Carroll 9712019 p. 217 –

$G_{\mu\nu}$. $g_{\mu\nu}$: dynamic variable vs $\eta_{\mu\nu}$ static
spacetime ⟺ matter

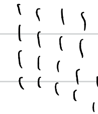
LHS of Einstein EQ.

Robertson–Walker metric

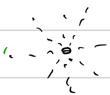
1920 ~ 30

* Homogeneity & isotropy : mathematical definition is there, but intuitively
cosmological principle
for 3-space, not for 4-spacetime.

homo. not iso



iso. not homo.



* time evolution is not symmetric. no off-diagonal

$\Rightarrow ds^2 = -dt^2 + a^2(t)\,\bar g_{ij}\,dx^i dx^j$ $i,j = 1,2,3$ a : scale factor x^i : comoving coord

↑ 3 metric w/o a

* maximally symmetric space $R^{(3)}_{ijkl} = K(\bar g_{ik}\bar g_{jl} - \bar g_{il}\bar g_{jk})$, $n(n+1)/2$ Killing vectors

largest possible # of

6 : for $n=3$. 3 rot. 3 trans.

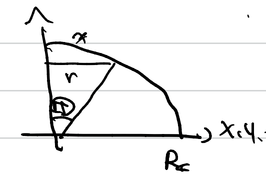
$2U_{\mu;[\rho\sigma]} = U_\nu R^\nu{}_{\mu\rho\sigma}$

How to find $\bar g_{ij}$? use 4

$\bar g_{ij}$ cannot depend on direction. depend only on r. $ds_3^2 = \bar g_{ij}\,dx^i dx^j = A(r)dr^2 + r^2 d\Omega^2$

intuitively, Euclidean. sphere or hyperbola

consider 3-sphere in 4D w/ radius R (fixed)



$0 < \frac{1}{K} = R_K^2 = x^2 + y^2 + z^2 + \omega^2$,

$0 > \frac{1}{K} = -R_K^2 = r^2 - \omega^2$

$x = R_K \sin\Theta \cdot \sin\theta \cdot \cos\phi$ ..

4th coord
$\omega = R_K \cos\Theta$

$r^2 = x^2 + y^2 + z^2 = (R_K \sin\Theta)^2 \le R_K^2$

$r_K \equiv R_K \Theta$ are length

$\therefore \bar g_{ij}\,dx^i dx^j = \underbrace{dx^2 + dy^2 + dz^2}_{= dr^2 + r^2 d\Omega^2} + d\omega^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{R^2 - r^2} = \frac{dr^2}{1 - r^2/R_K^2} + r^2 d\Omega^2$

$= \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 = dx_K^2 + r^2 d\Omega^2$

valid for $K<0$

Robertson–Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

K : constant $\sim L^{-2}$

→ solution of Einstein Eq for expanding universe

homo. iso expansion

$a(t)$ effect ds^2 : real

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ $g_{\mu\nu}^{RW}$

$g^{\mu\nu}$: inverse. $g^\mu_\nu = \delta^\mu_\nu$

$K \to \frac{K}{|K|L^2}$, $r \to \sqrt{|K|}\, r L$, $a \to a/\sqrt{|K|}\, L$ metric is invariant ∴ $K = 0, \pm 1$

for a unit length scale

but $K = \frac{1}{R_K^2} = \#\, a^{-2}$

open hyperbolic univ ($K<0$) $0 > \frac{1}{K} = -R_K^2 = x^2 + y^2 + z^2 - w^2$

$r = R_K \sinh\Theta$. $\chi_K = R_K \Theta$. $w = R_K \cosh\Theta$

Summary

$K=0$ flat. $ds^2 = -dt^2 + a^2(dr^2 + r^2 d\Omega^2)$ Euclidean 3-space

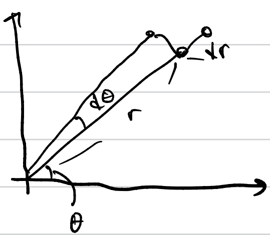
$K=\pm 1$. open. closed $ds^2 = -dt^2 + a^2(d\chi_K^2 + r^2 d\Omega^2)$

$r = \frac{1}{\sqrt{K}}\sin\sqrt{K}\chi_K$ 3-sphere in 4D

$r = \frac{1}{\sqrt{-K}}\sinh\sqrt{-K}\chi_K$ 3-hyperbole in 4D

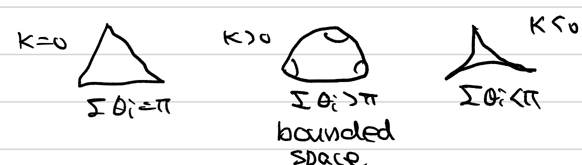
$d\chi_K^2 = \frac{dr^2}{1-Kr^2}$

$r = (x, y, z)$



radial length : $a\, dr$

transverse length : $a r\, d\Omega$



Energy - momentum tensor

RHS of Einstein eq.

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$ $\leftrightarrow$ perfect fluid (only ρ & p) not moving — observer dependent !

in a homogeneous & isotropic univ $\quad U^\mu = \frac{1}{a}(1, 0\,0\,0)$ $\quad U^\mu U_\mu = -1$ timelike

$\therefore T_{\mu\nu} = a^2 \begin{pmatrix} \rho & & & 0 \\ & p g_{11} & & \\ & & p g_{22} & \\ 0 & & & p g_{33} \end{pmatrix}$, $T^\mu{}_\nu = g^{\mu\rho} T_{\rho\nu} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ $\quad T_{\mu\nu}^{tot} = \sum_{i=1}^{N} T_{\mu\nu}^{(i)}$ each component

$T = T^\mu{}_\mu = -\rho + 3p$ $\quad$ with conformal time $\quad dt = a\, d\eta$

* Energy - momentum conservation $\quad T_{\mu\nu;\mu} = 0$ $\quad$ covariant derivative

$0 = T^\mu{}_{0;\mu} = \dot{\rho} + 3H(\rho + p)$ $\quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$

$\Gamma^0_{00} = H$. $\Gamma^0_{ij} = H \tilde{g}_{ij}$. $\Gamma^\alpha_{\beta 0} = H \delta^\alpha_\beta$ $\quad \Gamma^\alpha_{\beta\gamma}$

$\therefore \rho \propto a^{-3(1+\omega)}$ $\quad p \equiv \omega \cdot \rho$ $\quad$ Equation of state for perfect fluids

* Components

energy density mc^2/V

$\omega = 0$ dust , matter (pressureless) $\quad$ mde $\quad \rho \propto a^{-3}$, $p = 0$ $\quad$ also baryons

$\omega = 1/3$ radiation. relativistic $\quad$ rde $\quad \rho \propto a^{-4}$, $p = 1/3\, \rho$ $\quad n_\gamma \sim 1/a^3$. $E_\gamma \sim \frac{1}{a}$ $\therefore \rho \sim 1/a^4$

$\omega = -1$ $\quad p = -\rho$ (negative pressure) $\quad$ vacuum energy $\quad \rho$ = constant

not empty

$dQ = 0$ $\quad \therefore dU = \rho\, dV = -p\, dV$

Λ : Einstein's biggest blunder
static & infinite vs collapse
1917 before Hubble
$\Lambda \to p<0$ balance $\Rightarrow$ unstable

Friedmann - Lemaître equation

solution

Einstein eq. : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

$\uparrow$ ~ integral constant , satisfying the Bianchi identity $g_{\mu\nu;\mu} = 0$

metricity . length preservation

put it in RHS

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \qquad \therefore \rho = -p = \frac{\Lambda}{8\pi G} \qquad \omega_\Lambda = -1$$

Friedman equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} , \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$G^0_0 = 8\pi G T^0_0$ $\qquad$ $G^\alpha_\beta = 8\pi G T^\alpha_\beta$ & 1st. Fried.

$$2 u_{\mu;[\nu\rho]} = u_\varepsilon R^\varepsilon_{\ \mu\nu\rho}$$

$$R^\mu_{\ \nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\varepsilon_{\nu\sigma}\Gamma^\mu_{\rho\varepsilon} - \Gamma^\varepsilon_{\nu\rho}\Gamma^\mu_{\sigma\varepsilon}$$

$$R_{\mu\nu} = R^\rho_{\ \mu\rho\nu} \qquad R = R^\mu_\mu$$

$$R_{00} = -3(\dot H + H^2) \qquad R_{\alpha\beta} = a^2 \bar g_{\alpha\beta}\left(\dot H + 3H^2 + \frac{2K}{a^2}\right)$$

$$R = 6\left(\dot H + 2H^2 + \frac{K}{a^2}\right) \qquad \frac{\ddot a}{a} = H^2 + \dot H$$

how fast the expansion? $\qquad$ acceleration or deceleration
$\omega < -1/3$

for ordinary compo. $\rho \geq 0$. $p \geq 0 \xrightarrow{K=0} \ddot a < 0$. decelerating expansion
$\uparrow$ gravity $\qquad$ $\uparrow$ initial cond.

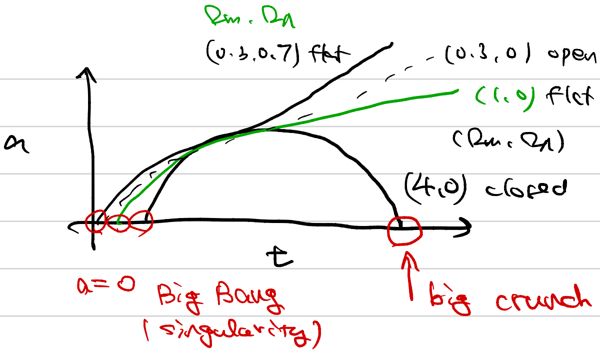
cosmological constant $\omega = -1 \to \ddot a > 0$

* density parameter $\Omega_i \equiv \frac{8\pi G}{3H^2}\rho_i \equiv \rho_i / \rho_{crit}$ $\qquad$ $\rho^{-1}_{crit} = \frac{8\pi G}{3H^2}$ $\qquad$ $H_0 = 100\, h$ km/s/Mpc

$$H^2 = H^2 \Omega_{tot} - \frac{K}{a^2} = H^2(\Omega_{tot} + \Omega_K) , \quad \Omega_K \equiv -\frac{K}{a^2 H^2} , \quad \Omega_{tot} + \Omega_K = 1 .$$

$\Omega_{tot} \ni \Omega_{dm} , \Omega_b . \Omega_{rad} , \Omega_\Lambda , \cdots$

$\nwarrow$ cannot be put in $T_{\mu\nu}$



$\Omega_{tot} + \Omega_k = 1$, $H^2 = H_0^2 \Omega_{tot} - k/a^2$ $\quad \Omega_k \sim 0$

today : $\Omega_m \sim 0.3$. $\Omega_\Lambda \sim 0.7$ → acceleration (dark energy) $\quad \Omega_r \sim 10^{-6}$

20 years ago : $\Omega_m \sim 0.3 - 1.0$. $\Omega_r \sim 10^{-6}$. $\Omega_k \sim \Omega_\Lambda \sim 0$

deceleration parameter

$k > 0$ closed → $\dot{a} = 0$ possible

$\therefore \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-n}$, $H = H_0 \left(\frac{t_0}{t}\right) = \frac{2}{nt}$

Solutions to Friedman equation

$k = 0$, $\rho \propto a^{-n}$, $n = 3(1+w)$, $H = \frac{\dot{a}}{a} = \frac{d \ln a}{dt}$ $\quad \therefore \sqrt{\frac{8\pi G \rho_0}{3}}\, dt = \left(\frac{a}{a_0}\right)^{n/2} d\ln a$

$\therefore a \sim t^{2/n} \sim \eta^{2/(2-n)}$, $H = H_0 (t_0/t) = \frac{2}{nt}$, $H = 2/3t$ for mde $\quad H = 1/2t$ for rde

de Sitter solution (empty univ.)

$\Lambda > 0$ $\quad k = 0$. $\quad a \sim \exp\left[\pm \sqrt{\frac{\Lambda}{3}}\, t\right]$

how can we tell which one describes our univ?