7 CMB Temperature Anisotropies

In the early Universe, the radiation dominates the overall energy density, and due to high pressure the fluctuations cannot grow within the horizon. In particular, the tight-coupling between the baryons and the photons leads to a single fluid, or the baryon-photon fluid, oscillating with a unique sound speed. As a fluid, the density (monopole) and the velocity (dipole) characterize the fluid, and the higher multipoles are negligible. Once the baryons recombine at later time, the photons are released and free-stream to the observer today. This free-streaming of the monopole and the dipole generates the temperature anisotropies we measure today, and they show the acoustic oscillations of the baryon-photon fluid at the recombination epoch.

In this chapter, we will ignore the vector and tensor perturbation, and choose the conformal Newtonian gauge:

$$\alpha \to \alpha_\chi \,, \qquad \varphi \to \varphi_\chi \,, \qquad \beta = \gamma = \chi \equiv 0 \,, \qquad \mathcal{U} \to v_\chi \,, \qquad V_\alpha = -v_{\chi,\alpha} + v^{(v)}_\alpha \to -v_{\chi,\alpha} \,. \tag{7.1}$$

7.1 Collisionless Boltzmann Equation

The Liouville theorem in GR states that the phase-space volume is conserved along the path parametrized by λ with momentum p^μ:

$$0 = \Delta(dN) = \left(\frac{\partial f}{\partial x^\mu}\Delta x^\mu + \frac{\partial f}{\partial p^\mu}\Delta p^\mu\right) dV_p = \left(p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu}\right) \Delta\lambda \, dV_p \,. \tag{7.2}$$

This translates into the relativistic collisionless Boltzmann equation:

$$0 = p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} \,, \qquad \text{or} \qquad 0 = p^\mu \frac{\partial f}{\partial x^\mu} + \Gamma^\rho_{\mu\sigma} p_\rho p^\sigma \frac{\partial f}{\partial p_\mu} \,, \tag{7.3}$$

where we used the geodesic equation

$$0 = \frac{d}{d\lambda} p^\mu + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = p^\nu p^\mu{}_{,\nu} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \,, \qquad 0 = \frac{d}{d\lambda} p_\mu - \Gamma^\rho_{\mu\sigma} p_\rho p^\sigma \,. \tag{7.4}$$

Despite the presence of the Christoffel symbol, the equation is indeed invariant under diffeomorphisms. We further need to impose the on-shell condition in the collisionless Boltzmann equation.

7.1.1 Geodesic Equation

To solve the Boltzmann equation (7.3), we need to know how the physical momentum p^μ in FRW coordinates is related to the physical quantities measured by an observer with four velocity u^μ.

$$[e_t]^\mu = u^\mu = \frac{1}{a}\left(1 - \alpha_\chi ,\; V^\alpha\right) \,, \qquad [e_i]^\mu = \frac{1}{a}\left[\delta^\beta_i V_\beta \,,\; \delta^\alpha_i \left(1 - \varphi_\chi\right)\right] \,, \tag{7.5}$$

where we ignored the rotation of tetrad vectors against FRW coordinates. The physical momentum is written in capital letters as

$$P^a = (E, P^i) \,, \qquad E = -p^\mu u_\mu \,, \qquad P^i = p^\mu e^i_\mu \,, \qquad E^2 = m^2 + P^2 \,. \tag{7.6}$$

Using the tetrad expression, the physical momentum $p^\mu = P^a e^\mu_a$ in FRW coordinates is then obtained as

$$p^\eta = \frac{(1-\alpha_\chi)E + V_j P^j}{a} \,, \qquad p^\alpha = \frac{1}{a}\left[P^\alpha + EV^\alpha - \varphi_\chi \, P^\alpha\right] \,, \tag{7.7}$$

and the covariant momentum is

$$p_\eta = -a(1+\alpha_\chi)E - a\, V_j P^j \,, \qquad p_\alpha = aV_\alpha E + aP_\alpha(1+\varphi_\chi) \,. \tag{7.8}$$

In the background, the physical momentum is redshifted as $P^\alpha \propto 1/a$ for both massless and massive particles.[1] So, it is convenient to define the "comoving momentum q" and "comoving energy ε" as

$$q := aP \,, \qquad \varepsilon := aE = \sqrt{q^2 + a^2 m^2} \,, \qquad q^i := qn^i \,. \tag{7.10}$$

[1] The geodesic equation yields

$$0 = p^\eta p^{\mu\prime} + \bar\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \quad \mapsto \quad 0 = p^\eta p^{\eta\prime} + \mathcal{H} p^\eta p^\eta + \mathcal{H} p^\alpha p_\alpha \,, \qquad 0 = p^\eta p^{\alpha\prime} + 2\mathcal{H} p^\eta p^\alpha \,. \tag{7.9}$$

The last equation says the spatial momentum $p^\alpha \propto 1/a^2$ in the background, i.e., the physical momentum $P^\alpha \propto 1/a$ for both massless and massive particles in the background. In the presence of perturbations, these relations change.

In the background, the comoving momentum and energy are constant, while the momentum in FRW coordinates redshifts as

$$\bar{p}^\eta = \frac{E}{a} = \frac{\varepsilon}{a^2} \,, \qquad \bar{p}^\alpha = \frac{1}{a}P^\alpha = \frac{1}{a^2}qn^\alpha \,, \qquad n^\alpha = n^i\delta_i^\alpha \,. \tag{7.11}$$

In terms of the comoving quantities, the momentum in FRW coordinates is now

$$p^\eta = \frac{(1-\alpha_\chi)\varepsilon + V_jq^j}{a^2} \,, \qquad p^\alpha = \frac{1}{a^2}\left(q^\alpha + \varepsilon V^\alpha - \varphi_\chi q^\alpha\right) \,. \tag{7.12}$$

To compute the change in the comoving momentum as the particle propagates, we need to solve the geodesic equation and obtain

$$\frac{dq_\alpha}{d\eta} = -\varepsilon\,\alpha_{\chi,\alpha} - V_\alpha'\varepsilon - q^\beta V_{\alpha,\beta} - \frac{a^2\mathcal{H}m^2}{\varepsilon}V_\alpha - \varphi_\chi' q_\alpha - \frac{q^\beta q^\gamma}{\varepsilon}\left(\varphi_{\chi,\gamma}\delta_{\alpha\beta} - \varphi_{\chi,\alpha}\delta_{\beta\gamma}\right) \,, \tag{7.13}$$

where we use the background geodesic equation (valid for massive & massless)

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \frac{q^\beta}{\varepsilon}\frac{\partial}{\partial x^\beta} \quad\to\quad \frac{d}{d\lambda} \,, \qquad \bar{\varepsilon}' = \frac{a^2\mathcal{H}m^2}{\varepsilon} \,. \tag{7.14}$$

At the linear order, the propagation direction is simply the straight path, and only the comoving momentum changes as

$$\frac{d\ln q}{d\eta} = -\frac{\varepsilon}{q}\left(\alpha_{\chi,\parallel} + V_\parallel'\right) - \varphi_\chi' - V_{\alpha,\beta}n^\alpha n^\beta - \frac{a^2\mathcal{H}m^2}{q\varepsilon}V_\parallel \,. \tag{7.15}$$

Indeed, the comoving momentum is constant in the background. For massless particles ($m = 0$), the comoving momentum is the comoving energy, changing as

$$\frac{d\ln q}{d\eta} = -\alpha_{\chi,\parallel} - \varphi_\chi' - V_\parallel' - V_{\alpha,\beta}n^\alpha n^\beta \,. \tag{7.16}$$

This can be further arranged as

$$\frac{d}{d\eta}\left(\ln q + \alpha_\chi + V_\parallel\right) = \left(\alpha_\chi - \varphi_\chi\right)' \,, \tag{7.17}$$

and the whole quantity in the bracket is affected by the structure growth along the path. Note that the gravitational potential α_χ becomes more negative as the structure grows in time.

7.1.2 Collisionless Boltzmann Equation for Massless Particles

The Boltzmann equation is further simplified, when we switch the variables (η, x^α, p^μ) to (η, x^α, q^i), where the on-shell condition removes one component of the physical momentum:

$$0 = \frac{df}{d\Lambda}\Delta\Lambda = \left(\frac{\partial f}{\partial x^\mu}\Delta x^\mu + \frac{\partial f}{\partial q^i}\Delta q^i\right)\Delta\Lambda = \left(p^\mu\frac{\partial f}{\partial x^\mu} + \frac{dq^i}{d\Lambda}\frac{\partial f}{\partial q^i}\right)\Delta\Lambda \,, \tag{7.18}$$

where the partial derivatives fix (x^μ, q^i), instead of (x^μ, p^μ) in Eq. (7.3). Splitting the distribution function F, we derive the Boltzmann equation in the background:

$$F := \bar{f} + f \,, \qquad 0 = \bar{f}' \,, \qquad \bar{f} = \bar{f}(q) \,, \tag{7.19}$$

i.e., the phase-space distribution is constant in time and space, but a function of the comoving momentum only. The perturbation equation can be derived as

$$0 = \left(\bar{p}^\eta f' + \bar{p}^\alpha f_{,\alpha} + \frac{dq^i}{d\Lambda}n_i\frac{d\bar{f}}{dq}\right)\Delta\Lambda + \mathcal{O}(2) = \bar{p}^\eta\left(f' + n^\alpha\frac{\partial f}{\partial x^\alpha} + \frac{d\bar{f}}{d\ln q}\frac{a^2 n_i}{q^2}\frac{dq^i}{d\Lambda}\right)\Delta\Lambda \,, \tag{7.20}$$

where the last term is

$$\frac{d}{d\eta} = \frac{1}{p^\eta}\frac{d}{d\Lambda} \,, \qquad \frac{dx^\alpha}{d\eta} = \frac{1}{p^\eta}\frac{dx^\alpha}{d\Lambda} = \frac{p^\alpha}{p^\eta} \,, \qquad \frac{d\bar{f}}{d\ln q}\frac{a^2n_i}{q^2}\frac{dq^i}{d\Lambda} = \frac{d\bar{f}}{d\ln q}\frac{n_i}{q}\frac{dq^i}{d\eta} \,. \tag{7.21}$$

Therefore, the collisionless Boltzmann equation is at the linear order in perturbations

$$0 = \frac{df}{d\eta} - \frac{d\bar{f}}{d\ln q}\left(\alpha_{\chi,\parallel} + \varphi_\chi' + V_\parallel' + V_{\alpha,\beta}n^\alpha n^\beta\right) \quad\to\quad \frac{d}{d\eta}\left[f - \frac{d\bar{f}}{d\ln q}\left(\alpha_\chi + V_\parallel\right)\right] = -\frac{d\bar{f}}{d\ln q}\left(\alpha_\chi - \varphi_\chi\right)' \,. \tag{7.22}$$

7.1.3 Convention for Multipole Decomposition

Now we will decompose the Boltzmann equation, but care must be taken in terms of what variables are decomposed. Schematically, we will perform the decomposition of the perturbations in the Boltzmann equation as

$$P(x^\mu, q^i) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\, P(\mathbf{k}, q^i, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{lm} (-i)^l \sqrt{\frac{4\pi}{2l+1}} P_{lm}(q, \mathbf{k}, \eta) Y_{lm}(\hat{n}) e^{i\mathbf{k}\cdot\mathbf{x}} \,, \qquad \mathbf{x} = \bar{r} n^i \,, \tag{7.23}$$

where the angular decomposition is (we suppress irrelevant arguments for clarity such as η and $\mathbf{k}$)

$$P(q^i) := \sum_{lm} (-i)^l \sqrt{\frac{4\pi}{2l+1}}\, P_{lm}(q)\, Y_{lm}(\hat{n}) \,, \qquad P_{lm}(q) \equiv i^l \sqrt{\frac{2l+1}{4\pi}} \int d^2n\, Y^*_{lm}(\hat{n}) P(q^i) \,. \tag{7.24}$$

Naturally, P_{lm} are helicity eigenstates, such that under a rotation $\phi \to \phi - \Phi$ in a coordinate ($\mathbf{k} /\!\!/ \boldsymbol{z}$) they transform as

$$\tilde{P}_{lm} = P_{lm} e^{im\Phi} \,. \tag{7.25}$$

Since we integrate over $\hat{n}$ and $\hat{k}$, we can simply set $\mathbf{k} /\!\!/ \boldsymbol{z}$.

In literature, there exists a different convention (up to $2l+1$ factor) for decomposition in terms of the Legendre polynomial, but it is in fact *valid only* for a scalar mode ($m = 0$):

$$P(\hat{k}\cdot\hat{n}) := \sum_l (-i)^l P_l\, L_l(\hat{n}\cdot\hat{k}) = \sum_l (-i)^l P_l \sum_m \frac{4\pi}{2l+1} Y_{lm}(\hat{n}) Y^*_{lm}(\hat{k}) \,, \tag{7.26}$$

and for a Fourier mode $\mathbf{k} /\!\!/ \boldsymbol{z}$ we derive the correspondence to our decomposition convention:

$$Y_{lm}(\boldsymbol{z}) = \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0} \,, \qquad P(\hat{k}\cdot\hat{n}) = \sum_L (-i)^l P_l\, \delta_{m0} \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\hat{n}) \,, \qquad \therefore\ P_l \to P_{l0} \,. \tag{7.27}$$

So the decomposition with the Legendre polynomial in Eq. (7.26) is *valid only for the scalar modes.*

• ***Notation convention in literature***.— A common notation convention: Seljak and Zaldarriaga (1996); Zaldarriaga and Seljak (1997); Dodelson (2003)

$$P(\hat{k}\cdot\hat{n}) := \sum_l (-i)^l (2l+1) \hat{P}_l\, L_l(\hat{n}\cdot\hat{k}) \,, \qquad \therefore\ (2l+1)\hat{P}_l \equiv P_l \equiv P_{l0} \,. \tag{7.28}$$

7.1.4 Multipole Expansion of the Boltzmann Equation

In general, the monopole f_0 changes only under diffeomorphisms and the dipole f_1 changes only under Lorentz transformations of the observer, while the higher multipoles are fully gauge-invariant. Therefore, we construct the fully gauge-invariant variables as

$$f_{\rm gi} := f_\chi - \frac{d\bar{f}}{d\ln q} V_\parallel \,, \qquad f_0^{\rm gi} := f_0^\chi \,, \qquad f_1^{\rm gi} := f_1^\chi - k v_\chi \frac{d\bar{f}}{d\ln q} \,, \qquad f_l^{\rm gi} := f_l \quad \text{for } l \geq 2 \,. \tag{7.29}$$

The collisionless Boltzmann equation is then

$$0 = f_0' + \frac{k}{3} f_1 - \frac{d\bar{f}}{d\ln q}\left(\varphi_\chi' + \frac{1}{3}k^2 v_\chi\right) = f_0^{\rm gi\prime} + \frac{k}{3} f_1^{\rm gi} - \frac{d\bar{f}}{d\ln q}\varphi_\chi' \,, \tag{7.30}$$

$$0 = f_1' + k\left(-f_0 + \frac{2}{5} f_2\right) + \frac{d\bar{f}}{d\ln q}\left(k\alpha_\chi - k v_\chi'\right) = f_1^{\rm gi\prime} + k\left(-f_0^{\rm gi} + \frac{2}{5} f_2^{\rm gi}\right) + \frac{d\bar{f}}{d\ln q} k\alpha_\chi \,. \tag{7.31}$$

The monopole and the dipole are affected by the scalar (and the vector) perturbations. The quadrupole moment is

$$0 = f_2' + k\left(-\frac{2}{3} f_1 + \frac{3}{7} f_3\right) - \frac{d\bar{f}}{d\ln q}\left(-\frac{2}{3}k^2 v_\chi\right) = f_2^{\rm gi\prime} + k\left(-\frac{2}{3} f_1^{\rm gi} + \frac{3}{7} f_3^{\rm gi}\right) \tag{7.32}$$

independent of scalar perturbations, but affected by the vector and the tensor perturbations. The higher-order multipoles for $l > 2$ are automatically gauge-invariant

$$0 = f_l' + k\left(-\frac{l}{2l-1} f_{l-1} + \frac{l+1}{2l+3} f_{l+1}\right) = f_l^{\rm gi\prime} + k\left(-\frac{l}{2l-1} f_{l-1}^{\rm gi} + \frac{l+1}{2l+3} f_{l+1}^{\rm gi}\right) \,, \tag{7.33}$$

and they are not source by any metric perturbations in the absence of collision (e.g., neutrino distribution), but they are not damped either.

7.1.5 Massless Particles

Here we consider photons and neutrinos. Though neutrinos are massive, massless neutrinos are in most cases a good approximation, with which equations are greatly simplified. We define the temperature anisotropies

$$\rho = aT^4 = a\bar{T}^4\left(1+4\frac{\delta T}{\bar{T}}\right)+\mathcal{O}(2)\,, \qquad \Theta(\hat{n}) := \frac{\delta T}{\bar{T}} = \frac{1}{4}\frac{\delta\rho}{\bar{\rho}}\,, \tag{7.34}$$

and its multipole decomposition

$$\Theta_l \equiv \frac{\int dq\, q^3 f_l}{4\int dq\, q^3 \bar{f}} = \frac{\pi g}{a^4\bar{\rho}}\int_0^\infty dq\, q^3 f_l\,. \tag{7.35}$$

Note that the temperature anisotropies Θ are related to the distribution function as

$$F = \left[\exp\left(\frac{q}{a\bar{T}(\eta)[1+\Theta]}\right)-1\right]^{-1} \simeq \bar{f}\left(1+\bar{f}e^{q/a\bar{T}}\frac{q}{a\bar{T}}\Theta\right) = \bar{f}\left(1-\frac{d\ln\bar{f}}{d\ln q}\Theta\right)\,, \qquad \therefore\;\; f = -\frac{d\bar{f}}{d\ln q}\,\Theta\,, \tag{7.36}$$

where we used the relation and the background quantities are

$$\frac{d\ln\bar{f}}{d\ln q} = -\bar{f}e^{q/a\bar{T}}\frac{q}{a\bar{T}}\,, \qquad \bar{\rho} = \frac{1}{3}\bar{p} = \frac{4\pi g}{a^4}\int dq\, q^3\bar{f}\,. \tag{7.37}$$

Therefore, the perturbation quantities for massless particles are

$$\delta\rho = \frac{1}{3}\delta p = \frac{4\pi g}{a^4}\int_0^\infty dq\, q^3 f_0 = 4\bar{\rho}\,\Theta_0\,, \qquad kv_\gamma = kv_{\rm obs}+\Theta_1\,, \tag{7.38}$$

$$\pi^{ij} = \frac{g}{a^4}\int dq\, q^3\int d\Omega\left(n^in^j-\frac{1}{3}\delta_{ij}\right)\left(-\frac{d\bar{f}}{d\ln q}\Theta\right) = 4\bar{\rho}\int\frac{d\Omega}{4\pi}\left(n^in^j-\frac{1}{3}\delta_{ij}\right)\Theta \equiv \Pi^{(0)}Q^{(0)}_{\alpha\beta}\,, \tag{7.39}$$

where the useful relations are from Eq. (**??**)

$$\Pi^{(0)} = \frac{k^2}{a^2}\Pi = \frac{4}{5}\bar{\rho}\Theta_2\,, \qquad Q^{(0)}_{\alpha\beta} := \frac{1}{k^2}Q^{(0)}_{|\alpha\beta}+\frac{1}{3}\bar{g}_{\alpha\beta}Q^{(0)}\,. \tag{7.40}$$

Finally, noting that

$$\frac{d}{d\eta}f = -\frac{d\bar{f}}{d\ln q}\frac{d\Theta}{d\eta}\,, \tag{7.41}$$

we show that the collisionless Boltzmann equation for massless particles becomes

$$\Theta_{\rm gi} = \Theta^\chi + V^{\rm obs}_\parallel\,, \qquad \frac{d}{d\eta}\left(\Theta_{\rm gi}+\alpha_\chi\right) = \left(\alpha_\chi-\varphi_\chi\right)'\,. \tag{7.42}$$

Note that the monopole Θ_0 changes *only under diffeomorphisms* and the dipole Θ_{1m} changes *only under Lorentz transformations*, while the higher-order multipoles ($l\geq 2$) are *fully gauge-invariant*. We construct the fully gauge-invariant variables as

$$\Theta^{\rm gi}_l \equiv \frac{\pi g}{a^4\bar{\rho}}\int dq\, q^3\, f^{\rm gi}_l\,, \qquad \Theta^{\rm gi}_0 = \Theta^\chi_0\,, \qquad \Theta^{\rm gi}_1 = \Theta^\chi_1 + kv^{\rm obs}_\chi = kv^\gamma_\chi\,, \qquad \Theta^{\rm gi}_l = \Theta_l\;\;\text{for } l\geq 2\,. \tag{7.43}$$

Now, in terms of the multipole coefficients of the temperature anisotropies, the Boltzmann equation becomes

$$0 = \Theta_0' + \frac{k}{3}\Theta_1 + \varphi_\chi' + \frac{1}{3}k^2v_\chi = \Theta^{\rm gi\prime}_0 + \frac{k}{3}\Theta^{\rm gi}_1 + \varphi_\chi'\,, \tag{7.44}$$

$$0 = \Theta_1' + k\left(-\Theta_0+\frac{2}{5}\Theta_2\right) - k\alpha_\chi + kv_\chi' = \Theta^{\rm gi\prime}_1 - k\Theta^{\rm gi}_0 + \frac{2}{5}k\Theta^{\rm gi}_2 - k\alpha_\chi\,, \tag{7.45}$$

$$0 = \Theta_2' + k\left(-\frac{2}{3}\Theta_1+\frac{3}{7}\Theta_3\right) + -\frac{2}{3}k^2v_\chi = \Theta^{\rm gi\prime}_2 + k\left(-\frac{2}{3}\Theta^{\rm gi}_1+\frac{3}{7}\Theta^{\rm gi}_3\right)\,, \tag{7.46}$$

where the derivative term $d\bar{f}/d\ln q$ is integrated by part to give minus sign. The higher-order multipoles for $l>2$ are again gauge-invariant

$$0 = \Theta_l' + k\left(-\frac{l}{2l-1}\,\Theta_{l-1}+\frac{l+1}{2l+3}\,\Theta_{l+1}\right) = \Theta^{\rm gi\prime}_l + k\left(-\frac{l}{2l-1}\,\Theta^{\rm gi}_{l-1}+\frac{l+1}{2l+3}\,\Theta^{\rm gi}_{l+1}\right)\,. \tag{7.47}$$

7.2 Collisions of the Baryon-Photon Fluid: Thompson Scattering

In the early Universe ($t \simeq 6$ sec), the temperature is already below the electron mass, so that the dominant interaction for the baryon-photon fluid is the Thompson scattering, low-energy limit of the Compton scattering:

$$\frac{d\sigma_T}{d\Omega} = \frac{3\sigma_T}{16\pi}\left[1 + (\hat{\boldsymbol{p}}_{\rm in}\cdot\hat{\boldsymbol{p}}_{\rm out})^2\right] , \qquad \sigma_T := \frac{8\pi}{3}r_e^2 = 6.651\times 10^{-25}{\rm cm}^2 , \tag{7.48}$$

where r_e is the effective radius of electrons ($m_ec^2 = e^2/r_e$) and $\hat{\boldsymbol{p}}_{\rm in,out}$ represents the incoming and outgoing directions of the photons in the rest frame of the electron. The Thompson scattering cross-section for protons is smaller by the mass ratio. Due to the directional dependence (quadrupole), the Thompson scattering generates the polarization of the scattered photons. Finally, the collisional term should be Lorentz transformed from the electron rest-frame to the FRW coordinate to be put in the collisional Boltzmann equation.

7.2.1 Collisional Boltzmann Equation for Photons

Using the multipole decomposition, the collisional term for the Boltzmann equation is given as

$$\mathfrak{C} \equiv \frac{d\bar f}{d\ln q}\,\Gamma\left[\Theta(\hat n) - \hat n\cdot v_b - \Theta_0 + \frac{1}{2}L_2(\mu)\left(\frac{1}{5}\Theta_2 + \frac{1}{5}\Theta_2^p + \Theta_0^p\right)\right] , \tag{7.49}$$

and the probability of the Thompson scattering until today is expressed in terms of the optical depth τ:

$$\tau(\eta) := \int_\eta^{\bar\eta_0} d\eta'\; an_e\sigma_T , \qquad \Gamma := an_e\sigma_T \equiv |\tau'| , \tag{7.50}$$

where n_e is the electron number density. Therefore, the collisional Boltzmann equation for photons is then

$$\frac{d}{d\eta}\left(f_{\rm gi} - \frac{d\bar f}{d\ln q}\alpha_\chi\right) = -\frac{d\bar f}{d\ln q}\left(\alpha_\chi - \varphi_\chi\right)' + \mathfrak{C} , \tag{7.51}$$

$$\frac{d}{d\eta}\left(\Theta_{\rm gi} + \alpha_\chi\right) = \left(\alpha_\chi - \varphi_\chi\right)' - \Gamma\left[\Theta(\hat n) - \hat n\cdot v_b - \Theta_0 + \frac{1}{2}L_2(\mu)\left(\frac{1}{5}\Theta_2 + \frac{1}{5}\Theta_2^p + \Theta_0^p\right)\right] . \tag{7.52}$$

Applying the multipole decomposition on both sides, we derive

$$0 = \Theta_0^{\rm gi\prime} + \frac{k}{3}\Theta_1^{\rm gi} + \varphi_\chi' , \qquad \Theta_0^{\rm gi} = \frac{1}{4}\delta_\chi^\gamma , \qquad \Theta_1^{\rm gi} = kv_\chi^\gamma , \tag{7.53}$$

$$0 = \Theta_1^{\rm gi\prime} - k\Theta_0^{\rm gi} + \frac{2}{5}k\Theta_2^{\rm gi} - k\alpha_\chi + \Gamma\left(\Theta_1^{\rm gi} - kv_\chi^b\right) , \tag{7.54}$$

$$0 = \Theta_2^{\rm gi\prime} + k\left(-\frac{2}{3}\Theta_1^{\rm gi} + \frac{3}{7}\Theta_3^{\rm gi}\right) + \Gamma\left(-\frac{9}{10}\Theta_2 + \frac{1}{8}\Theta_0^P + \frac{1}{8}\Theta_2^P\right) .$$

The higher-order multipoles for $l > 2$ are again gauge-invariant

$$0 = \Theta_l' + k\left(-\frac{l}{2l-1}\,\Theta_{l-1} + \frac{l+1}{2l+3}\,\Theta_{l+1}\right) + \Gamma\,\Theta_l = \Theta_l^{\rm gi\prime} + k\left(-\frac{l}{2l-1}\,\Theta_{l-1}^{\rm gi} + \frac{l+1}{2l+3}\,\Theta_{l+1}^{\rm gi}\right) + \Gamma\,\Theta_l^{\rm gi} . \tag{7.55}$$

In the early Universe, when the collision is efficient $\Gamma \gg 1$, the higher-multipoles are highly suppressed:

$$\Theta_l' \sim \frac{\Theta_l}{\eta} , \qquad \Gamma\Theta_l \sim \frac{\tau}{\eta}\Theta_l , \qquad \therefore\;\; \Theta_l' \ll \Gamma\Theta_l , \tag{7.56}$$

and from the multipole equations we derive

$$0 \sim 0 - k\Theta_{l-1} + 0 + \Gamma\Theta_l , \qquad \therefore\;\; \Theta_l \sim \frac{k\eta}{\tau}\Theta_{l-1} \approx 0 \quad \text{for } l \geq 2 . \tag{7.57}$$

7.2.2 Collisional Boltzmann Equation for Baryons

The collisional Boltzmann equation for baryons can be solved exactly the same way the Boltzmann equation for photons is derived. However, it simply reduces to the fluid equation. The number density of baryons is conserved, regardless of the Thompson scattering with photons:

$$\delta_\chi^{b\,\prime} + k^2 v_\chi^b = -3\varphi_\chi' \,, \tag{7.58}$$

but the momentum of the baryon fluid is affected by scattering off photons as

$$v_\chi^{b\,\prime} + \mathcal{H} v_\chi^b = \alpha_\chi + \frac{\Gamma}{R}\left(v_\chi^\gamma - v_\chi^b\right) \,, \tag{7.59}$$

where the baryon-to-photon momentum density ratio is

$$R := \frac{\dot{\bar{\rho}}_b}{\dot{\bar{\rho}}_\gamma} \equiv \frac{\bar\rho_b + \bar p_b}{\bar\rho_\gamma + \bar p_\gamma} = \frac{3\bar\rho_b}{4\bar\rho_\gamma} = 0.6\left(\frac{\omega_b}{0.02}\right)\left(\frac{a}{10^{-3}}\right) \,. \tag{7.60}$$

Note that the change in baryon velocity is proportional to the relative velocity between photons and baryons and inversely proportional to the momentum density ratio.

While the interaction between photons and protons is negligible, protons and electrons are tightly coupled by the Coulomb interaction, so that all three components (b, p, γ) are tightly coupled.

7.3 Initial Conditions for the Evolution

We would like to set up the initial conditions, with which the Boltzmann equations can be evolved in time. At early time in the radiation-dominated era, the set of the Boltzmann-Einstein equations can be greatly simplified on large scales as

$$0 = \Theta_0' + \varphi_\chi' \,, \qquad 0 = N_0' + \varphi_\chi' \,, \qquad 0 = \delta_\chi' + 3\varphi_\chi' \,, \qquad \therefore\ \Theta_0' = N_0' = \frac{1}{3}\delta_\chi' = -\varphi_\chi' \,, \tag{7.61}$$

where we used $k\eta \ll 1$. Using the adiabatic condition in Eq. (**??**), the initial conditions are set

$$\Theta_0 = N_0 = \frac{1}{3}\delta \,. \tag{7.62}$$

The Einstein equation (**??**) in the limit $k \to 0$ becomes

$$0 = 4\pi G a^2 \bar\rho \delta_\chi + a\mathcal{H}\kappa_\chi \,, \qquad \kappa_\chi = 3H\alpha_\chi \,, \qquad \therefore\ \Theta_0 = N_0 = -\frac{1}{2}\alpha_\chi \,, \tag{7.63}$$

where the matter density is ignored in the radiation-dominated era. Similarly, the Einstein equation (**??**) gives the initial conditions for the velocity scalars

$$v_\chi^\gamma = v_\chi^\nu = \frac{\alpha_\chi}{2\mathcal{H}} \,, \qquad v_\chi^b = v_\chi^m = v_\chi^\gamma \,. \tag{7.64}$$

While the photons are tightly coupled with baryons, the neutrinos decouple at $t \simeq 1$ sec ($T \simeq 1$ MeV), such that the neutrino distribution develops non-vanishing quadrupole moment and hence the anisotropic pressure. Using the relation of the anisotropic pressure to the quadrupole moment in Eq. (7.40) and the Einstein equation (**??**), we derive

$$8\pi G\Pi = \frac{32\pi G}{5}\frac{a^2}{k^2}\bar\rho_\nu N_2 = \frac{12}{5}\frac{\mathcal{H}^2}{k^2} f_\nu N_2 \,, \qquad \therefore\ N_2 = -\frac{5}{12}\frac{k^2}{\mathcal{H}^2}\frac{\alpha_\chi + \varphi_\chi}{f_\nu} \,, \tag{7.65}$$

where we used

$$8\pi G\bar\rho_\nu = 3H^2 f_\nu \,, \qquad f_\nu := \frac{\bar\rho_\nu}{\bar\rho_T} \,. \tag{7.66}$$

Finally, the Boltzmann equation for N_2 at the leading order of $k\eta$ becomes

$$N_2' \simeq \frac{2}{3}kN_1 = \frac{k^2}{3\mathcal{H}}\alpha_\chi \,, \qquad \therefore\ N_2 = \frac{k^2}{6\mathcal{H}^2}\alpha_\chi \,, \tag{7.67}$$

where we integrated the equation using $\mathcal{H} = 1/\eta$ in Eq. (**??**). Therefore, the relation between two gravitational potentials becomes

$$-\left(\alpha_\chi + \varphi_\chi\right) = 8\pi G\Pi = \frac{2}{5}f_\nu\alpha_\chi \,, \qquad \varphi_\chi = -\left(1 + \frac{2}{5}f_\nu\right)\alpha_\chi \,. \tag{7.68}$$

The comoving-gauge curvature perturbation is generated during the inflationary period, and it is conserved on super horizon scales. After the inflationary period ends, the Universe enters into the standard radiation dominated era (under the assumption that the reheating period is very short). The conformal Newtonian gauge potential is then related to the curvature perturbation on large scales as

$$\varphi_v = \varphi_\chi - \frac{1}{2}\left(\alpha_\chi - \frac{\dot{\varphi}_\chi}{H}\right) \simeq -\frac{3}{2}\alpha_\chi - \frac{2}{5}f_\nu \alpha_\chi \,, \qquad \alpha_\chi = -\frac{10}{15+4f_\nu}\,\varphi_v \,, \tag{7.69}$$

where we used $w = 1/3$ and ignored the time-derivative terms.

7.4 Observed CMB Power Spectrum

7.4.1 Acoustic Oscillation: Tight-Coupling Approximation

Before the recombination, the photon-baryon fluid is tightly coupled, so that the higher-order multipoles ($l \geq 2$) is greatly suppressed. Under this tight-coupling approximation ($\Theta_l \approx 0$ for $l \geq 2$), we can derive the master evolution equation for the baryon-photon fluid. From the baryon velocity equation in (7.59),

$$v_\chi^\gamma - v_\chi^b = \frac{R}{\Gamma}\left(v_\chi^{b\,\prime} + \mathcal{H}v_\chi^b - \alpha_\chi\right) \approx \frac{R}{\Gamma}\left[v_\chi^{\gamma\,\prime} + \mathcal{H}v_\chi^\gamma - \alpha_\chi + \mathcal{O}\left(\frac{R}{\Gamma}\right)\right] \,, \tag{7.70}$$

where R/Γ is a small number, so that we used $v_\chi^b = v_\chi^\gamma + \mathcal{O}(R/\Gamma)$. Using this tight-coupling approximation, the Boltzmann equation for the dipole becomes

$$0 = \Theta_1^{\rm gi\prime} - k\Theta_0^{\rm gi} + \frac{2}{5}k\Theta_2^{\rm gi} - k\alpha_\chi + \Gamma\left(\Theta_1^{\rm gi} - kv_\chi\right) \approx \Theta_1^{\rm gi\prime} - k\Theta_0^{\rm gi} + 0 - k\alpha_\chi + R\left(\Theta_1^{\rm gi\prime} + \mathcal{H}\Theta_1^{\rm gi} - k\alpha_\chi\right) \,, \tag{7.71}$$

such that the derivative of the dipole is

$$\Theta_1^{\rm gi\prime} = \frac{k}{1+R}\Theta_0^{\rm gi} - \frac{\mathcal{H}R}{1+R}\Theta_1^{\rm gi} + k\alpha_\chi \,. \tag{7.72}$$

Taking the derivative of the monopole equation and removing the dipole term, we arrive at the master governing equation for the monopole in the tight-coupling limit:

$$\left[\frac{d^2}{d\eta^2} + \frac{\mathcal{H}R}{1+R} + k^2c_s^2\right]\left(\Theta_0^{\rm gi} + \varphi_\chi\right) = k^2c_s^2\varphi_\chi - \frac{k^2}{3}\alpha_\chi \,, \tag{7.73}$$

where the (background) sound speed of the baryon-photon fluid is

$$c_s^2 := \frac{\delta\bar{P}_{b\gamma}}{\delta\bar{\rho}_{b\gamma}} = \frac{1}{3}\frac{\dot{\bar{\rho}}_\gamma}{\dot{\bar{\rho}}_\gamma + \dot{\bar{\rho}}_b} = \frac{1}{3}\frac{1}{1+R} \,, \qquad R' = \mathcal{H}R \,. \tag{7.74}$$

It is the simple harmonic oscillator for $\Theta_0^{\rm gi} + \varphi_\chi$ with the Hubble damping and the driving force.

On small scales, where we can ignore the Hubble expansion, the homogeneous solutions to the equation are

$$y_1 = \sin kr_s \,, \qquad y_2 = \cos kr_s \,, \qquad r_s := \int_0^\eta d\eta'\, c_s \,, \tag{7.75}$$

where r_s is the comoving sound horizon and we ignored the time derivative of the sound speed. Given the initial condition from the inflation, only the cosine mode y_2 is excited. The particular solution can be found by first computing the Wronskian

$$\mathbb{W} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -kc_s \,, \tag{7.76}$$

and by using the variation method as

$$\begin{aligned}\Theta_0^{\rm gi} + \varphi_\chi &= \left(\Theta_0^{\rm gi} + \varphi_\chi\right)_0 \cos kr_s + \int_0^\eta d\tilde\eta\, \frac{y_2(\eta)y_1(\tilde\eta) - y_1(\eta)y_2(\tilde\eta)}{\mathbb{W}}\,\mathbb{F}(\tilde\eta) \\ &= \left(\Theta_0^{\rm gi} + \varphi_\chi\right)_0 \cos kr_s + \frac{k}{\sqrt{3}}\int_0^\eta d\tilde\eta\, \left(\varphi_\chi - \alpha_\chi\right)\sin k(r_s - \tilde r_s) \,, \end{aligned} \tag{7.77}$$

where we approximated $c_s \approx 1/\sqrt{3}$ and the driving force as

$$\mathbb{F} := k^2 c_s^2 \varphi_\chi - \frac{k^2}{3}\alpha_\chi \approx \frac{k^2}{3}\left(\varphi_\chi - \alpha_\chi\right) . \tag{7.78}$$

The temperature fluctuation (plus the potential) is a simple oscillation (modulo correction due to the integration) with the frequency set by the sound horizon at each epoch. The solution for the dipole can be readily obtained as

$$\Theta_1^{\rm gi} = -\frac{3}{k}\left(\Theta_0^{\rm gi} + \varphi_\chi\right)' = \sqrt{3}\left(\Theta_0^{\rm gi} + \varphi_\chi\right)_0 \sin kr_s - k\int_0^\eta d\tilde\eta\ \left(\varphi_\chi - \alpha_\chi\right)\cos k(r_s - \tilde r_s) , \tag{7.79}$$

and it is out of phase with the monopole.

7.4.2 Diffusion Damping

To a good approximation, the baryon-photon fluid is indeed a fluid. However, this tight-coupling breaks down on small scales, where photons simply diffuse out without scattering off the electrons. So the fluctuations in the baryon-photon fluid are damped, and it is called the diffusion damping. This scale is closely related to the photon mean-free path, and on this scale our fluid equations are not valid.

In the tight coupling limit, the baryon-photon fluid oscillates with $\omega = kc_s$, so we look for a small deviation that describes the diffusion process as

$$v_\chi^b \sim \exp\left[i\int d\eta\ \omega\right] , \qquad \omega := \bar\omega + \delta\omega , \qquad \bar\omega^2 = k^2 c_s^2 , \tag{7.80}$$

where the background frequency is obtained in the tight coupling limit ($\Gamma \to \infty$). On small scales, we can simplify the photon Boltzmann equations by ignoring the gravitational potential fluctuations and again neglecting the higher multipoles ($l \geq 3$) and polarization:

$$0 = \Theta_0^{\rm gi\prime} + \frac{k}{3}\Theta_1^{\rm gi} , \qquad 0 = \Theta_1^{\rm gi\prime} - k\Theta_0^{\rm gi} + \frac{2}{5}k\Theta_2^{\rm gi} + \Gamma\left(\Theta_1^{\rm gi} - kv_\chi^b\right) , \qquad 0 = \Theta_2^{\rm gi\prime} - \frac{2}{3}k\Theta_1^{\rm gi} - \frac{9}{10}\Gamma\Theta_2^{\rm gi} , \tag{7.81}$$

so that we derive the relation among the multipole moments

$$\Theta_0^{\rm gi} = \frac{i}{3}\frac{k}{\omega}\Theta_1^{\rm gi} , \qquad \Theta_2^{\rm gi} \simeq -\frac{20}{27}\frac{k}{\Gamma}\Theta_1^{\rm gi} , \tag{7.82}$$

where we ignored $\Theta_2^{\rm gi\prime}$. Using the baryon velocity equation, we express the baryon velocity in terms of the photon velocity

$$v_\chi^\gamma - v_\chi^b = \frac{R}{\Gamma}\left(v_\chi^{b\,\prime} + \mathcal{H}v_\chi^b - \alpha_\chi\right) \approx \frac{R}{\Gamma}v_\chi^{\gamma\prime} = i\omega\frac{R}{\Gamma}v_\chi^b \quad \to \quad \therefore\ v_\chi^b = v_\chi^\gamma\left(1 + i\omega\frac{R}{\Gamma}\right)^{-1} \simeq v_\chi^\gamma\left(1 - i\omega\frac{R}{\Gamma} - \frac{R^2\omega^2}{\Gamma^2}\right) , \tag{7.83}$$

and the collisional term in the photon dipole equation becomes

$$\Gamma\left(\Theta_1^{\rm gi} - kv_\chi^b\right) \simeq \Theta_1^{\rm gi}\left(i\omega R + \frac{R^2\omega^2}{\Gamma}\right) . \tag{7.84}$$

By plugging these terms in the photon dipole equation, we derive the dispersion relation

$$\omega^2(1+R) - \frac{1}{3}k^2 = \frac{i\omega}{\Gamma}\left(R^2\omega^2 - \frac{8}{27}k^2\right) . \tag{7.85}$$

Since the background $\bar\omega^2 = k^2/3(1+R)$ in the limit $\Gamma \to \infty$, we obtain

$$\frac{2}{3}k^2\delta\omega = \frac{i\bar\omega^2}{\Gamma}\left(R^2\bar\omega^2 - \frac{8}{27}k^2\right) \quad \to \quad \therefore\ \delta\omega = \frac{ik^2}{6\Gamma(1+R)}\left(\frac{R^2}{1+R} + \frac{8}{9}\right) . \tag{7.86}$$

Therefore, the photon propagation is now described by

$$\exp\left[i\int d\eta\ \omega\right] = \exp\left[ik\int d\eta\ c_s\right]\exp\left[-k^2/k_d^2\right] , \tag{7.87}$$

and the diffusion scale is

$$1/k_d^2 \equiv \int d\eta\left[\frac{k^2}{6\Gamma(1+R)}\left(\frac{R^2}{1+R} + \frac{8}{9}\right)\right] . \tag{7.88}$$

7.4.3 Free Streaming: Line-of-Sight Integration

Here we derive a formal integral solution by performing the line-of-sight integration. The Boltzmann equation (7.52) for photons is

$$\frac{d}{d\eta}\left(\Theta_{\rm gi}+\alpha_\chi\right)=\left(\alpha_\chi-\varphi_\chi\right)'-\Gamma\left[\Theta(\hat n)-\hat n\cdot v_b-\Theta_0+\frac{1}{2}L_2(\mu)\left(\frac{1}{5}\Theta_2+\frac{1}{5}\Theta_2^p+\Theta_0^p\right)\right]~. \tag{7.89}$$

and noting that the derivative along the path in Fourier space becomes

$$\frac{d}{d\eta}\Theta_{\rm gi}+\Gamma\Theta_{\rm gi}=\Theta_{\rm gi}'+\left(ik\mu_k-\tau'\right)\Theta_{\rm gi}=e^{-ik\mu_k\eta+\tau(\eta)}\frac{d}{d\eta}\left(\Theta_{\rm gi}\,e^{ik\mu_k\eta-\tau(\eta)}\right)~, \tag{7.90}$$

the Boltzmann equation can be re-arranged and integrated to yield the line-of-sight integral solution

$$\Theta_{\rm gi}=\int_0^{\eta_0}d\eta\,e^{-ik\mu_k(\eta_0-\eta)-\tau(\eta)}\left[-ik\mu_k\alpha_\chi-\varphi_\chi'+\tau'\left(ik\mu_kv_\chi^b-\Theta_0^{\rm gi}+\frac{1}{2}L_2(\mu_k)\Pi\right)\right]~, \tag{7.91}$$

where $\Theta_{\rm gi}$ at the initial time is neglected due to large optical depth and we defined

$$\tau(0)=\infty~,\qquad\qquad \tau(\eta_0)=0~,\qquad\qquad \Pi:=\frac{1}{5}\Theta_2+\frac{1}{5}\Theta_2^p+\Theta_0^p~. \tag{7.92}$$

By replacing the angular dependence with the derivative

$$ik\mu_k\rightarrow\frac{d}{d\eta}~,\qquad\qquad \mu_k^2\rightarrow-\frac{1}{k^2}\frac{d^2}{d\eta^2}~, \tag{7.93}$$

the solution can be further simplified as

$$\Theta_{\rm gi}=\int_0^{\eta_0}d\eta\,\left[-e^{-\tau}\left(\varphi_\chi'+\tau'\Theta_0^{\rm gi}+\frac{1}{4}\tau'\Pi\right)-e^{-\tau}\left(\alpha_\chi-\tau'v_\chi^b\right)\frac{d}{d\eta}-\tau'e^{-\tau}\frac{3\Pi}{4k^2}\frac{d^2}{d\eta^2}\right]e^{-ik\mu_k(\eta_0-\eta)}~. \tag{7.94}$$

Expanding the exponential and performing the multipole decomposition in Eq. (7.23) on both sides, we derive

$$\begin{aligned}\frac{\Theta_l^{\rm gi}}{2l+1}&=\int_0^{\eta_0}d\eta\,\left[-e^{-\tau}\left(\varphi_\chi'+\tau'\Theta_0^{\rm gi}+\frac{1}{4}\tau'\Pi\right)-e^{-\tau}\left(\alpha_\chi-\tau'v_\chi^b\right)\frac{d}{d\eta}-\tau'e^{-\tau}\frac{3\Pi}{4k^2}\frac{d^2}{d\eta^2}\right]j_l[k(\eta_0-\eta)]\\&\approx\int_0^{\eta_0}d\eta\,\left[-e^{-\tau}\left(\varphi_\chi'+\tau'\Theta_0^{\rm gi}\right)+\frac{d}{d\eta}\left(e^{-\tau}\alpha_\chi\right)+e^{-\tau}\tau'v_\chi^b\frac{d}{d\eta}\right]j_l[k(\eta_0-\eta)]~,\end{aligned} \tag{7.95}$$

where we ignored the polarization for the moment and integrated by part for the second term in the square bracket. By defining the visibility

$$g(\eta):=-\tau'e^{-\tau}\equiv\Gamma e^{-\tau}~, \tag{7.96}$$

the integral solution can be rearranged as

$$\frac{\Theta^{\rm gi}}{2l+1}=\int_0^{\eta_0}d\eta\,\left[g\left(\Theta_0^{\rm gi}+\alpha_\chi\right)j_l(x)-g\,kv_\chi^bj_l'(x)+e^{-\tau}\left(\alpha_\chi-\varphi_\chi\right)'j_l(x)\right]~,\qquad\qquad x:=k(\eta_0-\eta)~. \tag{7.97}$$

Sine the visibility is close to a sharp Dirac delta function at the recombination time

$$g(\eta)\simeq\delta^D(\eta-\eta_\star)~, \tag{7.98}$$

the temperature anisotropies are

$$\frac{\Theta^{\rm gi}}{2l+1}\approx\left(\Theta_0^{\rm gi}+\alpha_\chi\right)_\star j_l[k(\eta_0-\eta_\star)]-kv_{\chi\star}^b\,j_l'[k(\eta_0-\eta_\star)]~, \tag{7.99}$$

where we ignored the time evolution of the potential term with the exponential damping. The observed temperature anisotropies are essentially the "monopole" and the "dipole" of the baryon-photon fluid at the recombination epoch $\eta_\star$, free-streaming to the observer after the recombination.

On large scales, we can derive the analytic solution for the observed temperature anisotropies. At $k\eta\ll1$, the Boltzmann equation yields

$$0=\Theta_0^{\rm gi\prime}+\varphi_\chi'~,\qquad\qquad \therefore~\Theta_0^{\rm gi}(k,\eta)=-\varphi_\chi(k,\eta)+C(k)\approx-\varphi_\chi(k,\eta)+\frac{3}{2}\varphi_\chi(k,0)~, \tag{7.100}$$

where the integral constant is fixed by the initial condition. The comoving-gauge curvature perturbation is conserved on large scales all the time, while the conformal Newtonian gauge curvature transitions its value from rde to mde. Using Eq. (**??**), we derive

$$\varphi_v = \frac{5+3w}{3(1+w)}\varphi_\chi = \frac{3}{2}\varphi_\chi(0) = \frac{5}{3}\varphi_\chi(\eta_\star) \, , \qquad \therefore \ \varphi_\chi(0) = \frac{10}{9}\varphi_\chi(\eta_\star) \, , \qquad \Theta_0^{\rm gi}(\eta_\star) = \frac{2}{3}\varphi_\chi(\eta_\star) \, . \tag{7.101}$$

The same calculation can be done for the matter density on large scales:

$$0 = \delta' + 3\varphi_\chi' \, , \qquad \therefore \ \delta(\eta_\star) = -3\varphi_\chi(\eta_\star) + \frac{9}{2}\varphi_\chi(0) = 2\varphi_\chi(\eta_\star) \, , \qquad \left(\Theta_0^{\rm gi} + \alpha_\chi\right)_\star = -\frac{1}{3}\varphi_\chi(\eta_\star) = -\frac{1}{6}\delta(\eta_\star) \, . \tag{7.102}$$

At the recombination, the overdense region with $\delta > 0$ corresponds to the hotter spot $\Theta_0 > 0$, but the observed temperature today is colder due to the energy loss by the gravitational redshift from the overdense region. Furthermore, given the level that the temperature anisotropies are $\sim 10^{-5}$, the density growth from the recombination epoch $z \sim 1100$ will lead only to $\delta \sim 10^{-2}$, unless it is further boosted by the nonlinear growth of dark matter prior to the recombination epoch.

The full line-of-sight integral solution is

$$\begin{aligned}\frac{\Theta_l^{\rm gi}}{2l+1} &= \int_0^{\eta_0} d\eta \left[-e^{-\tau}\left(\varphi_\chi' + \tau'\Theta_0^{\rm gi} + \frac{1}{4}\tau'\Pi\right) - e^{-\tau}\left(\alpha_\chi - \tau' v_\chi^b\right)\frac{d}{d\eta} - \tau' e^{-\tau}\frac{3\Pi}{4k^2}\frac{d^2}{d\eta^2}\right] j_l[k(\eta_0-\eta)] \\ &= \int_0^{\eta_0} d\eta \left[g\left(\Theta_0^{\rm gi} + \alpha_\chi + v_\chi^{b\prime} + \frac{\Pi}{4} + \frac{3\Pi''}{4k^2}\right) + e^{-\tau}\left(\alpha_\chi - \varphi_\chi\right)' + g'\left(v_\chi^b + \frac{3\Pi'}{2k^2}\right) + \frac{3g''\Pi}{4k^2}\right] . \end{aligned} \tag{7.103}$$

7.4.4 CMB Power Spectrum

Finally, we need to connect our theoretical predictions to the observation. The observed CMB temperature can be harmonically decomposed as

$$\Theta(\hat n) := \sum_{lm} a_{lm} Y_{lm}(\hat n) \, , \qquad a_{lm} \equiv \int d^2\hat n \, Y_{lm}^*(\hat n)\Theta(\hat n) \, , \tag{7.104}$$

and the observed CMB power spectrum can be obtained as

$$C_l = \frac{1}{2l+1}\sum_m |a_{lm}|^2 \, . \tag{7.105}$$

So far, our calculation has been performed by assuming $k//z$. In a given coordinate, we have $\hat k = (\theta_k, \phi_k)$ and $\hat n = (\theta, \phi)$, so the relation to the above expressions in a system with $k//z$ is obtained by rotating $\hat n$ into $\hat n'$ with $R_{\hat{\mathbf n}} \equiv R_{\mathbf k}$, where the above expressions are indeed with $\hat n'$, not with $\hat n$. The observed temperature fluctuation is therefore,

$$\Theta(\hat n) = \int \frac{d^3k}{(2\pi)^3}\sum_l \Theta_l(-i)^l\sqrt{\frac{4\pi}{2l+1}}\,\mathcal{D}(R_{\hat{\mathbf n}})Y_{l0}(\hat n') = \int \frac{d^3k}{(2\pi)^3}\sum_l \Theta_l(-i)^l\sqrt{\frac{4\pi}{2l+1}}\,\sum_{m'}\mathcal{D}_{0m'}^{l*}Y_{lm'}(\hat n) \, , \tag{7.106}$$

and the angular multipole and the power spectrum are then

$$\begin{aligned} a_{LM} &= \int \frac{d^3k}{(2\pi)^3}\,\Theta_L(-i)^L\sqrt{\frac{4\pi}{2L+1}}\,\mathcal{D}_{0M}^{L*} \, , \qquad |m| \le 2 \, , \\ C_L &= \left\langle |a_{LM}|^2\right\rangle = \frac{4\pi}{2L+1}\int d\ln k\,\Delta_\varphi^2(k)\,|\Theta_L|^2 \times \left(\int \frac{d\Omega_k}{4\pi}\left|\mathcal{D}_{0M}^L(R_{\mathbf k})\right|^2\right) = 4\pi\int d\ln k\,\Delta_\varphi^2(k)\frac{|\Theta_L|^2}{(2L+1)^2} \, , \end{aligned} \tag{7.107}$$

where $\mathcal{D}_{mm'}^l$ is the Wigner matrix. Compare to the expressions in literature, where the rotation is neglected:

$$a_{lm} \equiv \int \frac{d^3k}{(2\pi)^3}\,a_{lm}(\mathbf k) \, , \qquad a_{lm}(\mathbf k) = (-i)^l\sqrt{\frac{4\pi}{2l+1}}\,\Theta_l \, , \tag{7.108}$$

so that the resulting power spectrum C_l is identical.

We can derive a simple approximation to the observed power spectrum, or the Sachs-Wolfe plateau on large scales by assuming an instantaneous recombination and considering only the monopole contribution:

$$\frac{\Theta^{\rm gi}}{2l+1} \approx \left(\Theta_0^{\rm gi} + \alpha_\chi\right)_\star j_l[k(\eta_0 - \eta_\star)] \approx -\frac{1}{3}\varphi_\chi(\eta_\star)j_l(k\eta_0) \, , \qquad \int_0^\infty \frac{dk}{k}\,j_l^2(k) = \frac{1}{2l(l+1)} \, , \tag{7.109}$$

and the power spectrum is then

$$C_l^{\rm SW} \approx 4\pi\int d\ln k\,\Delta_\varphi^2(k)j_l^2 k(\eta_0)\left[\frac{T_{\varphi_\chi}(k,\eta_*)}{3}\right]^2 \propto \frac{1}{2l(l+1)} \, , \qquad l(l+1)C_l^{\rm SW} = \text{constant} \, , \tag{7.110}$$

where the transfer function and the initial power spectrum Δ_φ^2 are both constant at low k.