3 Nonlinear Relativistic Dynamics: ADM Formalism and its Cosmological Applications

3.1 Arnowitt-Deser-Misner (ADM) Formalism

3.1.1 Basics

The ADM (Arnowitt et al., 1962) equations are based on splitting the spacetime into the spatial and the temporal parts using a normal vector field n_μ and a time-like vector $t^\mu = (1,0,0,0)$ defined by the coordinate system (i.e., coordinate observer). *It is a fully nonlinear description of GR.* The metric is written as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) \\ &= (N^\alpha N_\alpha - N^2)dt^2 + 2N_\alpha dx^\alpha dt + h_{\alpha\beta}dx^\alpha dx^\beta \ , \end{aligned} \tag{3.1}$$

and the individual components are

$$g_{00} =: -N^2 + N^\alpha N_\alpha \ , \qquad g_{0\alpha} =: N_\alpha \ , \qquad g_{\alpha\beta} =: h_{\alpha\beta} \ , \qquad N^\alpha := h^{\alpha\beta}N_\beta \ , \tag{3.2}$$

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0\alpha} = \frac{N^\alpha}{N^2} \ , \qquad g^{\alpha\beta} = h^{\alpha\beta} - \frac{N^\alpha N^\beta}{N^2} \ , \tag{3.3}$$

where N_α is based on $h_{\alpha\beta}$ as the metric and $h^{\alpha\beta}$ is an inverse metric of $h_{\alpha\beta}$. In fact, the ADM metric can be derived by introducing a normal vector n^μ, or a vector normal to a hypersurface dx_3^μ:

$$n_\mu =: (-N,0) \ , \qquad dx_3^\mu = (0, dx^\alpha) \ , \qquad 0 = n_\mu dx_3^\mu \ . \tag{3.4}$$

The lapse function N represents the alignment of the normal vector and a time coordinate direction $dt^\mu = (dt, 0)$:

$$d\tau = g_{\mu\nu}n^\mu dt^\nu = n_\mu dt^\mu = -N dt \ . \tag{3.5}$$

Similarly, wecan define a spatial lapse N_α in terms of the alignment to a hypersurface:

$$ds_3 = g_{\mu\nu}dt^\mu dx_3^\nu = g_{0\alpha}dt\, dx^\alpha =: N_\alpha dt\, dx^\alpha \ . \tag{3.6}$$

Given that the normal vector is timelike, we recover the metric:

$$n_0 =: -N, \qquad n_\alpha \equiv 0 \ , \qquad n^0 = \frac{1}{N} \ , \qquad n^\alpha = -\frac{1}{N}N^\alpha \ , \qquad h_{\alpha\beta} = \mathcal{H}^\mu_\alpha \mathcal{H}^\nu_\beta g_{\mu\nu} = g_{\alpha\beta} \ , \tag{3.7}$$

where

$$\mathcal{H}_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu \ , \tag{3.8}$$

is the projection tensor. The 3-metric $h_{\alpha\beta}$ is in fact the induced 3-metric on the hypersurface.

The fluid quantities in the ADM formalism are the energy density E, the momentum flux J_α, the stress tensor $S_{\alpha\beta}$ (its trace S and traceless part $\bar{S}_{\alpha\beta}$), and they are what an observer flowing with the normal vector n^μ (or normal observer) would measure:

$$\begin{aligned} &E = n_\mu n_\nu T^{\mu\nu} = N^2 T^{00} \ , \qquad J_\alpha = -n_\mu T^\mu_\alpha = N T^0_\alpha \ , \qquad S_{\alpha\beta} = T_{\alpha\beta} \ , \\ &S := h^{\alpha\beta}S_{\alpha\beta} \ , \qquad \bar{S}_{\alpha\beta} := S_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}S \ , \qquad T_{\mu\nu} = E n_\mu n_\nu + \frac{1}{3}S\mathcal{H}_{\mu\nu} + J_\mu n_\nu + J_\nu n_\mu + \bar{S}_{\mu\nu} \ , \end{aligned} \tag{3.9}$$

where J_α and $S_{\alpha\beta}$ are based on $h_{\alpha\beta}$. The ADM formulation is based on the normal observer $u_\mu = n_\mu$, which is defined in relation to metric, completely independent of fluid components with different velocities $u^\mu_{(i)}$. Hence in the multi-component situation, e.g.,

$$E_{\rm tot} = \sum_i E_{(i)} = n_\mu n_\nu \sum_i T^{\mu\nu}_{(i)} \ . \tag{3.10}$$

Connection and Curvature

The extrinsic curvature tensor (of 3-geometry in 4-D spacetime) is introduced as

$$K_{\alpha\beta} := \frac{1}{2N}\left(N_{\alpha:\beta} + N_{\beta:\alpha} - h_{\alpha\beta,0}\right) = -n_{\alpha;\beta} = -N\Gamma^0_{\alpha\beta} , \qquad K := h^{\alpha\beta}K_{\alpha\beta} , \tag{3.11}$$

$$K^{\alpha\beta} = h^{\alpha\gamma}h^{\beta\delta}K_{\gamma\delta} = \frac{1}{2N}\left(N^{\alpha:\beta} + N^{\beta:\alpha} + h^{\alpha\beta}{}_{,0}\right) , \qquad \bar{K}_{\alpha\beta} := K_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}K , \tag{3.12}$$

where $K_{\alpha\beta}$ is based on $h_{\alpha\beta}$ and a colon ':' denotes a covariant derivative based on $h_{\alpha\beta}$. The full 4D connections become:

$$\Gamma^0_{00} = \frac{1}{N}\left(N_{,0} + N_{,\alpha}N^\alpha - K_{\alpha\beta}N^\alpha N^\beta\right) , \qquad \Gamma^0_{0\alpha} = \frac{1}{N}\left(N_{,\alpha} - K_{\alpha\beta}N^\beta\right) , \qquad \Gamma^0_{\alpha\beta} = -\frac{1}{N}K_{\alpha\beta},$$

$$\Gamma^\alpha_{00} = \frac{1}{N}N^\alpha\left(-N_{,0} - N_{,\beta}N^\beta + K_{\beta\gamma}N^\beta N^\gamma\right) + NN^{,\alpha} + N^\alpha{}_{,0} - 2NK^{\alpha\beta}N_\beta + N^{\alpha:\beta}N_\beta,$$

$$\Gamma^\alpha_{0\beta} = -\frac{1}{N}N_{,\beta}N^\alpha - NK^\alpha_\beta + N^\alpha{}_{:\beta} + \frac{1}{N}N^\alpha N^\gamma K_{\beta\gamma}, \qquad \Gamma^\alpha_{\beta\gamma} = \Gamma^{(h)\alpha}{}_{\beta\gamma} + \frac{1}{N}N^\alpha K_{\beta\gamma}, \tag{3.13}$$

where $\Gamma^{(h)\alpha}{}_{\beta\gamma}$ is the 3D connection based on $h_{\alpha\beta}$ as the metric:

$$\Gamma^{(h)\alpha}{}_{\beta\gamma} := \frac{1}{2}h^{\alpha\delta}\left(h_{\beta\delta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}\right) . \tag{3.14}$$

In the same way, the intrinsic curvatures (of 3-geometry) are based on $h_{\alpha\beta}$ as the metric:

$$R^{(h)\alpha}{}_{\beta\gamma\delta} = \Gamma^{(h)\alpha}{}_{\beta\delta,\gamma} - \Gamma^{(h)\alpha}{}_{\beta\gamma,\delta} + \Gamma^{(h)\epsilon}{}_{\beta\delta}\Gamma^{(h)\alpha}{}_{\gamma\epsilon} - \Gamma^{(h)\epsilon}{}_{\beta\gamma}\Gamma^{(h)\alpha}{}_{\delta\epsilon} , \tag{3.15}$$

$$R^{(h)}_{\alpha\beta} = R^{(h)\gamma}{}_{\alpha\gamma\beta} , \qquad R^{(h)} = h^{\alpha\beta}R^{(h)}_{\alpha\beta} , \qquad \bar{R}^{(h)}_{\alpha\beta} := R^{(h)}_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}R^{(h)} . \tag{3.16}$$

The Gauss-Codazzi equation relates the geometric quantities of a 3-hypersurface to the 4D intrinsic curvature tensor as

$$\tilde{R} = R^{(h)} + K^{\alpha\beta}K_{\alpha\beta} + K^2 + \frac{2}{N}\left(-K_{,0} + K_{,\alpha}N^\alpha - N^{:\alpha}{}_\alpha\right) . \tag{3.17}$$

They are fully nonlinear equations.

• ***Background FRW Metric***.— In a homogeneous and isotropic universe, we derive

$$h_{\alpha\beta} = a^2\bar{g}_{\alpha\beta} , \qquad K_{\alpha\beta} = -H\,h_{\alpha\beta} , \qquad K = -3H , \qquad R^{(h)} = R^{(3)}_{\alpha\beta} = 2\hat{K}\bar{g}_{\alpha\beta} , \qquad a^2R^{(h)} = R^{(3)} = 6\hat{K} , \tag{3.18}$$

where $\hat{K} = 0, \pm1$ is the normalized spatial curvature. These are curvatures of 3D space.

• ***Notation convention***.— In the ADM formalism, the spacetime coordinate is simply (t, x^α). When applied to the FLRW universe, the zeroth time coordinate can be the cosmic time $dt = a\,d\eta$, but in that approach one needs conversion to compare with quantities derived in the usual perturbation analysis, where 0-th component is the conformal time. Another approach is to put $dt = ad\eta$ in Eq. (3.2), but the metric components (e.g., $\tilde{g}_{00}$) carry not only the ADM variables, but also the expansion factor a. The other approach *we adopt here* is that the zeroth time component in the ADM formalism is simply considered as the conformal time η, in which easy comparison can be made. *Last*, one has to be careful in lowering and raising indicies in the ADM formalism, as they are all based on $h_{\alpha\beta} = a^2(\bar{g}_{\alpha\beta} + 2C_{\alpha\beta})$. Note, however, it is quite often the case in literature that the 0-th component is a proper time.

When proper time coordinate (with carat below) is used for 0-th component instead of conformal time:

$$N = a\hat{N} , \qquad N_\alpha = a\hat{N}_\alpha , \qquad N^\alpha = a\hat{N}^\alpha , \qquad h_{\alpha\beta} = \hat{h}_{\alpha\beta} , \qquad K_{\alpha\beta} = \hat{K}_{\alpha\beta} , \qquad \frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial\tau} , \tag{3.19}$$

and the Christoffel symbols are related as

$$\Gamma^\eta_{\eta\eta} = \mathcal{H} + a\Gamma^t_{tt} , \qquad \Gamma^\eta_{\eta\alpha} = \Gamma^t_{t\alpha} , \qquad \Gamma^\eta_{\alpha\beta} = \frac{1}{a}\Gamma^t_{\alpha\beta} , \qquad \Gamma^\alpha_{\eta\eta} = a^2\Gamma^\alpha_{tt} , \qquad \Gamma^\alpha_{\eta\beta} = a\Gamma^\alpha_{t\beta} , \tag{3.20}$$

and of course the expressions for spatial quantities like $R^{(h)}$ and 4D quantities R remain unchanged. Note that when written in terms of ADM quantities, their functional form remains unchanged regardless whether the time coordinate is dt or $d\eta$.

3.1.2 Hamiltonian Approach

• see CPT.pdf for details

The Hamiltonian formalism treats q_a and $\dot{q}_a$ of the Lagrangian on a more symmetric footing with the canonical momentum:

$$p_a := \frac{\partial L}{\partial \dot{q}_a} , \qquad H(q_a, p_a, t) \equiv p_a\dot{q}_a - L(q_a, \dot{q}_a, t) . \tag{3.21}$$

This way, instead of second-order differential equations, it deals with first-order differential equations. This mathematical trick to replace one independent variable $\dot{q}_a$ with another independent variable p_a is called the *Legendre transformation*:

$$\dot{p}_a = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = -\frac{\partial H}{\partial q_a} , \qquad \dot{q}_a = \frac{\partial H}{\partial p_a} , \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} = \frac{dH}{dt} , \tag{3.22}$$

where in the first equation we used the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 . \tag{3.23}$$

If the Hamiltonian does not explicitly depend on time, it is a constant of motion.

If the Lagrangian has $\ddot{q}$, it is not in a canonical form. One can introduce $\dot{p}\ddot{q}$ term, in addition to $p\dot{q}$ term in the canonical Hamiltonian approach, but another variable is often introduced to remove $\ddot{q}$ terms. In general, the Hamiltonian approach is applied to canonical systems.

• ***Application to GR***.— In general relativity a Lagrangian formulation is spacetime covariant: An action is specified on a spacetime manifold. However, a Hamiltonian formulation requires a breakup of spacetime into space and time. The first step is to choose a time-like vector t^μ and its hypersurface Σ_t. In the ADM formalism the Ricci scalar is

$$R = 2\left(G_{\mu\nu} - R_{\mu\nu}\right) n^\mu n^\nu . \tag{3.24}$$

The first term can be computed by using the Gauss-Codazzi equation as

$$G_{\mu\nu} n^\mu n^\nu = \frac{1}{2}\left(R^{(h)} - K_{\mu\nu}K^{\mu\nu} + K^2\right) , \tag{3.25}$$

and the second term is

$$R_{\mu\nu} n^\mu n^\nu = R^\rho_{\mu\rho\nu} n^\mu n^\nu = K^2 - K_{\mu\nu}K^{\mu\nu} - \nabla_\mu(n^\mu \nabla_\nu n^\nu) + \nabla_\nu(n^\mu \nabla_\mu n^\nu) , \tag{3.26}$$

where we used the definition of the Riemann tensor in terms of covariant derivatives (Wald E.2.28). Therefore, the Ricci scalar is

$$R = R^{(h)} + K_{\alpha\beta}K^{\alpha\beta} - K^2 + 2\left[\nabla_\mu(n^\mu \nabla_\nu n^\nu) - \nabla_\nu(n^\mu \nabla_\mu n^\nu)\right] , \tag{3.27}$$

where the last two divergent terms will be boundary terms and hence discared. So the Einstein-Hilbert action is now

$$\mathcal{L}_{\rm grav} = \frac{\sqrt{-g}\,R}{16\pi G} = \frac{N\sqrt{h}}{16\pi G}\left[R^{(h)} + K_{\alpha\beta}K^{\alpha\beta} - K^2\right] = \frac{\sqrt{h}}{16\pi G}\left[NR^{(h)} + \frac{E_{\alpha\beta}E^{\alpha\beta} - E^2}{N}\right] , \qquad \sqrt{-g} = N\sqrt{h} , \tag{3.28}$$

and the extrinsic curvature is

$$K_{\alpha\beta} = \frac{1}{2N}\left(N_{\alpha:\beta} + N_{\beta:\alpha} - h_{\alpha\beta,0}\right) = -n_{\alpha;\beta} =: -\frac{E_{\alpha\beta}}{N} , \tag{3.29}$$

where $E_{\alpha\beta}$ is independent of N. The canonical variables are N, N_α, and $h_{\alpha\beta}$, but since the action is independent of $\dot{N}$ or $\dot{N}_\alpha$, they are not dynamical variables.

The equation of motion in the Lagrangian formalism is with respect to $g_{\mu\nu}$ (rather than N, N_α, and $h_{\alpha\beta}$), and this yields the Einstein equation. Here, $g_{\mu\nu}$ and $\partial g_{\mu\nu}$ are independent in the Lagrangian formalism. In the Hamiltonian formalism, we define the canonical conjugate momentum

$$\pi^{\alpha\beta} := \frac{\delta\mathcal{L}}{\delta(h_{\alpha\beta,0})} = \frac{\sqrt{h}}{16\pi G}\left[h^{\alpha\beta}K - K^{\alpha\beta}\right] = \frac{\sqrt{h}}{16\pi GN}\left[E^{\alpha\beta} - h^{\alpha\beta}E\right] , \tag{3.30}$$

in addition to $\pi_N = \pi^\alpha = 0$ for N and N_α. Now the Hamiltonian needs to be written in terms of the canonical variables. Using

$$\pi := h_{\alpha\beta}\pi^{\alpha\beta} = -\frac{\sqrt{h}\,E}{8\pi GN}\,, \qquad E_{\alpha\beta} = \frac{8\pi GN}{\sqrt{h}}\left(2\pi_{\alpha\beta} - \pi\, h_{\alpha\beta}\right)\,, \qquad \frac{\sqrt{h}}{16\pi GN}\left(\frac{E_{\alpha\beta}E^{\alpha\beta} - E^2}{N}\right) = \frac{16\pi GN}{\sqrt{h}}\left(\pi_{\alpha\beta}\pi^{\alpha\beta} - \frac{1}{2}\pi^2\right)\,, \tag{3.31}$$

and also noting from the definition of $E_{\alpha\beta}$

$$\dot{h}_{\alpha\beta} = 2E_{\alpha\beta} + 2N_{(\alpha:\beta)} = \frac{16\pi GN}{\sqrt{h}}\left(2\pi_{\alpha\beta} - \pi h_{\alpha\beta}\right) + 2N_{(\alpha:\beta)}\,, \tag{3.32}$$

the Hamiltonian is then

$$\begin{aligned}\mathcal{H}_{\rm grav} &:= \pi^{\alpha\beta}\dot{h}_{\alpha\beta} - \mathcal{L}_{\rm grav} = \pi^{\alpha\beta}\dot{h}_{\alpha\beta} - \frac{\sqrt{h}NR^{(h)}}{16\pi G} - \frac{16\pi GN}{\sqrt{h}}\left(\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}\pi^2\right) \\ &= N\sqrt{h}\left[-\frac{R^{(h)}}{16\pi G} + \frac{16\pi G}{h}\left(\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}\pi^2\right)\right] - 2N_\beta\pi^{\alpha\beta}{}_{:\alpha} + 2\left(N_\beta\pi^{\alpha\beta}\right)_{:\alpha} =: NH_N + N_\alpha H^\alpha\,,\end{aligned} \tag{3.33}$$

where the last two terms come from $\dot{h}_{\alpha\beta}$

$$N_{(\alpha:\beta)}\pi^{\alpha\beta} = N_{\alpha,\beta}\pi^{\alpha\beta} - \Gamma^{(h)\gamma}_{\alpha\beta}N_\gamma\pi^{\alpha\beta} = \left(N_\alpha\pi^{\alpha\beta}\right)_{:\beta} - N_\alpha\pi^{\alpha\beta}{}_{:\beta}\,, \tag{3.34}$$

and the last term is a spatial boundary term that will be discarded. Therefore, we have only two terms that are linear in the lapse and the shift, defining the constraint equations

$$H_N := \sqrt{h}\left[-\frac{R^{(h)}}{16\pi G} + \frac{16\pi G}{h}\left(\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}\pi^2\right)\right] \equiv 0\,, \tag{3.35}$$

$$H^\alpha := -2\pi^{\alpha\beta}{}_{:\beta} \equiv 0\,. \tag{3.36}$$

Finally, the dynamical equations are

$$\dot{h}_{\alpha\beta} = \frac{\delta\mathcal{H}}{\delta\pi^{\alpha\beta}} = \frac{2N}{\sqrt{h}}\left(\pi_{\alpha\beta} - \frac{1}{2}\pi h_{\alpha\beta}\right) + 2N_{(\alpha:\beta)}\,, \tag{3.37}$$

$$\begin{aligned}\dot{\pi}^{\alpha\beta} &= -\frac{\delta\mathcal{H}}{\delta h_{\alpha\beta}} = -N\sqrt{h}\left(R^{\alpha\beta} - \frac{1}{2}Rh^{\alpha\beta}\right) + \frac{1}{2}N\frac{h^{\alpha\beta}}{\sqrt{h}}\left(\pi^{\gamma\delta}\pi_{\gamma\delta} - \frac{1}{2}\pi^2\right) - \frac{2N}{\sqrt{h}}\left(\pi^{\alpha\gamma}\pi^\beta_\gamma - \frac{1}{2}\pi\pi^{\alpha\beta}\right) \\ &\quad -\sqrt{h}\left(N^{,\alpha:\beta} - h^{\alpha\beta}N_{,\gamma}{}^{:\gamma}\right) + \left(\pi^{\alpha\beta}N^\gamma\right)_{:\gamma} - N^\alpha{}_{:\gamma}\pi^{\gamma\beta} - N^\beta{}_{:\gamma}\pi^{\gamma\alpha}\,.\end{aligned} \tag{3.38}$$

See Appendix E in **?** or Goodnote for details.

3.1.3 FRW Metric and its Connection to ADM variables up to 2nd order

We use the following convention for the metric variables:

$$g_{00} := -a^2\left(1 + 2A\right)\,, \qquad g_{0\alpha} := -a^2B_\alpha\,, \qquad g_{\alpha\beta} := a^2\left(\bar{g}_{\alpha\beta} + 2C_{\alpha\beta}\right), \tag{3.39}$$

where A, B_α and $C_{\alpha\beta}$ are perturbed order variables and are *assumed* to be based on $\bar{g}_{\alpha\beta}$ as the metric. To the second-order, we can write the perturbation variables explicitly as:

$$A \equiv A^{(1)} + A^{(2)}\,, \qquad B_\alpha \equiv B^{(1)}_\alpha + B^{(2)}_\alpha\,, \qquad C_{\alpha\beta} \equiv C^{(1)}_{\alpha\beta} + C^{(2)}_{\alpha\beta}\,. \tag{3.40}$$

The inverse metric expanded to the second-order in perturbation variables is (note $g^{ac}g_{cb} = \delta^a_b$ holds to all orders):

$$\begin{aligned}&g^{00} = \frac{1}{a^2}\left(-1 + 2A - 4A^2 + B_\alpha B^\alpha\right)\,, \qquad g^{0\alpha} = \frac{1}{a^2}\left(-B^\alpha + 2AB^\alpha + 2B_\beta C^{\alpha\beta}\right)\,, \\ &g^{\alpha\beta} = \frac{1}{a^2}\left(\bar{g}^{\alpha\beta} - 2C^{\alpha\beta} - B^\alpha B^\beta + 4C^\alpha_\gamma C^{\beta\gamma}\right)\,.\end{aligned} \tag{3.41}$$

The components of the frame four-vector u^μ are introduced as:

$$\begin{aligned}&u^0 = \frac{1}{a}\left(1 - A + \frac{3}{2}A^2 + \frac{1}{2}V^\alpha V_\alpha - V^\alpha B_\alpha\right)\,, \qquad u^\alpha =: \frac{1}{a}V^\alpha\,, \qquad u_0 = -a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}V^\alpha V_\alpha\right)\,, \\ &u_\alpha = a\left(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta}\right) =: a\,v_\alpha =: a(-v_{,\alpha} + v^{(v)}_\alpha)\,,\end{aligned} \tag{3.42}$$

where V^α is based on $\bar{g}_{\alpha\beta}$. The connections are:

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + A' - 2AA' - A_{,\alpha}B^\alpha + B_\alpha\left(B^{\alpha\prime} + \frac{a'}{a}B^\alpha\right) , \qquad (3.43)\\
\Gamma^0_{0\alpha} &= A_{,\alpha} - \mathcal{H}B_\alpha - 2AA_{,\alpha} + 2\mathcal{H}AB_\alpha - B_\beta C^{\beta\prime}_\alpha + B^\beta B_{[\beta|\alpha]} ,\\
\Gamma^\alpha_{00} &= A^{|\alpha} - B^{\alpha\prime} - \mathcal{H}B^\alpha + A'B^\alpha - 2A_{,\beta}C^{\alpha\beta} + 2C^\alpha_\beta\left(B^{\beta\prime} + \mathcal{H}B^\beta\right) ,\\
\Gamma^0_{\alpha\beta} &= \mathcal{H}(1-2A)\bar{g}_{\alpha\beta} + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta} + \mathcal{H}\bar{g}_{\alpha\beta}\left(4A^2 - B_\gamma B^\gamma\right) - 2A\left(B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta}\right) - B_\gamma\left(2C^\gamma_{(\alpha|\beta)} - C_{\alpha\beta}{}^{|\gamma}\right)\\
\Gamma^\alpha_{0\beta} &= \mathcal{H}\delta^\alpha_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right) + C^{\alpha\prime}_\beta + B^\alpha\left(A_{,\beta} - \mathcal{H}B_\beta\right) + 2C^{\alpha\gamma}\left(B_{[\gamma|\beta]} - C'_{\gamma\beta}\right) ,\\
\Gamma^\alpha_{\beta\gamma} &= \bar{\Gamma}^\alpha_{\beta\gamma} + \mathcal{H}\bar{g}_{\beta\gamma}B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\alpha} - 2C^\alpha_\delta\left(2C^\delta_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\delta}\right) - 2\mathcal{H}\bar{g}_{\gamma\beta}\left(AB^\alpha + B^\delta C^\alpha_\delta\right) + B^\alpha\left(B_{(\beta|\gamma)} + C'_{\beta\gamma} + 2\mathcal{H}C_{\beta\gamma}\right) ,
\end{aligned}$$

where a vertical bar indicates a covariant derivative based on $\bar{g}_{\alpha\beta}$. An index 0 indicates the conformal time η, and a prime indicates a time derivative with respect to η.

The normal-frame vector n^μ has a property $n_\alpha \equiv 0$. Thus we have

$$n^0 = \frac{1}{a}\left(1 - A + \frac{3}{2}A^2 - \frac{1}{2}B^\alpha B_\alpha\right) , \qquad n^\alpha = \frac{1}{a}\left(B^\alpha - AB^\alpha - 2B^\beta C^\alpha_\beta\right) , \qquad n_0 = -a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha\right) . \qquad (3.44)$$

Therefore, the normal frame condition can be derived by imposing $u_\alpha = 0 = V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta} = 0$. Using Eqs. (3.3) and (3.7) the ADM metric variables become:

$$N = a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha\right) , \qquad N_\alpha \equiv -a^2 B_\alpha , \qquad N^\alpha = -B^\alpha + 2B^\beta C^\alpha_\beta , \qquad (3.45)$$

$$h_{\alpha\beta} \equiv g_{\alpha\beta} , \qquad h^{\alpha\beta} = \frac{1}{a^2}\left(\bar{g}^{\alpha\beta} - 2C^{\alpha\beta} + 4C^\alpha_\gamma C^{\beta\gamma}\right) . \qquad (3.46)$$

The connection becomes

$$\Gamma^{(h)\gamma}{}_{\alpha\beta} = \bar{\Gamma}^\gamma_{\alpha\beta} + \left(\bar{g}^{\gamma\delta} - 2C^{\gamma\delta}\right)\left(C_{\delta\alpha|\beta} + C_{\delta\beta|\alpha} - C_{\alpha\beta|\delta}\right) . \qquad (3.47)$$

The extrinsic curvature in Eq. (3.12) gives:

$$\begin{aligned}
K_{\alpha\beta} &= -a\left[\left(\mathcal{H}\bar{g}_{\alpha\beta} + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta}\right)(1-A) + \frac{1}{2}\mathcal{H}\bar{g}_{\alpha\beta}\left(3A^2 - B_\gamma B^\gamma\right) - B_\gamma\left(2C^\gamma_{(\alpha|\beta)} - C_{\alpha\beta}{}^{|\gamma}\right)\right] , \qquad (3.48)\\
K^\alpha_\beta &= -\frac{1}{a}\left[\left(\mathcal{H}\delta^\alpha_\beta + B_{(\beta|\gamma)}\bar{g}^{\alpha\gamma} + C^{\alpha\prime}_\beta\right)(1-A) + \frac{1}{2}\mathcal{H}\delta^\alpha_\beta(3A^2 - B_\gamma B^\gamma) - B_\gamma\left(2C^\gamma_{(\delta|\beta)}\bar{g}^{\alpha\delta} - C^{\alpha|\gamma}_\beta\right) - 2C^{\alpha\gamma}\left(B_{(\gamma|\beta)} + C'_{\gamma\beta}\right)\right] ,\\
K^{\alpha\beta} &= -\frac{1}{a^3}\left[\left(\mathcal{H}\bar{g}^{\alpha\beta} + B^{(\alpha|\beta)} + C'^{\alpha\beta} - 2\mathcal{H}C^{\alpha\beta}\right)(1-A) + \frac{1}{2}\mathcal{H}\bar{g}^{\alpha\beta}\left(3A^2 - B_\gamma B^\gamma\right) + 4\mathcal{H}C^\alpha_\gamma C^{\beta\gamma} - B_\gamma\left(2C^{\gamma(\alpha|\beta)} - C^{\alpha\beta|\gamma}\right)\right.\\
&\qquad \left. -4C^{\gamma(\alpha}C'^{\beta)}_\gamma - C^{\alpha\gamma}\left(B^\beta{}_{|\gamma} + B_\gamma{}^{|\beta}\right) - C^{\beta\gamma}\left(B^\alpha{}_{|\gamma} + B_\gamma{}^{|\alpha}\right)\right] ,\\
K &= -\frac{1}{a}\left[\left(3\mathcal{H} + B^\alpha{}_{|\alpha} + C^{\alpha\prime}_\alpha\right)(1-A) + \frac{3}{2}\mathcal{H}\left(3A^2 - B^\alpha B_\alpha\right) - B^\beta\left(2C^\alpha_{\beta|\alpha} - C^\alpha_{\alpha|\beta}\right) - 2C^{\alpha\beta}\left(C'_{\alpha\beta} + B_{\alpha|\beta}\right)\right]\\
&=: -3H + \kappa = -3H + \left[3(-\dot{\varphi} + H\alpha) - \frac{\Delta}{a^2}\chi\right] + \mathcal{O}(2) ,\\
\bar{K}_{\alpha\beta} &= -a\left\{\left(B_{(\alpha|\beta)} + C'_{\alpha\beta}\right)(1-A) - B_\gamma\left(2C^\gamma_{(\alpha|\beta)} - C_{\alpha\beta}{}^{|\gamma}\right) - \frac{2}{3}C_{\alpha\beta}\left(B^\gamma{}_{|\gamma} + C^{\gamma\prime}_\gamma\right)\right.\\
&\qquad \left. -\frac{1}{3}\bar{g}_{\alpha\beta}\left[\left(B^\gamma{}_{|\gamma} + C^{\gamma\prime}_\gamma\right)(1-A) - B^\gamma\left(2C^\delta_{\gamma|\delta} - C^\delta_{\delta|\gamma}\right) - 2C^{\gamma\delta}\left(C'_{\gamma\delta} + B_{\gamma|\delta}\right)\right]\right\}\\
&\quad \underset{\text{No VT}}{\longrightarrow} -(1-\alpha)\,\chi_{,\alpha|\beta} + 2\chi_{,(\alpha}\varphi_{,\beta)} - \frac{1}{3}\bar{g}_{\alpha\beta}\left[-(1-\alpha)\,\Delta\chi + 2\chi^{,\gamma}\varphi_{,\gamma}\right] \underset{\chi\to 0}{\longrightarrow} 0 ,
\end{aligned}$$

where $\bar{K}_{\alpha\beta} = \sigma_{\alpha\beta}$ of the normal observer and vanishes if $\chi = 0$ (ignoring vector and tensor). To the background and linear order, we have

$$R = 6\left(2H^2 + \dot{H} + \frac{K}{a^2}\right) , \qquad \delta R = 2\left[-\dot{\kappa} - 4H\kappa + \left(\frac{k^2}{a^2} - 3\dot{H}\right)\alpha + 2\,\frac{k^2 - 3K}{a^2}\varphi\right] . \tag{3.49}$$

3.2 Energy-Momentum Tensor in ADM Formalism

3.2.1 General Decomposition

For our purposes, we are not interested in the microscopic states of the systems, but interested in their macroscopic states, often described by the density, the pressure, the temperature, and so on. The energy-momentum tensor for a fluid can be expressed in terms of the fluid quantities measured by an observer with four velocity u^μ as (*the most general decomposition*)

$$T_{\mu\nu} := \rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu} , \qquad 0 = \mathcal{H}_{\mu\nu} u^\nu , \tag{3.50}$$

where $\mathcal{H}_{\mu\nu}$ is the projection tensor and

$$\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu , \qquad \mathcal{H}^\mu_\mu = 3 , \qquad u^\mu q_\mu = 0 = u^\mu \pi_{\mu\nu} , \qquad \pi_{\mu\nu} = \pi_{\nu\mu} , \qquad \pi^\mu_\mu = 0 . \tag{3.51}$$

The variables ρ, p, q_μ and $\pi_{\mu\nu}$ are the energy density, the isotropic pressure (including the entropic one), the (spatial) energy flux and the anisotropic pressure measured by the observer with u_μ, respectively, i.e.,

$$\rho = T_{\mu\nu} u^\mu u^\nu , \qquad p = \frac{1}{3} T_{\mu\nu}\mathcal{H}^{\mu\nu} , \qquad q_\mu = -T_{\rho\sigma} u^\rho \mathcal{H}^\sigma_\mu , \qquad \pi_{\mu\nu} = T_{\rho\sigma}\mathcal{H}^\rho_\mu \mathcal{H}^\sigma_\nu - p\mathcal{H}_{\mu\nu} . \tag{3.52}$$

Remember that these fluid quantities are observer-dependent.

3.2.2 Normal Frame and Its Relation to Energy Frame

The fluid quantities are observer dependent quantities, and the choice of observers to write the energy-momentum tensor (or fluid quantities) is called a choice of frame. This choice is independent of a coordinate choice (or gauge choice). The fluid quantities are best described by the observer moving together with the fluid, i.e., *fluid rest frame*, rather than an observer moving relative to the fluid:

$$T^f_{\mu\nu} = \rho_f u^f_\mu u^f_\nu + p_f \mathcal{H}^f_{\mu\nu} + \pi^f_{\mu\nu} , \qquad 0 = \mathcal{H}^f_{\mu\nu} u^\nu_f , \tag{3.53}$$

where u^μ_f is the fluid velocity and the fluid quantities (ρ_f, p_f, etc) are the those in the fluid rest frame (defined by no spatial energy flux $q^\mu_f = 0$). In the presence of multiple fluids with different fluid velocities, the energy momentum tensor needs to be summed over the fluid components:

$$T^{\rm tot}_{\mu\nu} = \sum_f T^f_{\mu\nu} , \qquad \rho^{\rm tot}_{\rm obs} = T_{\mu\nu} u^\mu_{\rm obs} u^\nu_{\rm obs} \neq \sum_f \rho_f . \tag{3.54}$$

The ADM normal observer is set by a coordinate, not by any fluids. Hence, the fluid quantities (E, S, etc) described by the ADM observer (or in the ADM formalism) are different from the fluid quantities in the rest frame (ρ_f, p_f, etc):

$$T^{(i)}_{\mu\nu} = E_{(i)} n_\mu n_\nu + \frac{1}{3} S_{(i)}\mathcal{H}_{\mu\nu} + J^{(i)}_\mu n_\nu + J^{(i)}_\nu n_\mu + \bar{S}^{(i)}_{\mu\nu} , \tag{3.55}$$

where $E_{(i)}$, $S_{(i)}$, etc are the fluid quantities measured by the normal observer. Hence in the multi-component situation, the total energy density measured by the normal observer is simply.

$$E_{\rm tot} = \sum_i E_{(i)} = n_\mu n_\nu \sum_i T^{\mu\nu}_{(i)} . \tag{3.56}$$

Given that the general observer velocity u_α can be set equal to the normal observer by requiring that

$$u_\alpha = a\left(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta}\right) = a\, v_\alpha = 0 , \tag{3.57}$$

we derive the relation for the fluid quantities in two frames, *normal frame* vs *energy frame* (or rest frame), up to the second-order in perturbations:

$$Q^N_\alpha = (\rho+p)\left(V^E_\alpha - B_\alpha\right) + (\rho+p)\left(AB_\alpha + 2V^{E\beta}C_{\alpha\beta}\right) + \left(\delta\rho^E + \delta p^E\right)\left(V^E_\alpha - B_\alpha\right) + \left(V^{E\beta} - B^\beta\right)\Pi^E_{\alpha\beta} \,, \tag{3.58}$$

$$\delta\rho^N = \delta\rho^E + (\rho+p)\left(V^{E\alpha} - B^\alpha\right)\left(V^E_\alpha - B_\alpha\right) = \delta\rho^E + \frac{1}{\rho+p}Q^{N\alpha}Q^N_\alpha \,, \tag{3.59}$$

$$\delta p^N = \delta p^E + \frac{1}{3}(\rho+p)\left(V^{E\alpha} - B^\alpha\right)\left(V^E_\alpha - B_\alpha\right) = \delta p^E + \frac{1}{3}\frac{1}{\rho+p}Q^{N\alpha}Q^N_\alpha \,, \tag{3.60}$$

$$\begin{aligned}\Pi^N_{\alpha\beta} &= \Pi^E_{\alpha\beta} + (\rho+p)\left(V^E_\alpha - B_\alpha\right)\left(V^E_\beta - B_\beta\right) - \frac{1}{3}\bar g_{\alpha\beta}(\rho+p)\left(V^{E\gamma} - B^\gamma\right)\left(V^E_\gamma - B_\gamma\right) \\ &= \Pi^E_{\alpha\beta} + \frac{1}{\rho+p}\left(Q^N_\alpha Q^N_\beta - \frac{1}{3}\bar g_{\alpha\beta}Q^{N\gamma}Q^N_\gamma\right) \,,\end{aligned} \tag{3.61}$$

where the spatial energy flux is zero by definition ($q^E_\alpha =: aQ^E_\alpha = 0$).

3.3 Fully Nonlinear Einstein Equations

3.3.1 ADM Equations

Spacetime is unified in general relativity, but to follow the evolution of a system at some initial time, we need to *undo* the unification and split space and time, i.e., need to cast the Einstein's equation into Cauchy (initial-value) problem — construct initial data consistent with the constraint equations, then solve the evolution (dynamical) equations. A simple analogy to E&M is that the Maxwell's equation is composed of two constraint equations (no time evolution)

$$C_E \equiv \nabla\cdot\mathbf{E} - 4\pi\rho = 0 \,, \qquad C_B \equiv \nabla\cdot\mathbf{B} = 0 \,, \tag{3.62}$$

and two dynamical equations

$$\frac{\partial}{\partial t}\mathbf{E} = \nabla\times\mathbf{B} - 4\pi\mathbf{j} \,, \qquad \frac{\partial}{\partial t}\mathbf{B} = -\nabla\times\mathbf{E} \,. \tag{3.63}$$

The source-free Lagrangian for E&M is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(B^2 - E^2) = \frac{1}{2}\left(\dot{\mathbf{A}} + \nabla\phi\right)\cdot\left(\dot{\mathbf{A}} + \nabla\phi\right) - \frac{1}{2}(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{A}) \,, \tag{3.64}$$

where we used the four vector potential A^μ

$$A^\mu = (\phi, \mathbf{A}) \,, \qquad \mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi \,, \qquad \mathbf{B} = \nabla\times\mathbf{A} \,. \tag{3.65}$$

Given the canonical momentum

$$\pi_\phi := \frac{\partial\mathcal{L}}{\partial\dot\phi} = 0 \,, \qquad \pi_{\mathbf{A}} := \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}} = -\mathbf{E} \,, \tag{3.66}$$

it is clear that the electric potential ϕ is not a dynamical variable, as no time-derivative appears in the Lagrangian. The Hamiltonian for the source-free E&M is

$$H = \dot{\mathbf{A}}\cdot\boldsymbol{\pi}_{\mathbf{A}} - \mathcal{L} = \frac{1}{2}\Big[\boldsymbol{\pi}_{\mathbf{A}}\cdot\boldsymbol{\pi}_{\mathbf{A}} + (\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{A})\Big] - \boldsymbol{\pi}_{\mathbf{A}}\cdot\nabla\phi \,, \tag{3.67}$$

and the last term can be manipulated to yield the surface term and the constraint equation:

$$\boldsymbol{\pi}_{\mathbf{A}}\cdot\nabla\phi = \nabla\cdot(\phi\boldsymbol{\pi}_{\mathbf{A}}) - \phi\nabla\cdot\boldsymbol{\pi}_{\mathbf{A}} \,. \tag{3.68}$$

Using the Hamiltonian formalism, the constrain equation is obtained as

$$0 = \dot\pi_\phi = -\frac{\partial H}{\partial\phi} = \nabla\cdot\boldsymbol{\pi}_{\mathbf{A}} = C_E \,, \tag{3.69}$$

(the other constraint $C_B = 0$ is trivially satisfied, given $\mathbf{B} = \nabla\times\mathbf{A}$). Two dynamical equations yield

$$\dot{\mathbf{A}} = \frac{\partial H}{\partial\boldsymbol{\pi}_{\mathbf{A}}} = \boldsymbol{\pi}_{\mathbf{A}} - \nabla\phi = -\mathbf{E} - \nabla\phi \,, \qquad \therefore\ \dot{\mathbf{B}} = -\nabla\times\mathbf{E} \,, \tag{3.70}$$

$$\dot{\boldsymbol{\pi}}_{\mathbf{A}} = -\frac{\partial H}{\partial\mathbf{A}} = -\frac{1}{2}\frac{\partial}{\partial\mathbf{A}}B^2 \,, \qquad \therefore\ \dot{\mathbf{E}} = \nabla\times\mathbf{B} \,. \tag{3.71}$$

A complete set of the ADM equations is the following Bardeen (1980). The Einstein equation ($G_{ab} \propto T_{ab}$) is split: 00-part and 0α-part, involving non-dynamical quantities N and N^α (the Lagrangian is independent of their time derivatives). These give two constraint equations that relate the energy-momentum to the extrinsic and the intrinsic geometry. $\alpha\beta$-part involves dynamics of $h_{\alpha\beta}$: Two propagation equations (i.e., time derivatives, how they evolve); Finally, there exist two usual conservation equations from the energy-momentum tensor. Energy constraint and momentum constraint equations:

$$R^{(h)} = \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{2}{3}K^2 + 16\pi GE + 2\Lambda \,, \qquad \bar{K}^\beta_{\alpha:\beta} - \frac{2}{3}K_{,\alpha} = 8\pi GJ_\alpha \,. \tag{3.72}$$

Trace and tracefree ADM propagation equations:

$$K_{,0}N^{-1} - K_{,\alpha}N^\alpha N^{-1} + N^{:\alpha}{}_\alpha N^{-1} - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{1}{3}K^2 - 4\pi G\left(E + S\right) + \Lambda = 0 \,, \tag{3.73}$$

$$\bar{K}^\alpha_{\beta,0}N^{-1} - \bar{K}^\alpha_{\beta:\gamma}N^\gamma N^{-1} + \bar{K}_{\beta\gamma}N^{\alpha:\gamma}N^{-1} - \bar{K}^\alpha_\gamma N^\gamma{}_{:\beta}N^{-1} = K\bar{K}^\alpha_\beta - \left(N^{:\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta N^{:\gamma}{}_\gamma\right)N^{-1} + \bar{R}^{(h)\alpha}{}_\beta - 8\pi G\bar{S}^\alpha_\beta \,.$$

Energy and momentum conservation equations:

$$E_{,0}N^{-1} - E_{,\alpha}N^\alpha N^{-1} - K\left(E + \frac{1}{3}S\right) - \bar{S}^{\alpha\beta}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2J^\alpha\right)_{:\alpha} = 0 \,,$$

$$J_{\alpha,0} - J_{\alpha:\beta}N^\beta - J_\beta N^\beta{}_{:\alpha} - KJ_\alpha N + EN_{,\alpha} + NS^\beta_{\alpha:\beta} + S^\beta_\alpha N_{,\beta} = 0 \,. \tag{3.74}$$

In the multi-component system, two conservation equations hold separately for each component, and interaction terms should be considered if there is any between components.

3.3.2 Second-Order Equations

The basic set of the ADM equations is derived with fluid quantities based on the normal-frame. By using Eq. (3.58) we can recover the equations with fluid quantities based on the energy frame. *The fluid quantities are based on the energy frame here!*

The definition of the extrinsic curvature $K = \bar{K} + \delta K$:

$$\bar{K} + 3H + \delta K - 3HA + \dot{C}^\alpha_\alpha + \frac{1}{a}B^\alpha{}_{|\alpha} \equiv n_0 \,, \tag{3.75}$$

where the quadratic terms n_0 can be read from Eq. (3.48).

Energy constraint equation:

$$16\pi G\mu + 2\Lambda - 6H^2 - \frac{6}{a^2}\hat{K} + 16\pi G\delta\mu + 4H\delta K - \frac{1}{a^2}\left(2C^{\beta|\alpha}_\alpha{}_\beta - 2C^{\alpha|\beta}_\alpha{}_\beta - 4\hat{K}\,C^\alpha_\alpha\right) \equiv n_1 \,. \tag{3.76}$$

Momentum constraint equation:

$$\left[\dot{C}^\beta_\alpha + \frac{1}{2a}\left(B^\beta{}_{|\alpha} + B_\alpha{}^{|\beta}\right)\right]_{|\beta} - \frac{1}{3}\left(\dot{C}^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right)_{,\alpha} + \frac{2}{3}\delta K_{,\alpha} + 8\pi Ga(\mu + p)(-v_{,\alpha} + v^{(v)}_\alpha) \equiv n_{2\alpha} \,. \tag{3.77}$$

Trace of the ADM propagation equation:

$$-\left[3\dot{H} + 3H^2 + 4\pi G\left(\mu + 3p\right) - \Lambda\right] + \delta\dot{K} + 2H\delta K - 4\pi G\left(\delta\mu + 3\delta p\right) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)A \equiv n_3 \,. \tag{3.78}$$

Trace-free ADM propagation equation:

$$\left[\dot{C}^\alpha_\beta + \frac{1}{2a}\left(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha}\right)\right]^{\cdot} + 3H\left[\dot{C}^\alpha_\beta + \frac{1}{2a}\left(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha}\right)\right] - \frac{1}{a^2}A^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\left[\left(\dot{C}^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right)^{\cdot} + 3H\left(\dot{C}^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right) - \frac{1}{a^2}A^{|\gamma}{}_\gamma\right]$$
$$+\frac{1}{a^2}\left[C^{\alpha\gamma}{}_{|\beta\gamma} + C^{\gamma|\alpha}_\beta{}_\gamma - C^{\alpha|\gamma}_\beta{}_\gamma - C^{\gamma|\alpha}_\gamma{}_\beta - 4\hat{K}\,C^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta\left(2C^{\delta|\gamma}_\gamma{}_\delta - 2C^{\gamma|\delta}_\gamma{}_\delta - 4\hat{K}\,C^\gamma_\gamma\right)\right] - 8\pi G\,\Pi^\alpha_\beta \equiv n_{4\beta}^{\ \alpha} \,. \tag{3.79}$$

Energy conservation equation:

$$[\dot{\mu}+3H(\mu+p)]+\delta\dot{\mu}+3H(\delta\mu+\delta p)-(\mu+p)(\delta K-3HA)+\frac{1}{a}(\mu+p)\left[V^\alpha-B^\alpha+AB^\alpha+2V^\beta C^\alpha_\beta\right]_{|\alpha}\equiv n_5\,. \quad (3.80)$$

Momentum conservation equation:

$$\frac{1}{a^4}\left[a^4(\mu+p)\left(V_\alpha-B_\alpha+AB_\alpha+2V^\beta C_{\alpha\beta}\right)\right]^{\cdot}+\frac{1}{a}(\mu+p)A_{,\alpha}+\frac{1}{a}\left(\delta p_{,\alpha}+\Pi^\beta_{\alpha|\beta}\right)\equiv n_{6\alpha}\,. \quad (3.81)$$

For the multi-fluid system, the conservation equations become modified with energy transfer among different fluids. Energy conservation equation for the i-th component:

$$\left[\dot{\mu}_{(i)}+3H\left(\mu_{(i)}+p_{(i)}\right)+\frac{1}{a}I_{(i)0}\right]+\delta\dot{\mu}_{(i)}+3H\left(\delta\mu_{(i)}+\delta p_{(i)}\right)-\left(\mu_{(i)}+p_{(i)}\right)(\delta K-3HA)$$
$$+\frac{1}{a}\left(\mu_{(i)}+p_{(i)}\right)\left[V^\alpha_{(i)}-B^\alpha+AB^\alpha+2V^\beta_{(i)}C^\alpha_\beta\right]_{|\alpha}+\frac{1}{a}\delta I_{(i)0}\equiv n_{(i)5}\,. \quad (3.82)$$

Momentum conservation equation for the i-th component:

$$\frac{1}{a^4}\left[a^4\left(\mu_{(i)}+p_{(i)}\right)\left(V_{(i)\alpha}-B_\alpha+AB_\alpha+2V^\beta_{(i)}C_{\alpha\beta}\right)\right]^{\cdot}+\frac{1}{a}\left(\mu_{(i)}+p_{(i)}\right)A_{,\alpha}+\frac{1}{a}\left(\delta p_{(i),\alpha}+\Pi_{(i)\alpha|\beta}^{\ \ \ \ \ \beta}-\delta I_{(i)\alpha}\right)\equiv n_{(i)6\alpha}\,. \quad (3.83)$$

All the quadratic terms n_i can be found in Hwang and Noh (2007).