n^j\,\delta_D^3(\mathbf{x}-\mathbf{x}_n)
\end{aligned}$$

The Newtonian potential is sensitive to an infrared quadratic divergence in an infinite homogeneous universe. To isolate this divergence we introduce an exponential infrared regulator with cutoff μ (a mass term) and will take the $\mu\to0$ limit whenever it is allowed. The Newtonian potential ϕ is

$$\phi_n(\mathbf{x}) = -G\,a^2\int d^3\mathbf{x}'\,\frac{\rho_n(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}e^{-\mu|\mathbf{x}-\mathbf{x}'|}=-\frac{mG}{a|\mathbf{x}-\mathbf{x}_n|}e^{-\mu|\mathbf{x}-\mathbf{x}_n|} , \qquad (2.231)$$

$$\phi(\mathbf{x}) = -G\,a^2\int d^3\mathbf{x}'\,\frac{\rho(\mathbf{x}')-\rho_b}{|\mathbf{x}-\mathbf{x}'|}e^{-\mu|\mathbf{x}-\mathbf{x}'|}=\sum_n\phi_n+\frac{4\pi Ga^2\rho_b}{\mu^2} , \quad \mu^2\sum_n\phi_n\to-4\pi Ga^2\rho_b \quad \text{as} \quad \mu\to0 . \qquad (2.232)$$

The k-space version of the Newtonian potential is

$$\phi_n(\mathbf{k}) = -\frac{4\pi mG}{a(k^2+\mu^2)}e^{-i\mathbf{k}\cdot\mathbf{x}_n} , \quad \phi(\mathbf{k}) = \sum_n \phi_n(\mathbf{k}) + \frac{4\pi Ga^2\rho_b}{\mu^2}(2\pi)^3\delta^3_D(\mathbf{k}) , \quad \nabla^2\phi = 4\pi Ga^2(\rho(\mathbf{x},t)-\rho_b(t)) , \tag{2.233}$$

where the final term evidently subtracts out the zero-mode. The collisionless Boltzmann equation becomes in the Newtonian limit:

$$\left(p^\mu\frac{\partial}{\partial x^\mu} + \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta\frac{\partial}{\partial p^\mu}\right)f_n = 0 \ \rightarrow\ 0 = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m\,a^2}\cdot\frac{\partial f}{\partial \mathbf{x}} - m\sum_{\bar n\neq n}\frac{\partial\phi_{\bar n}}{\partial\mathbf{x}}\cdot\frac{\partial f_n}{\partial\mathbf{p}} , \tag{2.234}$$

where the final term now involves a double summation over $\bar n$ and n.

Let us define the following Gaussian smoothing function

$$W_\Lambda(\mathbf{x}) \equiv \left(\frac{\Lambda}{\sqrt{2\pi}}\right)^3\exp\left(-\frac{\Lambda^2\mathbf{x}^2}{2}\right) , \qquad W_\Lambda(k) = \exp\left(-\frac{k^2}{2\,\Lambda^2}\right) , \qquad \int d^3\mathbf{x}\,W(\mathbf{x}) = 1 . \tag{2.235}$$

Of course our results will not depend on the choice of smoothing function, but the Gaussian is chosen for convenience. For certain observables $O(\mathbf{x})$, we will define the smoothed value by the convolution

$$O_l(\mathbf{x}) = [O]_\Lambda(\mathbf{x}) \equiv \int d^3\mathbf{x}'\,W_\Lambda(\mathbf{x}-\mathbf{x}')O(\mathbf{x}') . \tag{2.236}$$

The smoothed versions of the phase space density f_l, mass density ρ_l, momentum density π^i_l, stress-tensor σ^{ij}_l, derivative of Newtonian potential $\partial_i\phi_l$ are

$$f_l(\mathbf{x},\mathbf{p}) = \sum_n W_\Lambda(\mathbf{x}-\mathbf{x}_n)\,\delta^3_D(\mathbf{p}-m\,a\,\mathbf{v}_n) , \qquad \rho_l(\mathbf{x}) = \sum_n m\,a^{-3}\,W_\Lambda(\mathbf{x}-\mathbf{x}_n) , \tag{2.237}$$

$$\pi^i_l(\mathbf{x}) = \sum_n m\,a^{-3}\,v^i_n\,W_\Lambda(\mathbf{x}-\mathbf{x}_n) , \qquad \sigma^{ij}_l(\mathbf{x}) = \sum_n m\,a^{-3}\,v^i_n\,v^j_n\,W_\Lambda(\mathbf{x}-\mathbf{x}_n) , \tag{2.238}$$

where the l-subscript indicates that these only depend on the *long* modes. Similarly, the smoothed version of the Newtonian potential ϕ_l is

$$\phi_{l,n}(\mathbf{x}) = -\frac{mG}{a|\mathbf{x}-\mathbf{x}_n|}\mathrm{Erf}\left[\frac{\Lambda|\mathbf{x}-\mathbf{x}_n|}{\sqrt{2}}\right]e^{-\mu|\mathbf{x}-\mathbf{x}_n|} , \qquad \phi_l(\mathbf{x}) = \sum_n\phi_{l,n} + \frac{4\pi Ga^2\rho_b}{\mu^2} . \tag{2.239}$$

We now write down the smoothed version of the Boltzmann equation by multiplying it by W_Λ and integrating over space

$$0 = \left[\frac{Df}{Dt}\right]_\Lambda = \frac{\partial f_l}{\partial t} + \frac{\mathbf{p}}{m\,a^2}\cdot\frac{\partial f_l}{\partial\mathbf{x}} - m\sum_{n\neq\bar n}\int d^3\mathbf{x}'\,W_\Lambda(\mathbf{x}-\mathbf{x}')\frac{\partial\phi_n}{\partial\mathbf{x}'}(\mathbf{x}')\cdot\frac{\partial f_{\bar n}}{\partial\mathbf{p}}(\mathbf{x}',\mathbf{p}) , \tag{2.240}$$

where we applied the smoothing kernel and integrated (integration by part is needed for the spatial derivative). The zeroth moment gives the continuity equation:

$$\dot\rho_l + 3\,H\rho_l + \frac{1}{a}\partial_i(\rho_l\,v^i_l) = 0 , \qquad v^i_l(\mathbf{x}) \equiv \frac{\pi^i_l(\mathbf{x})}{\rho_l(\mathbf{x})} = \frac{\sum_n v^i_n\,W_\Lambda(\mathbf{x}-\mathbf{x}_n)}{\sum_n W_\Lambda(\mathbf{x}-\mathbf{x}_n)} . \tag{2.241}$$

The first moment gives the Euler equation:

$$\dot v^i_l + Hv^i_l + \frac{1}{a}v^j_l\partial_j v^i_l + \frac{1}{a}\partial_i\phi_l = -\frac{1}{a\,\rho_l}\partial_j\left[\tau^{ij}\right]_\Lambda \equiv -J^i_l , \qquad [\tau^{ij}]_\Lambda \equiv \kappa^{ij}_l + \Phi^{ij}_l , \tag{2.242}$$

where we defined the source term, which is explicitly

$$a\,\rho_b\,J^i_l = \partial_j(\sigma^{ij}_l - \rho_l v^i_l v^j_l) + \sum_{n\neq\bar n}[\rho_n\,\partial_i\phi_{\bar n}]_\Lambda - \rho_l\partial_i\phi_l , \quad [\rho_n\,\partial_i\phi_{\bar n}]_\Lambda = \frac{m^2G(x^i_n - x^i_{\bar n})}{a^4|\mathbf{x}_n-\mathbf{x}_{\bar n}|^3}(1+\mu|\mathbf{x}_n-\mathbf{x}_{\bar n}|)e^{-\mu|\mathbf{x}_n-\mathbf{x}_{\bar n}|}W_\Lambda(\mathbf{x}-\mathbf{x}_n) , \tag{2.243}$$

which requires one to perform a double summation over n and $\bar n$ (which can be computationally expensive). Furthermore, we split the effective-stress tensor into two, where κ^{ij}_l is a type of kinetic dispersion and Φ^{ij}_l is a type of gravitational dispersion:

$$\kappa^{ij}_l = \sigma^{ij}_l - \rho_l v^i_l v^j_l , \qquad \Phi^{ij}_l = -\frac{w^{kk}_l\delta^{ij} - 2w^{ij}_l}{8\pi G\,a^2} + \frac{\partial_k\phi_l\partial_k\phi_l\delta^{ij} - 2\partial_i\phi_l\partial_j\phi_l}{8\pi G\,a^2} , \tag{2.244}$$

where the former describes the dispersion around the mean velocity and the latter describes the higher gravitational multipoles, and we have defined

$$w_l^{ij}(\mathbf{x}) = \int d^3\mathbf{x}'\, W_\Lambda(\mathbf{x}-\mathbf{x}')\Big[\partial_{i'}\phi(\mathbf{x}')\,\partial_{j'}\phi(\mathbf{x}') - \sum_n \partial_{i'}\phi_n(\mathbf{x}')\,\partial_{j'}\phi_n(\mathbf{x}')\Big]\,. \tag{2.245}$$

Note that we have subtracted out the self term in w_l^{ij}, and used $\nabla^2\phi = 4\pi G a^2(\rho-\rho_b)$ and $\nabla^2\phi_l = 4\pi G a^2(\rho_l-\rho_b)$ to express Φ_l in terms of ϕ and ϕ_l. In the limit in which there are no short modes, it is simple to see from the definition of κ_l and Φ_l that they vanish in this limit.

2.5.2 Effective Stress Tensor

The derivation of the effective stress-tensor is as follows. We define the short modes to be

$$\begin{aligned}
\sigma_s^{ij} &\equiv m^{-1}a^{-5}\int d^3\mathbf{p}\,(p^i - p_l^i(\mathbf{x}))(p^j - p_l^j(\mathbf{x}))f(\mathbf{x},\mathbf{p}) = \sum_n \frac{m}{a^3}(v_n^i - v_l^i(\mathbf{x}_n))(v_n^j - v_l^j(\mathbf{x}_n))\,\delta_D^3(\mathbf{x}-\mathbf{x}_n)\,,\\
\phi_{s,n} &\equiv \phi_n - \phi_{l,n}\,,\quad \partial_i\phi_s = \sum_n \partial_i\phi_{s,n}\,,\quad w_s^{ij} \equiv \partial_i\phi_s\,\partial_j\phi_s - \sum_n \partial_i\phi_{s,n}\,\partial_j\phi_{s,n}\,,\quad p_l^i(\mathbf{x}) \equiv m\,a\,v_l^i(\mathbf{x})\,,\\
\sigma_l^{ij} &= \big[\sigma_s^{ij}\big]_\Lambda + \Big[\rho_m v_l^i v_l^j\Big]_\Lambda + \Big[v_l^i(\pi^j - \rho_m v_l^j) + v_l^j(\pi^i - \rho_m v_l^i)\Big]_\Lambda\,,\quad \sigma_s^{ij} \neq \sigma^{ij} - \sigma_l^{ij}\,.
\end{aligned} \tag{2.246}$$

Therefore, we have

$$\begin{aligned}
\kappa_l^{ij} &\simeq \big[\sigma_s^{ij}\big]_\Lambda + \frac{\rho_l \partial_k v_l^i \partial_k v_l^j}{\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\,,\\
\Phi_l^{ij} &\simeq -\frac{[w_s^{kk}]_\Lambda \delta^{ij} - 2[w_s^{ij}]_\Lambda}{8\pi G\, a^2} + \frac{\partial_m\partial_k\phi_l\partial_m\partial_k\phi_l\delta^{ij} - 2\partial_m\partial_i\phi_l\partial_m\partial_j\phi_l}{8\pi G\, a^2\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\,,
\end{aligned} \tag{2.247}$$

where some terms are ignored and expanded (not really derived). Now, we re-arrange the effective stress-tensor as

$$\big[\tau^{ij}\big]_\Lambda = \kappa_l^{ij} + \Phi_l^{ij} \equiv \big[\tau_s^{ij}\big]_\Lambda + \big[\tau^{ij}\big]^{\partial^2}\,,\quad \big[\tau_s^{ij}\big]_\Lambda = \big[\sigma_s^{ij}\big]_\Lambda - \frac{[w_s^{kk}]_\Lambda\delta^{ij} - 2[w_s^{ij}]_\Lambda}{8\pi G\, a^2}\,, \tag{2.248}$$

$$\big[\tau^{ij}\big]^{\partial^2} = \frac{\rho_l\partial_k v_l^i\partial_k v_l^j}{\Lambda^2} + \frac{\partial_m\partial_k\phi_l\partial_m\partial_k\phi_l\delta^{ij} - 2\partial_m\partial_i\phi_l\partial_m\partial_j\phi_l}{8\pi G\, a^2\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\,, \tag{2.249}$$

and by taking the derivative ∂_j this leading piece becomes

$$\partial_j[\tau_s^{ij}]_\Lambda = \partial_j\big[\sigma_s^{ij}\big]_\Lambda + [\rho_s\partial_i\phi_s]_\Lambda\,,\quad [\rho_s\partial_i\phi_s]_\Lambda = \sum_{n\neq\bar n} m\,a^{-3}\partial_i\phi_{s,\bar n}(\mathbf{x}_n)W_\Lambda(\mathbf{x}-\mathbf{x}_n) - [\rho_l\partial_i\phi_s]_\Lambda\,, \tag{2.250}$$

where we have $[\rho_l\partial_i\phi_s]_\Lambda = -\frac{1}{2\Lambda^2}\rho_l\partial_i\partial^2\phi_l + \cdots$ and

$$\sum_{n\neq\bar n} m\,a^{-3}\partial_i\phi_{s,\bar n}(\mathbf{x}_n)W_\Lambda(\mathbf{x}-\mathbf{x}_n) = \sum_{n\neq\bar n}\frac{m^2 G}{a^4}\frac{(x_n - x_{\bar n})^i}{|\mathbf{x}_n-\mathbf{x}_{\bar n}|^3}\left(\text{Erfc}\left[\frac{\Lambda|\mathbf{x}_n-\mathbf{x}_{\bar n}|}{\sqrt{2}}\right] + \frac{4\pi|\mathbf{x}_n-\mathbf{x}_{\bar n}|}{\Lambda^2}W_\Lambda(\mathbf{x}_n-\mathbf{x}_{\bar n})\right)W_\Lambda(\mathbf{x}-\mathbf{x}_n)\,. \tag{2.251}$$

It is of some interest to compute the trace of the stress-tensor, so-called mechanical pressure. This includes the gravitational piece

$$\Phi_l^{ii} = -\frac{w_l^{kk}}{8\pi G\, a^2} + \frac{\partial_k\phi_l\partial_k\phi_l}{8\pi G\, a^2}\,, \tag{2.252}$$

where the first term is approximately given by

$$\begin{aligned}
-\frac{w_l^{kk}}{8\pi G\, a^2} &\approx \frac{1}{2}\int d^3\mathbf{x}'\, W_\Lambda(\mathbf{x}-\mathbf{x}')\Big[\delta\rho(\mathbf{x}')\phi(\mathbf{x}') - \sum_n\delta\rho_n(\mathbf{x}')\phi_n(\mathbf{x}')\Big] \qquad (\nabla\phi)^2 = -\phi\,\nabla^2\phi + \frac{1}{2}\nabla^2(\phi^2)\\
&= -\frac{1}{2}\sum_{n\neq\bar n}\frac{Gm^2}{a^4|\mathbf{x}_n-\mathbf{x}_{\bar n}|}e^{-\mu|\mathbf{x}_n-\mathbf{x}_{\bar n}|}W_\Lambda(\mathbf{x}-\mathbf{x}_n) + \frac{1}{2}\sum_n\frac{4\pi G m\rho_b}{a\mu^2}W_\Lambda(\mathbf{x}-\mathbf{x}_n)\,,
\end{aligned}$$

and we dropped all terms that are suppressed by the ratio of low k-modes to high k-modes. Therefore, the trace of the stress-tensor is roughly

$$[\tau]_\Lambda \approx \int d^3\mathbf{x}'\, W_\Lambda(\mathbf{x}-\mathbf{x}')\Big[\rho(\mathbf{x}')\Big(v_s(\mathbf{x}')^2 + \frac{1}{2}\phi_s(\mathbf{x}')\Big) - \frac{1}{2}\sum_n\rho_{s,n}(\mathbf{x}')\phi_n(\mathbf{x}')\Big]\,. \tag{2.253}$$

The background pressure has the zero mode contribution

$$p_b = \frac{1}{3}\langle[\tau]_\Lambda\rangle \,, \tag{2.254}$$

where we have ignored a correction from the bulk viscosity. There are also stochastic fluctuations to the pressure. Now, since the density field $\rho(\mathbf{x})$ can be arbitrarily large for dense objects on small scales, it suggests that each of the contributions to the renormalized pressure, both the kinetic and the gravitational, can be quite large. However, for virialized structures, these two terms cancel each other. Hence the only significant contribution to the integral comes from modes of order $k \sim k_{\rm nl}$ which have yet to virialize.

The effective stress-tensor $[\tau^{ij}]_\Lambda$ in the Euler equation (first moment) comes from smoothing over the short modes (and the second-moment) and therefore is sourced by the short modes. When it comes to n-point correlation functions, coupling between long modes is connected to the non-linear terms in the continuity and Euler equations, while coupling between long and short modes is connected to the stress-tensor, which generates non-zero correlation functions $\langle[\tau^{ij}]_\Lambda\,\delta_l\rangle$ and $\langle[\tau^{ij}]_\Lambda\,v_l^k\rangle$. In order to make further progress, we write the stress-tensor as an expansion in terms of the long fields, whose coefficients are determined by various correlation functions. This will involves a type of *pressure perturbation* term $\propto \delta^{ij}\,\delta_l$, *a shear viscosity* term $\propto \partial^j v_l^i + \partial^i v_l^j - \frac{2}{3}\delta^{ij}\,\partial_k v_l^k$, and *a bulk viscosity* term $\propto \delta^{ij}\,\partial_k v_l^k$. Demanding rotational symmetry, we write a type of effective field theory expansion for the stress-tensor as

$$[\tau^{ij}]_\Lambda = \rho_b\left[c_s^2\,\delta^{ij}(\gamma^{-1}+\delta_l) - \frac{c_{bv}^2}{Ha}\delta^{ij}\,\partial_k v_l^k - \frac{3}{4}\frac{c_{sv}^2}{Ha}\left(\partial^j v_l^i + \partial^i v_l^j - \frac{2}{3}\delta^{ij}\,\partial_k v_l^k\right)\right] + \Delta\tau^{ij} \,, \tag{2.255}$$

where γ would correspond to the ratio of specific heats in an ordinary fluid (e.g., $\gamma = 5/3$ for an ideal monatonic gas) but here it just parameterizes the background pressure term, c_s is a sound speed, and $c_{sv}\,, c_{bv}$ are viscosity coefficients with units of speed. Note that c_s, c_{sv}, c_{bv} are allowed to depend on time, but not space. Our fluid coefficients are related to the conventional fluid quantities: background pressure p_b, pressure perturbation δp, shear viscosity η, and bulk viscosity ζ by

$$p_b = \frac{c_s^2\rho_b}{\gamma} \,, \qquad \delta p = c_s^2\rho_b\delta_l \,, \qquad \eta = \frac{3\rho_b c_{sv}^2}{4H} \,, \qquad \zeta = \frac{\rho_b c_{bv}^2}{H} \,. \tag{2.256}$$

In addition to those, there is an entire tower of higher order corrections carrying the appropriate rotational symmetry, guaranteed to exist by the principles of effective field theory. These will be parametrically suppressed at low wave number k compared to the non-linear wavenumber $k_{\rm nl}$. Here $\Delta\tau^{ij}$ represents stochastic fluctuations due to fluctuations in the short modes, with $\langle\Delta\tau^{ij}\rangle = 0$.

For later convenience, let's introduce various quantities:

$$a\,J_l^i = \frac{1}{\rho_b}\partial_j\left[\tau^{ij}\right]_\Lambda = c_s^2\,\partial_i\delta_l + \frac{3}{4}c_{sv}^2\,\partial_j\Theta_l^{ji} + \left(\frac{c_{sv}^2}{4}+c_{bv}^2\right)\partial_i\Theta_l \,, \qquad \Theta_l \equiv -\frac{\partial_k v_l^k}{Ha} \,, \quad \Theta_l^{ki} \equiv -\frac{\partial_k v_l^i}{Ha} \,, \tag{2.257}$$

$$a^2 A_l^{ki} \equiv \frac{1}{\rho_b}\partial_k\partial_j\left[\tau^{ij}\right]_\Lambda = a\partial_k J_l^i = c_s^2\,\partial_k\partial_i\delta_l + \frac{3}{4}c_{sv}^2\,\partial_k\partial_j\Theta_l^{ji} + \left(\frac{c_{sv}^2}{4}+c_{bv}^2\right)\partial_k\partial_i\Theta_l \,, \tag{2.258}$$

$$a^2 A_l \equiv \frac{1}{\rho_b}\partial_i\partial_j\left[\tau^{ij}\right]_\Lambda = a\partial_i J_l^i = c_s^2\,\partial^2\delta_l + (c_{sv}^2+c_{bv}^2)\partial^2\Theta_l \,, \qquad a^2 B_l \equiv \frac{1}{\rho_b}\left(\partial_i\partial_j - \frac{\delta^{ij}}{3}\partial^2\right)\left[\tau^{ij}\right]_\Lambda = c_{sv}^2\,\partial^2\Theta_l \,.$$

With these definitions, the EFT parameters are

$$c_s^2 = \frac{P_{A\,\Theta}(x)\,\partial^2 P_{\delta\,\Theta}(x) - P_{A\,\delta}(x)\,\partial^2 P_{\Theta\,\Theta}(x)}{(\partial^2 P_{\delta\,\Theta}(x))^2/a^2 - \partial^2 P_{\delta\,\delta}(x)\,\partial^2 P_{\Theta\,\Theta}(x)/a^2} \,, \quad c_v^2 = \frac{P_{A\,\delta}(x)\,\partial^2 P_{\delta\,\Theta}(x) - P_{A\,\Theta}(x)\,\partial^2 P_{\delta\,\delta}(x)}{(\partial^2 P_{\delta\,\Theta}(x))^2/a^2 - \partial^2 P_{\delta\,\delta}(x)\,\partial^2 P_{\Theta\,\Theta}(x)/a^2} \,,$$

$$c_{sv}^2 = \frac{4}{3}\frac{P_{A^{ki}\,\Theta^{ki}}(x) - P_{A\,\Theta}(x)}{\partial^2 P_{\Theta^{ki}\,\Theta^{ki}}(x)/a^2 - \partial^2 P_{\Theta\,\Theta}(x)/a^2} = \frac{P_{B\,\Theta}(x)}{\partial^2 P_{\Theta\,\Theta}(x)/a^2} \,, \qquad c_v^2 \equiv c_{sv}^2 + c_{bv}^2 \,, \qquad P_{AB}(x) = \langle A(\mathbf{x}+\mathbf{x}')B(\mathbf{x}')\rangle \,,$$

where we defined various correlation function $P_{AB}(x)$. Similarly, in Fourier space, we have

$$c_s^2 = \frac{P_{A\,\Theta}(k)\,P_{\delta\,\Theta}(k) - P_{A\,\delta}(k)\,P_{\Theta\,\Theta}(k)}{-k^2 P_{\delta\,\Theta}(k)^2/a^2 + k^2 P_{\delta\,\delta}(k)\,P_{\Theta\,\Theta}(k)/a^2} \,, \quad c_v^2 = \frac{P_{A\,\delta}(k)\,P_{\delta\,\Theta}(k) - P_{A\,\Theta}(k)\,P_{\delta\,\delta}(k)}{-k^2 P_{\delta\,\Theta}(k)^2/a^2 + k^2 P_{\delta\,\delta}(k)\,P_{\Theta\,\Theta}(k)/a^2} \,,$$

$$c_{sv}^2 = \frac{4}{3}\frac{P_{A^{ki}\,\Theta^{ki}}(k) - P_{A\,\Theta}(k)}{-k^2 P_{\Theta^{ki}\,\Theta^{ki}}(k)/a^2 + k^2 P_{\Theta\,\Theta}(k)/a^2} = \frac{P_{B\,\Theta}(k)}{-k^2 P_{\Theta\,\Theta}(k)/a^2} \,.$$

Let us now compare the relative size of the terms that appear in the Euler equation. We use the Hubble friction term Hv_l^i as the quantity to compare to.

$$\begin{aligned}\frac{\text{Pressure,Viscosity}}{\text{Hubble Friction}} &\sim \frac{c_{s,v}^2\,k\,\delta_l/a}{Hv_l}\sim c_{s,v}^2\left(\frac{k}{Ha}\right)^2\sim\frac{c_{s,v}^2}{10^{-5}c^2}\,\delta_l\sim\delta_l\;,\\ \frac{\text{non-linear Velocity}}{\text{Hubble Friction}} &\sim \frac{k\,v_l^2/a}{Hv_l}\sim 10^{-5}\left(\frac{ck}{Ha}\right)^2\sim\delta_l\;.\end{aligned}\tag{2.259}$$

2.5.3 One-Loop Power Spectrum

In the absence of vorticity and ignoring stochastic fluctuation $\Delta\tau^{ij}$, we can solve the continuity equation and the Euler equation

$$\dot v_l^i+Hv_l^i+v_l^j\partial_jv_l^i+\frac{1}{a}\partial_i\phi_l=-\frac{1}{a}c_s^2\,\partial_i\delta_l+\frac{3c_{sv}^2}{4Ha^2}\partial^2v_l^i+\frac{4c_{bv}^2+c_{sv}^2}{4Ha^2}\partial_i\partial_jv_l^j-\Delta J^i\;,\quad \Delta J^i\equiv\rho_b^{-1}\partial_j\Delta\tau^{ij}/a\;,\tag{2.260}$$

using the standard procedure:

$$\begin{aligned}\frac{d\delta_l}{d\tau}+\theta_l &= -\int\frac{d^3k'}{(2\pi)^3}\alpha(\mathbf{k},\mathbf{k}')\delta_l(\mathbf{k}-\mathbf{k}')\theta_l(\mathbf{k}')\;,\qquad \alpha(\mathbf{k},\mathbf{k}')\equiv\frac{\mathbf{k}\cdot\mathbf{k}'}{(k')^2}\;,\quad \beta(\mathbf{k},\mathbf{k}')\equiv\frac{k^2\,\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{2|\mathbf{k}'|^2|\mathbf{k}-\mathbf{k}'|^2}\;,\\ \frac{d\theta_l}{d\tau}+\mathcal{H}\theta_l+\frac{3}{2}\mathcal{H}^2\Omega_m\delta_l &= -\int\frac{d^3k'}{(2\pi)^3}\beta(\mathbf{k},\mathbf{k}')\theta_l(\mathbf{k}-\mathbf{k}')\theta_l(\mathbf{k}')+c_s^2k^2\delta_l-\frac{c_v^2k^2}{\mathcal{H}}\theta_l\;.\end{aligned}\tag{2.261}$$

The time dependence is removed by assuming the EdS-like behavior, but we have to consider the additional EFT terms, which are approximated as

$$c_s^2(a,\Lambda)=a\,c_{s,0}^2(\Lambda)\;,\quad c_v^2(a,\Lambda)=a\,c_{v,0}^2(\Lambda)\;,\quad C_{s,0}(k)\equiv\frac{c_{s,0}^2k^2}{\mathcal{H}_0^2}\;,\quad C_{v,0}(k)\equiv\frac{c_{v,0}^2k^2}{\mathcal{H}_0^2}\;,\tag{2.262}$$

where the Λ dependence is implied.

For $n>1$ we find the following set of relationships between the fields at different orders

$$\begin{aligned}A_n(\mathbf{k}) &= n\,\delta_n(\mathbf{k})-\theta_n(\mathbf{k})\;,\qquad B_n(\mathbf{k})=3\delta_n(\mathbf{k})-(2n+1)\theta_n(\mathbf{k})-2C_{s,0}(k)\delta_{n-2}(\mathbf{k})-2C_{v,0}(k)\theta_{n-2}(\mathbf{k})\;,\\ A_n(\mathbf{k}) &= \int\frac{d^3k_1}{(2\pi)^3}\int d^3k_2\,\delta_D^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\alpha(\mathbf{k},\mathbf{k}_1)\sum_{m=1}^{n-1}\theta_m(\mathbf{k}_1)\delta_{n-m}(\mathbf{k}_2)\;,\\ B_n(\mathbf{k}) &= -\int\frac{d^3k_1}{(2\pi)^3}\int d^3k_2\,\delta_D^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})2\beta(\mathbf{k},\mathbf{k}_1)\sum_{m=1}^{n-1}\theta_m(\mathbf{k}_1)\theta_{n-m}(\mathbf{k}_2)\;,\end{aligned}$$

and then

$$\begin{aligned}\delta_n(\mathbf{k}) &= \frac{1}{(2n+3)(n-1)}\Big[(2n+1)A_n(\mathbf{k})-B_n(\mathbf{k})-2C_{s,0}(k)\delta_{n-2}(\mathbf{k})-2C_{v,0}(k)\theta_{n-2}(\mathbf{k})\Big]\\ &= \sum_{j=1}^n\int\frac{d^3q_1}{(2\pi)^3}\dots\int d^3q_j\,\delta_D^3(\mathbf{q}_1+\cdots+\mathbf{q}_j-\mathbf{k})F_{n,j}(\mathbf{q}_1,\dots,\mathbf{q}_j)\,\delta_1(\mathbf{q}_1)\dots\delta_1(\mathbf{q}_j)\;,\end{aligned}\tag{2.263}$$

$$\begin{aligned}\theta_n(\mathbf{k}) &= \frac{1}{(2n+3)(n-1)}\Big[3A_n(\mathbf{k})-nB_n(\mathbf{k})-2nC_{s,0}(k)\delta_{n-2}(\mathbf{k})-2nC_{v,0}(k)\theta_{n-2}(\mathbf{k})\Big]\\ &= \sum_{j=1}^n\int\frac{d^3q_1}{(2\pi)^3}\dots\int d^3q_j\,\delta_D^3(\mathbf{q}_1+\cdots+\mathbf{q}_j-\mathbf{k})G_{n,j}(\mathbf{q}_1,\dots,\mathbf{q}_j)\,\delta_1(\mathbf{q}_1)\dots\delta_1(\mathbf{q}_j)\;,\end{aligned}\tag{2.264}$$

where the kernels are

$$\begin{aligned}F_{n,n}(\mathbf{q}_1,\dots,\mathbf{q}_n) &= \sum_{m=1}^{n-1}\frac{G_{m,m}(\mathbf{q}_1,\dots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[(2n+1)\alpha(\mathbf{k},\mathbf{k}_1)F_{n-m,n-m}(\mathbf{q}_{m+1},\dots,\mathbf{q}_n)+2\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m,n-m}(\mathbf{q}_{m+1},\dots,\mathbf{q}_n)\Big]\;,\\ G_{n,n}(\mathbf{q}_1,\dots,\mathbf{q}_n) &= \sum_{m=1}^{n-1}\frac{G_{m,m}(\mathbf{q}_1,\dots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[3\alpha(\mathbf{k},\mathbf{k}_1)F_{n-m,n-m}(\mathbf{q}_{m+1},\dots,\mathbf{q}_n)+2n\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m,n-m}(\mathbf{q}_{m+1},\dots,\mathbf{q}_n)\Big]\;,\end{aligned}$$

and also $F_{n,1}=G_{n,1}=0$ for n even ($G_{n,1}=n\,F_{n,1}$) and

$$F_{n,1}(k) = \prod_{m=3,5,\ldots}^{n}\frac{-2(C_s(k)+(m-2)C_v(k))}{(2m+3)(m-1)}, \qquad \text{for } n \text{ odd} \tag{2.265}$$

$$G_{n,1}(k) = n\prod_{m=3,5,\ldots}^{n}\frac{-2(C_s(k)+(m-2)C_{v,0}(k))}{(2m+3)(m-1)}, \quad \text{for } n \text{ odd}\,. \tag{2.266}$$

At the one-loop order, we have the usual SPT kernels $F_{2,2}^{(s)}$, $G_{2,2}^{(s)}$ ($F_{2,1}=G_{2,1}=0$), and

$$\begin{aligned} F_{3,3}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) &= \frac{1}{18}\Big[7\alpha(\mathbf{k},\mathbf{q}_1)F_{2,2}(\mathbf{q}_2,\mathbf{q}_3)+2\beta(\mathbf{q}_1,\mathbf{q}_2+\mathbf{q}_3)G_{2,2}(\mathbf{q}_2,\mathbf{q}_3)+(7\alpha(\mathbf{k},\mathbf{q}_1+\mathbf{q}_2)+2\beta(\mathbf{q}_1+\mathbf{q}_2,\mathbf{q}_3)G_{2,2}(\mathbf{q}_1,\mathbf{q}_2)\Big]\,, \\ G_{3,3}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) &= \frac{1}{18}\Big[3\alpha(\mathbf{k},\mathbf{q}_1)F_{2,2}(\mathbf{q}_2,\mathbf{q}_3)+6\beta(\mathbf{q}_1,\mathbf{q}_2+\mathbf{q}_3)G_{2,2}(\mathbf{q}_2,\mathbf{q}_3)+(3\alpha(\mathbf{k},\mathbf{q}_1+\mathbf{q}_2)+6\beta(\mathbf{q}_1+\mathbf{q}_2,\mathbf{q}_3)G_{2,2}(\mathbf{q}_1,\mathbf{q}_2)\Big]\,, \\ F_{3,1}(k) &= -\frac{1}{9}(C_{s,0}(k)+C_{v,0}(k))\,, \quad G_{3,1}(k)=-\frac{1}{3}(C_{s,0}(k)+C_{v,0}(k))\,. \end{aligned} \tag{2.267}$$

The final result for P_{13} and P_{22} have two contributions: the contributions from IR modes and the contribution from UV modes, which we write in the following obvious notation

$$P_{13}(k)=P_{13,IR}(k,\Lambda)+P_{13,UV}(k,\Lambda)\,, \qquad P_{22}(k)=P_{22,IR}(k,\Lambda)+P_{22,UV}(k,\Lambda)\,. \tag{2.268}$$

The IR contribution is the usual SPT but with the cutoff

$$\begin{aligned} P_{13,IR}(k,\Lambda) &= 3\,P_{11}(k)\int^{\Lambda}\frac{d^3q}{(2\pi)^3}F_{3,3}^{(s)}(\mathbf{q},-\mathbf{q},\mathbf{k})P_{11}(q) \\ &= \frac{1}{504}\frac{k^3}{4\pi^2}P_{11}(k)\int_0^{\Lambda/k}dr\,P_{11}(k\,r)\left(\frac{12}{r^2}-158+100r^2-42r^4+\frac{3}{r^3}(r^2-1)^3(7r^2+2)\ln\left|\frac{1+r}{1-r}\right|\right)\,, \\ P_{22,IR}(k,\Lambda) &= 2\int^{\Lambda}\frac{d^3q}{(2\pi)^3}\left[F_{2,2}^{(s)}(\mathbf{q},\mathbf{k}-\mathbf{q})\right]^2P_{11}(q)P_{11}(|\mathbf{k}-\mathbf{q}|) \\ &= \frac{1}{98}\frac{k^3}{4\pi^2}\int_0^{\Lambda/k}dr\int_{-1}^{1}dx\,P_{11}(k\,r)P_{11}(k\sqrt{1+r^2-2rx})\frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2}\,, \end{aligned}$$

and the UV contributions are

$$P_{13,UV}(k,\Lambda)=F_{3,1}(k,\Lambda)P_{11}(k)=-\frac{(c_{s,0}^2(\Lambda)+c_{v,0}^2(\Lambda))k^2}{9\mathcal{H}_0^2}P_{11}(k)\,, \qquad P_{22,UV}(k,\Lambda)\equiv\Delta P_{22}(k,\Lambda)\,, \tag{2.269}$$

where P_{13} is set by the (Λ dependent) sound speed and viscosity, and ΔP_{22} is set by the stochastic fluctuations (ignored later). In order to extract the Λ dependence of $P_{13}(k,\Lambda)$ we take the large r limit inside the integrand ($k\ll\Lambda_1<\Lambda$)

$$P_{13,IR}(k,\Lambda)=P_{13,IR}(k,\Lambda_1)-\frac{488}{5}\frac{1}{504}\frac{k^2}{4\pi^2}P_{11}(k)\int_{\Lambda_1}^{\Lambda}dq\,P_{11}(q)\,, \tag{2.270}$$

and this Λ-dependence should be cancelled by the UV piece with the same k-dependence as

$$c_{s,0}^2(\Lambda)+c_{v,0}^2(\Lambda)=\left(\frac{488}{5}\frac{1}{504}\frac{9\mathcal{H}_0^2}{4\pi^2}\int_{\Lambda}dq\,P_{11}(q)\right)+c_{s,0}^2(\infty)+c_{v,0}^2(\infty) \tag{2.271}$$

The constant contributions are determined by explicit matching to numerical simulations. This structure is sort of expected as follows. For the cutoff in the perturbative regime ($\Lambda\lesssim k_{\rm nl}$), the Λ dependence of the fluid parameters is adequately described by the linear theory. This allows us to estimate the value of the fluid parameters c_s^2 and c_v^2 and their time dependence as a function of the linear power spectrum. The sound speed is roughly given by the velocity dispersion, so in linear theory we estimate the sound speed by an integral over the velocity dispersion of the short modes. The linear theory is not applicable at very high k, so we shall include a constant correction as follows

$$c_s^2(a,\Lambda)=\alpha\int_{\Lambda}d\ln q\;\Delta_v^2(q)+c_s^2(a,\infty)\,, \tag{2.272}$$

where Δ_v^2 is the velocity dispersion, α is an $\mathcal{O}(1)$ constant of proportionality (which is fixed as above), since the sound speed arises from integrating out the *short* modes. In the $\Lambda\to\infty$ limit, which we shall eventually take once we cancel the Λ dependence, we find that $c_s^2(\Lambda)$ is non-zero (due to the UV dependence), which we account for with the $c_s^2(\infty)$ constant correction.

For viscous fluids there is a famous dimensionless number which captures its tendency for laminar or turbulent flow; the Reynolds number. The Reynolds number is defined as

$$R_e \equiv \frac{\rho\, vL}{\eta} \sim \frac{HvL}{c_{sv}^2} \sim \frac{H^2 a^2}{c_{sv}^2 k^2}\delta \lesssim 10 \tag{2.273}$$

where η is shear viscosity, ρ is density, v is a characteristic velocity, and L is a characteristic length scale. The Reynolds number is not very large, and the system is therefore not turbulent. Furthermore, if we were to estimate the viscosity by Hubble friction, then we would have $R_e \sim \delta$ and so the Reynolds number would be even smaller in the linear or weakly non-linear regime. For cosmological parameters $\rho_b \sim 3\times10^{-30}$ [g/cm^3], $H = 70$ [km/s/Mpc], and if we take a plausible value for the shear viscosity of $c_{sv}^2 \sim 2\times10^{-7}c^2$, then the viscosity coefficient is found to be $\eta \sim 20\,\text{Pa s}$ which is perhaps surprisingly not too far from unity in SI units. (For instance, it is somewhat similar to the viscosity of some everyday items, such as chocolate syrup.)