Cosmological constant & vacuum energy

* **Einstein – Hilbert action**

$$S_g = \int d^4x \sqrt{-g}\, \frac{R c^4}{16\pi G}$$

Lorentz invariant . in fact diffeomorphism invariance

$[R] = L^{-2}$ Λ^2_{QG} $\frac{\partial x^\mu}{\partial x^\nu}$ (or general covariance)

other actions $R^{\mu\nu}R_{\mu\nu}$, R^n , $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ not all ind.

* **Matter Lagrangian (density) $\mathcal{L}_m$**

$S_m = \int d^4x \sqrt{-g}\, \mathcal{L}_m$ minimally coupled. $T_{\mu\nu} \equiv -2\frac{\delta S_m}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\, \mathcal{L}_m$

* **variations wrt. $g_{\mu\nu}$. integration by part**

$$\delta S = \int d^4x \sqrt{-g}\, \delta g_{\mu\nu}\left(-\frac{1}{16\pi G}G^{\mu\nu} + \frac{1}{2}T^{\mu\nu}\right) \qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$\therefore\ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^2}T_{\mu\nu}$ $\qquad G_{\mu\nu}{}^{;\mu} = 0 = T_{\mu\nu}{}^{;\mu}$

LHS : geometry . RHS : matter conserved . Bianchi identity

* **Cosmological constant or vacuum energy**

$G_{\mu\nu} \to G_{\mu\nu} + \Lambda g_{\mu\nu}$ or $T_{\mu\nu} \to T_{\mu\nu} + T^\Lambda_{\mu\nu}$, $T^\Lambda_{\mu\nu} \equiv -\frac{\Lambda c^2}{8\pi G}g_{\mu\nu}$

$g_{\mu\nu;\mu} = 0$ at the moment Λ is just a constant of theory

or constant density everywhere (⇒ vacuum energy)

in terms of perfect fluid $T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$ $\qquad \rho_\Lambda = -p_\Lambda = \frac{\Lambda c^2}{8\pi G}$

in fact the only choice for vacuum energy density → $\rho_{vac}\eta_{\mu\nu}$ → $\rho_{vac}g_{\mu\nu}$ with constant ρ_{vac}

Lambda as vacuum energy

all physics except gravity : independent of absolute energy but only difference

gravity: separate kinetic energy by going to negative. peculiar! zero can be shifted

GR responds to a constant energy $\Lambda g_{\mu\nu}$

* Quantum Mechanics : A Simple harmonic oscillator

$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$ $\quad E = \frac{1}{2}\hbar\omega\,(1+2n)$ ground state ($n=0$) has energy $\frac{1}{2}\hbar\omega$ compared to classical vacuum

* Quantum field : $q \to \phi(x)$, $p \to \pi(x) \simeq \partial\mathcal{L}/\partial\dot{\phi}$ infinite dof QM

$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(\underbrace{\hat{N}_p}_{=a^\dagger a} + \frac{1}{2}[q_p, q_p^\dagger] \right)$ $\quad E_p^2 = m^2 + p^2$ all frequency

$\Rightarrow E_{vacuum} = \int \frac{d^3p}{(2\pi)^3} E_p \, \frac{1}{2}\cdot(2\pi)^3\delta^3(p-p) = \frac{1}{2}V\int \frac{d^3p}{(2\pi)^3} E_p$

$\therefore \rho_{vac} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} E_p \sim \int dp\, p^2 p \sim p_{max}^4$

* Cut-off scales

Planck scale $m_p^2 = \hbar c/G \sim (10^{19}\,GeV)^2 \sim (10^{-5}\,g)^2$. $\quad \ell_p \sim 10^{-33}\,cm$

GUT $m_{GUT} \sim 10^{16}\,GeV$ $\quad t_p \sim 10^{-44}\,sec$

Standard model $m_{std} \sim 1\,TeV \sim 10^3\,GeV$

* observations : $\rho_\Lambda \sim (10^{-12}\,GeV)^4$ $\Rightarrow$ $(10^{19+12})^4 \sim 120$ orders

$\rho_c \sim 10^{-5}\,mpc^2/cm^3$ or ~ 60 orders (standard)

Cosmological constant problem

120 order difference. even with std model. 60 orders!
really problem? nobody knows.

* A toy model in QFT

$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 - \rho_0$ bare quantities with 0

bare quantities are renormalized & only the renormalized match to experiments or observations → bare quantities : not predicted. no physical meaning

* Matching $\rho_{renorm} \sim \rho_0(m_{cut}) + m_{cut}^4 \equiv \rho_{obs}$

implies 60 - 120 order digit fine tuning

* QED electron mass : $m_{renorm} \sim m_0\left(1 + \frac{3\alpha}{4\pi}\log\frac{m_{cut}^2}{m_0^2}\right)$

~ 0.1 for M_{pl} & $m_0 \sim$ MeV

* Higgs mass (hierarchy prob.) : $m_{Higgs}^2 \sim m_0^2 + m_{cut}^2$

$= (10^2\,\text{GeV})^2$ $\sim ((10^{19}\,\text{GeV})^2$

↳ depends on particle contents if $m \gg m_{std}$ (vs. vacuum)

still all this fine tuning argument may not be real, as bare has no physical meaning

Cosmological constant as Einstein's biggest blunder

Einstein formulated GR in 1917 & applied to the Universe (or MW galaxy) before the discovery of the Hubble expansion (extra galaxies)

gravity of matter pulls things together & collapse
Einstein wanted some static & infinite universe, so add positive Λ to counter balance the pull

prob.: Λ is constant, but matter distribution is not → unstable
Hubble expansion was discovered