4 Relativistic Perturbation Theory

4.1 Metric Decomposition and Gauge Transformation

4.1.1 FRW Metric and its Perturbations

We describe the background for a spatially homogeneous and isotropic universe with the FRW metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -a^2(\eta)\, d\eta^2 + a^2(\eta)\, \bar{g}_{\alpha\beta}\, dx^\alpha dx^\beta \,, \tag{4.1}$$

where $a(\eta)$ is the scale factor and $\bar{g}_{\alpha\beta}$ is the metric tensor for a three-space with a constant spatial curvature $K = -H_0^2\,(1-\Omega_{\rm tot})$. We use the Greek indices $\alpha, \beta, \cdots$ for 3D spatial components and $\mu, \nu, \cdots$ for 4D spacetime components, respectively. To describe the real (inhomogeneous) universe, we parametrize the perturbations to the homogeneous background metric as

$$g_{00} := -a^2\,(1+2A)\,, \qquad g_{0\alpha} := -a^2 B_\alpha, \qquad g_{\alpha\beta} := a^2\,(\bar{g}_{\alpha\beta} + 2C_{\alpha\beta})\,, \tag{4.2}$$

where 3-tensor A, B_α and $C_{\alpha\beta}$ are perturbation variables and they are based on the 3-metric $\bar{g}_{\alpha\beta}$. Due to the symmetry of the metric tensor, we have ten components, capturing the deviation from the background. The inverse metric tensor can be obtained by using $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ and expanding to the linear order as

$$g^{00} = \frac{1}{a^2}\,(-1+2A)\,, \qquad g^{0\alpha} = -\frac{1}{a^2}B^\alpha\,, \qquad g^{\alpha\beta} = \frac{1}{a^2}\left(\bar{g}^{\alpha\beta} - 2C^{\alpha\beta}\right). \tag{4.3}$$

For later convenience, we also introduce a time-like four-vector, describing the motion of an observer ($-1 = u_\mu u^\mu$):

$$u^0 = \frac{1}{a}\,(1-A)\,, \qquad u^\alpha := \frac{1}{a}U^\alpha\,, \qquad u_0 = -a\,(1+A)\,, \tag{4.4}$$

$$u_\alpha = a\,(U_\alpha - B_\alpha) := a\, v_\alpha := a(-v_{,\alpha} + v^{(v)}_\alpha)\,, \qquad v := U + \beta\,, \qquad v^{(v)}_\alpha = U^{(v)}_\alpha - B^{(v)}_\alpha\,, \tag{4.5}$$

where U^α is again based on $\bar{g}_{\alpha\beta}$.

4.1.2 Scalar-Vector-Tensor Decomposition

Given the splitting of the spatial hypersurface and the symmetry associated with it, we decompose the perturbation variables (*to all orders*) as

$$A := \alpha\,, \qquad B_\alpha := \beta_{,\alpha} + B^{(v)}_\alpha\,, \qquad C_{\alpha\beta} := \varphi\bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}\,, \qquad U^\alpha := -U^{,\alpha} + U^{(v)\alpha}\,, \tag{4.6}$$

subject to the transverse and traceless conditions:

$$B^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \qquad C^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \qquad v^{(v)\alpha}{}_{|\alpha} \equiv 0\,, \qquad C^{(t)\alpha}{}_\alpha \equiv 0\,, \qquad C^{(t)\beta}{}_{\alpha|\beta} \equiv 0\,, \qquad U^{(v)\alpha}{}_{|\alpha} = 0\,, \tag{4.7}$$

where the vertical bar represents the covariant derivative with respect to the 3-metric $\bar{g}_{\alpha\beta}$:

$$X^\alpha{}_{|\beta} = X^\alpha{}_{,\beta} + \bar{\Gamma}^\alpha_{\beta\gamma}X^\gamma\,, \qquad X_{\alpha|\beta} = X_{\alpha,\beta} - \bar{\Gamma}^\gamma_{\alpha\beta}X_\gamma\,. \tag{4.8}$$

This simply implies that the scalar perturbations describe the longitudinal modes and the vector (v) and the tensor (t) perturbations describe the transverse modes. Furthermore, the tensor perturbation is traceless. The decomposed scalar perturbations can be obtained as

$$\beta = \Delta^{-1}\nabla^\alpha B_\alpha\,, \qquad \gamma = \frac{1}{2}\left(\Delta + \frac{1}{2}\bar{R}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta C_{\alpha\beta} - C^\alpha_\alpha\right)\,, \tag{4.9}$$

$$\varphi = \frac{1}{3}C^\alpha_\alpha - \frac{1}{6}\Delta\left(\Delta + \frac{1}{2}\bar{R}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta C_{\alpha\beta} - C^\alpha_\alpha\right)\,,$$

where ∇_α is the covariant derivative based on $\bar{g}_{\alpha\beta}$ (i.e., vertical bar) and $\Delta = \nabla^\alpha\nabla_\alpha$ is the Laplacian operator. The presence of the Ricci scalar ($\bar{R} = 6K$) for the three-space indicates that covariant derivatives are non-commutative.

$$\bar{R}_{\alpha\beta\gamma\delta} = 2K\ \bar{g}_{\alpha[\gamma}\bar{g}_{\delta]\beta}\ . \tag{4.10}$$

The decomposed vector and tensor components are computed in a similar manner as

$$\begin{aligned} B^{(v)}_\alpha &= B_\alpha - \nabla_\alpha\Delta^{-1}\nabla^\beta B_\beta\ , \qquad C^{(v)}_\alpha = 2\left(\Delta + \frac{1}{3}\bar{R}\right)^{-1}\left[\nabla^\beta C_{\alpha\beta} - \nabla_\alpha\Delta^{-1}\nabla^\beta\nabla^\gamma C_{\beta\gamma}\right]\ , \\ C^{(t)}_{\alpha\beta} &= C_{\alpha\beta} - \frac{1}{3}C^\gamma_\gamma\bar{g}_{\alpha\beta} - \frac{1}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}\bar{g}_{\alpha\beta}\Delta\right)\left(\Delta + \frac{1}{2}\bar{R}\right)^{-1}\left[3\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\gamma\delta} - C^\gamma_\gamma\right] \\ &\quad -2\nabla_{(\alpha}\left(\Delta + \frac{1}{3}\bar{R}\right)^{-1}\left[\nabla^\gamma C_{\beta)\gamma} - \nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\gamma\delta}\right]\ , \end{aligned} \tag{4.11}$$

and they satisfy the transverse condition $B^{(v)\alpha}{}_{|\alpha} = C^{(v)\alpha}{}_{|\alpha} = C^{(t)\beta}_{\alpha|\beta} = 0$ and the traceless condition $C^{(t)\alpha}_\alpha = 0$.

4.1.3 Comparison in Notation Convention

Bardeen convention:

$$A \to \alpha\ , \qquad B^{(0)} \to -k\beta\ , \qquad H_L \to \varphi + \frac{1}{3}\Delta\gamma\ , \qquad H_T \to -\Delta\gamma\ , \tag{4.12}$$

$$B^{(1)}Q^{(1)}_\alpha \to B_\alpha\ , \qquad H^{(1)}_T Q_\alpha \to -kC_\alpha\ , \qquad H^{(2)}_T Q_{\alpha\beta} \to C_{\alpha\beta}\ . \tag{4.13}$$

Weinberg convention:

$$\Phi \to \alpha_\chi\ , \qquad \Psi \to -\varphi_\chi\ , \qquad \delta u \to -av_\chi \qquad \mathcal{R} \to \varphi_v\ , \qquad \zeta \to \varphi_\delta\ , \tag{4.14}$$

$$\delta p \to \delta p - \frac{1}{3a^2}\Delta\Pi\ , \qquad \pi^S := \delta\sigma \to \frac{\Pi}{a^2}\ , \qquad \pi^V_i \to \frac{1}{2a}\Pi_\alpha\ , \qquad \pi^T_{ij} \to \Pi^{(t)}_{\alpha\beta}\ . \tag{4.15}$$

Dodelson convention:

$$\psi \to \alpha_\chi\ , \qquad \phi \to \varphi_\chi\ , \qquad ikv \to k^2 v_\chi\ , \qquad v \to -ikv_\chi\ . \tag{4.16}$$

Ma & Bertschinger:

$$\psi \to \alpha_\chi\ , \qquad \phi \to -\varphi_\chi\ , \qquad h \to 6\varphi_v + 2\Delta\gamma\ , \qquad \eta \to -\varphi_v\ , \qquad \theta \to k^2 v_\chi\ . \tag{4.17}$$

CLASS Boltzmann code:

$$\psi \to \alpha_\chi\ , \qquad \phi \to -\varphi_\chi\ , \qquad \theta \to k^2 v\ , \tag{4.18}$$

where θ_i and δ_i depend on the choice of gauge condition.

4.1.4 Gauge Transformation

The general covariance of general relativity guarantees that any coordinate system can be used to describe the physics and it has to be independent of coordinate systems. This is known as the diffeomorphism symmetry in general relativity. However, when we split the metric into the background and the perturbations around it by choosing a coordinate system, we explicitly change the correspondence of the physical Universe to the background homogeneous and isotropic Universe. Hence, the metric perturbations transform non-trivially (or gauge transform), and the diffeomorphism invariance implies that the physics should be gauge-invariant.

The gauge group of general relativity is the group of diffeomorphisms. A diffeomorphism corresponds to a differentiable coordinate transformation. The coordinate transformation on the manifold $\mathcal{M}$ can be considered as one generated by a smooth vector field ζ^μ. Given the vector field ζ^μ, consider the solution of the differential equation

$$\left.\frac{d\chi^\mu(\lambda)}{d\lambda}\right|_P = \zeta^\mu\left[\chi^\nu_P(\lambda)\right]\ , \qquad \chi^\mu_P(\lambda = 0) = x^\mu_P\ , \qquad \frac{d}{d\lambda} = \zeta^\mu\partial_\mu\ , \tag{4.19}$$

defines the parametrized integral curve $x^\mu(\lambda)=\chi^\mu_P(\lambda)$ with the tangent vector $\zeta^\mu(x_P)$ at P. Therefore, given the vector field ζ^μ on $\mathcal{M}$ we can define an associated coordinate transformation on $\mathcal{M}$ as $x^\mu_P\rightarrow\tilde x^\mu_P=\chi^\mu_P(\lambda=1)$ for any given P. Assuming that ζ^μ is small one can use the perturbative expansion for the solution of equation to obtain

$$\tilde x^\mu_P=\chi^\mu_P(\lambda=1)=\chi^\mu_P(\lambda=0)+\frac{d}{d\lambda}\chi^\mu_P\bigg|_{\lambda=0}+\frac{1}{2}\frac{d^2}{d\lambda^2}\chi^\mu_P\bigg|_{\lambda=0}+\cdots=x^\mu_P+\zeta^\mu(x_P)+\frac{1}{2}\zeta^\mu{}_{,\nu}\zeta^\nu+\mathcal{O}(\zeta^3)=e^{\zeta^\nu\partial_\nu}x^\mu\,. \tag{4.20}$$

This parametrization corresponds to the gauge-transformation with ζ^μ.

In general, any gauge-transformation of tensor $\mathbf{T}$ for an infinitesimal change ζ can be expressed in terms of the Lie derivative (valid to all orders of $\mathbf{T}$)

$$\delta_\zeta\mathbf{T}:=\tilde{\mathbf{T}}-\mathbf{T}=-\pounds_\zeta\mathbf{T}+\mathcal{O}(\zeta^2)\,,\qquad \pounds_\zeta A^\mu=A^\mu{}_{,\nu}\zeta^\nu-\zeta^\mu{}_{,\nu}A^\nu\,,\qquad \pounds_\zeta T_{\mu\nu}=T_{\mu\nu,\rho}\zeta^\rho+T_{\rho\nu}\zeta^\rho{}_{,\mu}+T_{\mu\rho}\zeta^\rho{}_{,\nu}\,, \tag{4.21}$$

where they are all evaluated at the same coordinate and the derivatives in the Lie derivatives can be replaced with covariant derivatives (Lie derivatives are tensorial). To all orders in ζ, we have

$$\tilde{\mathbf{T}}(x)=\mathbf{T}(x)-\pounds_\zeta\mathbf{T}+\frac{1}{2}\pounds_\zeta^2\mathbf{T}+\cdots=\exp\left[-\pounds_\zeta\right]\mathbf{T}\,. \tag{4.22}$$

Therefore, the gauge-transformation in perturbation theory is simply

$$\delta_\zeta\bar{\mathbf{T}}=0\,,\qquad \delta_\zeta\mathbf{T}^{(1)}=-\pounds_\zeta\bar{\mathbf{T}}\,,\qquad \delta_\zeta\mathbf{T}^{(n)}=-\pounds_\zeta\mathbf{T}^{(n-1)}\,, \tag{4.23}$$

where we used that ζ is also a perturbation.

In fact, there are two ways of looking at the transformation in perturbation theory. For example, the metric tensor has to transform as a tensor. But once we split it into the background and the perturbations, there exist two ways

$$g_{\mu\nu}=\bar g_{\mu\nu}+h_{\mu\nu}\,,\qquad \delta_\zeta g=-\pounds_\zeta g\ :\ \ (1)\ \delta_\zeta\bar g=-\pounds_\zeta\bar g\,,\quad \delta_\zeta h=-\pounds_\zeta h\,,\qquad (2)\ \delta_\zeta\bar g=0\,,\quad \delta_\zeta h=-\pounds_\zeta h-\pounds_\zeta\bar g\,, \tag{4.24}$$

where we suppressed the tensor indices. In (1), the background and the perturbation transform altogether like tensors (at the same coordinates), such that the sum transforms like a tensor. In perturbation theory, we do not use this, because the infinitesimal transformation ζ is always considered as a perturbation. However, for example we can consider some general spatial rotation ζ, such that the background metric also changes.[1]

4.1.5 Linear-Order Gauge Transformation

At the linear order, the Lie derivative is trivial, and the the most general coordinate transformation in Eq. (4.20) becomes

$$\tilde x^\mu=x^\mu+\xi^\mu\,,\qquad \xi^\mu:=(T,\mathcal{L}^\alpha)\,,\qquad \mathcal{L}^\alpha:=L^{,\alpha}+L^{(v)\alpha}\,, \tag{4.25}$$

where we now use $\xi^\mu=\zeta^\mu$. The transformation of the metric tensor at the leading order in ξ is then

$$\delta_\xi g_{\mu\nu}(x):=\tilde g_{\mu\nu}(x)-g_{\mu\nu}(x)=-\pounds_\xi g_{\mu\nu}=-\left(\xi_{\mu;\nu}+\xi_{\nu;\mu}\right)\,,\qquad \xi_\mu=g_{\mu\nu}\xi^\nu=a^2(-T,\ \mathcal{L}_\alpha)\,, \tag{4.26}$$

where the semi-colon represents the covariant derivative with respect to the full metric $g_{\mu\nu}$. The transformation equations are explicitly

$$\delta_\xi g_{00}\ =\ -2a^2\,\delta_\xi A=2a^2\left[\frac{1}{a}\left(aT\right)'\right]\,,\qquad \delta_\xi g_{0\alpha}=-a^2\,\delta_\xi B_\alpha=a^2\left(T_{,\alpha}-\mathcal{L}'_\alpha\right)\,, \tag{4.27}$$

$$\delta_\xi g_{\alpha\beta}\ =\ 2a^2\,\delta_\xi C_{\alpha\beta}=-2a^2\left[\mathcal{H}T\bar g_{\alpha\beta}+\mathcal{L}_{(\alpha|\beta)}\right]\,. \tag{4.28}$$

Using the scalar-vector-tensor decomposition, we derive that the scalar quantities gauge-transform as

$$\tilde\alpha=\alpha-\frac{1}{a}(aT)'\,,\qquad \tilde\beta=\beta-T+L'\,,\qquad \tilde\varphi=\varphi-\mathcal{H}T\,,\qquad \tilde\gamma=\gamma-L\,, \tag{4.29}$$

$$\tilde U=U-L'\,,\qquad \tilde v=v-T\,,\qquad \tilde\chi=\chi-aT\,,\qquad \tilde\kappa=\kappa+3\dot HaT+\frac{\Delta}{a}T\,, \tag{4.30}$$

[1] In FRW, we use the spatial metric $\bar g_{\alpha\beta}$ unspecified, implying we can do a further spatial transformation to e.g., spherical coordinate and so on and change the background metric (while it remains covariant). The time component is fixed, otherwise it ruins the FRW symmetry ($g_{0\alpha}$ component in the background or different coefficient in time component for example.

the vector metric perturbations gauge-transform as

$$\tilde{B}_\alpha^{(v)} = B_\alpha^{(v)} + L'_\alpha \,, \qquad \tilde{C}_\alpha^{(v)} = C_\alpha^{(v)} - L_\alpha \,, \qquad \tilde{U}_\alpha^{(v)} = U_\alpha^{(v)} + L_\alpha^{(v)\prime} \,, \tag{4.31}$$

and the tensor perturbations are gauge-invariant at the linear order, where we defined the scalar shear $\chi := a\,(\beta + \gamma')$ of the normal observer ($n_\alpha = 0$) and the extrinsic 3-curvature $K := -3H + \kappa$ and its perturbation $\kappa := 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi$.

Based on the above gauge transformation properties, we can construct linear-order gauge-invariant quantities. The gauge-invariant variables are

$$\varphi_v := \varphi - aHv \,, \qquad \varphi_\chi := \varphi - H\chi \,, \qquad v_\chi := v - \frac{1}{a}\chi \,, \qquad \delta_v := \delta - a\frac{\dot{\rho}}{\rho}v \,, \tag{4.32}$$

$$\alpha_\chi := \alpha - \frac{1}{a}\chi' \,, \qquad \varphi_\delta := \varphi + \frac{\delta\rho}{3(\rho + p)} , \qquad \Psi_\alpha^{(v)} := B_\alpha^{(v)} + C_\alpha^{(v)\prime} \,, \qquad v_\alpha^{(v)} := U_\alpha^{(v)} - B_\alpha^{(v)} \,.$$

These gauge-invariant variables ($\alpha_\chi, \varphi_\chi, v_\chi, \Psi_\alpha, v_\alpha^{(v)}$) correspond to Φ_A, Φ_H, $v_s^{(0)}$, Ψ, and v_c in Bardeen (1980).

4.1.6 Popular Choices of Gauge Condition

By a suitable choice of coordinates, we can set $T = L = 0$, simplifying the metric. For simplicity, we only consider the scalar perturbations in the following two cases.

• ***The conformal Newtonian Gauge***.— in which we choose the spatial and the temporal gauge conditions:

$$\tilde{\gamma} = \gamma = 0 \;\; \rightarrow \;\; L = 0 \,, \qquad \tilde{\beta} = \beta = 0 \;\; \rightarrow \;\; T = 0 \,, \qquad \chi = 0 \,. \tag{4.33}$$

All the gauge modes are fixed, and the metric in this gauge condition is

$$ds^2 = -a^2\,(1 + 2\psi)\,d\eta^2 + a^2\,(1 + 2\phi)\,\bar{g}_{\alpha\beta}dx^\alpha dx^\beta \,, \qquad \psi := \alpha = \alpha_\chi \,, \qquad \phi := \varphi = \varphi_\chi \,, \tag{4.34}$$

and the velocity vector is then

$$U = v_\chi \,, \qquad \boldsymbol{v} = -\nabla U \,. \tag{4.35}$$

The metric and its equations appear more like the Newtonian equations, and hence the name. We will use this gauge condition to illustrate and simplify the problems.

• ***Synchronous-Comoving Gauge***.— in which we choose the spatial and the temporal gauge conditions:

$$\tilde{\alpha} = \alpha = 0 \;\; \rightarrow \;\; (aT)' = 0 \,, \qquad \tilde{\beta} = \beta = 0 \;\; \rightarrow \;\; T = L' \,, \tag{4.36}$$

such that the metric becomes

$$ds^2 = -a^2 d\eta^2 + a^2\,(\bar{g}_{\alpha\beta} + 2C_{\alpha\beta})\,dx^\alpha dx^\beta \,. \tag{4.37}$$

All the metric perturbations in this gauge condition are included in the spatial metric tensor. However, as apparent from the above gauge condition, the gauge freedoms are not completely fixed:

$$T = L' = \frac{1}{a}F(\mathbf{x}) \,, \qquad L = \int d\eta\,\frac{F(\mathbf{x})}{a} + G(\mathbf{x}) \,, \tag{4.38}$$

where F and G are two arbitrary functions of spatial coordinates. Typically, this issue is resolved by assuming additional condition at the initial epoch

$$v = 0 \;\; \rightarrow \;\; F(\mathbf{x}) = 0 \,, \quad T = 0 \,. \tag{4.39}$$

This condition is indeed the temporal comoving gauge condition, and hence the whole choice is often referred to as the comoving-synchronous gauge (or synchronous-comoving). The comoving gauge is often chosen with a different spatial gauge condition ($\gamma = 0$). Note, however, that the spatial function $G(\mathbf{x})$ is still left unspecified, and hence γ is a gauge mode, whereas U for example is physical. Due to this deficiency, we will not use this gauge condition in the following.

The notation convention in Ma and Bertschinger (1995):

$$h_{ij} := \hat{k}_i\hat{k}_j h + \left(\hat{k}_i\hat{k}_j - \frac{1}{2}\delta_{ij}\right) 6\eta \;\; \rightarrow \;\; 2C_{ij} \,, \qquad h \rightarrow 6\varphi + 2\Delta\gamma \,, \qquad \eta \rightarrow -\varphi_v \,. \tag{4.40}$$

4.2 Energy-Momentum Tensor

4.2.1 Formal Definition

We will consider a simple action of the matter sector, in addition to the gravity described by the Einstein-Hilbert action:

$$S =: S_g + S_m =: \int \sqrt{-g}\, d^4x \left[\frac{R}{16\pi G} - \frac{\Lambda}{8\pi G} + \mathcal{L}_m\right] , \tag{4.41}$$

where the matter Lagrangian includes the cosmological fluids and other matter fields such as scalars and so on. The variation with respect to the metric,

$$0 = \frac{\delta S}{\delta g^{\mu\nu}} = \frac{M_{\rm pl}^2}{2}\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right] + \frac{\delta S_m}{\delta g^{\mu\nu}} , \tag{4.42}$$

yields the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{1}{M_{\rm pl}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = 8\pi G\, T_{\mu\nu} , \tag{4.43}$$

where the energy-momentum tensor defined by the action

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_m - 2\,\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\, \mathcal{L}_m , \tag{4.44}$$

and it is indeed the conserved current (tensor) of the action under the space-time translation invariance. The Noether theorem says that when there exists a (global) symmetry, there exists a conserved current. The space-time translation invariance is the symmetry of general relativity, and the Noether current associated with this symmetry is the energy-momentum tensor:

$$T_{\mu\nu;\nu} = 0 . \tag{4.45}$$

Note that we can repeat the calculations with upper indicies and obtain

$$T^{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} = g^{\mu\nu}\mathcal{L}_m + 2\,\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\, \mathcal{L}_m , \tag{4.46}$$

but mind the subtlety, for example, for the scalar field action,

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi , \qquad \delta\mathcal{L}_\phi = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{2}\delta g_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi . \tag{4.47}$$

4.2.2 General Decomposition for Cosmological Fluids

For our purposes, we are not interested in the microscopic states of the systems, but interested in their macroscopic states, often described by the density, the pressure, the temperature, and so on. The energy-momentum tensor for a fluid can be expressed in terms of the fluid quantities measured by an observer with four velocity u^μ as (*the most general decomposition*)

$$T_{\mu\nu} := \rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu} , \qquad 0 = \mathcal{H}_{\mu\nu}u^\nu , \tag{4.48}$$

where $\mathcal{H}_{\mu\nu}$ is the projection tensor and

$$\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu , \qquad \mathcal{H}^\mu_\mu = 3 , \qquad u^\mu q_\mu = 0 = u^\mu\pi_{\mu\nu} , \qquad \pi_{\mu\nu} = \pi_{\nu\mu} , \qquad \pi^\mu_\mu = 0 . \tag{4.49}$$

The variables ρ, p, q_μ and $\pi_{\mu\nu}$ are the energy density, the isotropic pressure (including the entropic one), the (spatial) energy flux and the anisotropic pressure. Given the energy momentum tensor $T_{\mu\nu}$, an observer with $u^\mu_{\rm obs}$ measures

$$\rho_{\rm obs} = T_{\mu\nu}u^\mu_{\rm obs}u^\nu_{\rm obs} , \qquad p_{\rm obs} = \frac{1}{3}T_{\mu\nu}\hat{\mathcal{H}}^{\mu\nu} , \qquad q^{\rm obs}_\mu = -T_{\rho\sigma}u^\rho_{\rm obs}\hat{\mathcal{H}}^\sigma_\mu , \qquad \pi^{\rm obs}_{\mu\nu} = T_{\rho\sigma}\hat{\mathcal{H}}^\rho_\mu\hat{\mathcal{H}}^\sigma_\nu - p_{\rm obs}\hat{\mathcal{H}}_{\mu\nu} , \tag{4.50}$$

where $\hat{\mathcal{H}}_{\mu\nu}$ is the projection tensor in terms of $u^\mu_{\rm obs}$. Remember that these fluid quantities are observer-dependent, hence it would be ideal to provide the energy momentum tensor for each fluid in terms of the fluid quantities that would be measured for a fictitious observer moving together with the fluid $u^\mu_f = u^\mu_{\rm obs}$ (hence without spatial flux $q_\mu = 0$).

In general, we decompose the fluid quantities into the background and the perturbations:

$$\rho := \bar\rho + \delta\rho \,, \qquad p := \bar p + \delta p \,, \qquad \delta p := c_s^2\delta\rho + e \,, \qquad c_s^2 := \frac{\dot p}{\dot\mu} \,, \tag{4.51}$$

$$q_\alpha := aQ_\alpha \,, \qquad \pi_{\alpha\beta} := a^2\Pi_{\alpha\beta} \,, \qquad e := \dot p\Gamma \,, \qquad \Gamma = \frac{\delta p}{\dot p} - \frac{\delta\rho}{\dot\rho} \,, \tag{4.52}$$

where Q_α and $\Pi_{\alpha\beta}$ are based on $\bar g_{\alpha\beta}$. At the background level, all the above fluid quantities vanish, except $\rho = \bar\rho$ and $p = \bar p$. For the adiabatic perturbations, we have $e = 0$ and the sound speed is $c_s^2 = \delta p/\delta\rho = \dot p/\dot\rho$. For multiple fluids, we can add up the individual energy-momentum tensor to derive the total energy-momentum tensor. In the case of multiple fluids, their fluid velocities are not necessarily identical, and there exist non-vanishing energy flux. Non-vanishing e and Γ parametrize the entropic perturbations of the fluids.

Though these relations are exact, we will be concerned with linear-order perturbations. Raising the index of the energy momentum tensor, we derive

$$T^0_0 = -\rho + \mathcal{O}(2) \,, \qquad T^0_\alpha = (\bar\rho + \bar p)\,(U_\alpha - B_\alpha) + Q_\alpha + \mathcal{O}(2) \,, \tag{4.53}$$

$$T^\alpha_\beta = p\,\delta^\alpha_\beta + \Pi^\alpha_\beta + \mathcal{O}(2) \,, \qquad T^\alpha_0 = -(\bar\rho + \bar p)U^\alpha - Q^\alpha + \mathcal{O}(2) \,, \tag{4.54}$$

where all quantities are those appearing in $T_{\mu\nu}$. Given the conditions $0 = u^\mu q_\mu = u^\mu\pi_{\mu\nu}$, the (spatial) energy flux and the anisotropic pressure should satisfy

$$q_0 = 0 + \mathcal{O}(2) \,, \qquad \pi_{0\mu} = 0 + \mathcal{O}(2) \,. \tag{4.55}$$

• ***Gauge transformation properties of the fluid quantities.***—

$$\tilde\delta = \delta - \frac{\bar\rho'}{\bar\rho}\,T \,, \qquad \widetilde{\delta p} = \delta p - \bar p'\,T \,, \qquad \tilde Q_\alpha = Q_\alpha \,, \qquad \tilde\pi_{\mu\nu} = \pi_{\mu\nu} \,, \qquad \tilde e = e \,. \tag{4.56}$$

Note that the spatial energy flux is gauge-invariant, but dependent upon the observer choice.

4.2.3 Tetrad Approach

Given the observer with u^μ, one can define a local Lorentz frame (where the metric is Minkowski) by constructing three spacelike orthonormal vectors $[e_i]^\mu$. For example, one can construct three rectangular basis vectors $[e_x]^\mu$, $[e_y]^\mu$, $[e_z]^\mu$ and of course $[e_t]^\mu = u^\mu = -[e^t]^\mu$, where the component index a of $[e_a]^\mu$ represents their coordinates. The orthonormality condition and the spacelike normalization is

$$\eta_{ab} = g_{\mu\nu}[e_a]^\mu[e_b]^\nu \;\;\rightarrow\;\; \delta_{ij} = g_{\mu\nu}[e_i]^\mu[e_j]^\nu \,, \qquad 0 = g_{\mu\nu}[e_i]^\mu[e_t]^\nu \,, \qquad -1 = g_{\mu\nu}[e_t]^\mu[e_t]^\nu \,, \tag{4.57}$$

where $a, b, \dots = t, x, y, z$ represent the local Lorentz indices.

Assuming that the four velocity of the observer is indeed the fluid velocity, we can go to the rest-frame of the fluid by using the tetrad expressions as

$$\rho = T_{\mu\nu}[e_t]^\mu[e_t]^\nu \,, \qquad p = \frac{1}{3}T_{\mu\nu}\sum_{i=1}^3[e_i]^\mu[e_i]^\nu \,, \qquad \mathcal{H}_{\mu\nu} = \sum_{i=1}^3[e_i]_\mu[e_i]_\nu \,, \qquad q_\mu = \sum_{i=1}^3 q_i[e^i]_\mu \,. \tag{4.58}$$

The other fluid quantities can be readily expressed in terms of the tetrad basis, and the energy-momentum tensor in the rest-frame of the fluid is then

$$T_{ab} = \begin{pmatrix} \rho & -q_x & -q_y & -q_z \\ -q_x & p + \pi_{xx} & \pi_{xy} & \pi_{xz} \\ -q_y & \pi_{yx} & p + \pi_{yy} & \pi_{yz} \\ -q_z & \pi_{zx} & \pi_{zy} & p + \pi_{zz} \end{pmatrix} \,. \tag{4.59}$$

The orthogonality condition for the energy-flux and the anisotropic stress implies

$$0 = u^\mu q_\mu = q_t \,, \qquad 0 = u^\mu \pi_{\mu\nu} = \pi_{ta} \,, \qquad 0 = \pi^\mu_\mu = \pi^t_t + \pi^i_i = \pi_{ii} \,. \tag{4.60}$$

However, we should pay attention to the difference in the quantities expressed in the rest-frame and in the FRW coordinate:

$$q_i := q_\mu [e_i]^\mu = Q^{(1)}_\alpha \delta^\alpha_i + \mathcal{O}(2) \,, \qquad \pi_{ij} := \pi_{\mu\nu}[e_i]^\mu [e_i]^\nu = \Pi^{(1)}_{\alpha\beta}\delta^\alpha_i \delta^\beta_j + \mathcal{O}(2) \,. \tag{4.61}$$

When the fluid velocity is the same as the observer velocity, the spatial flux $q_i \equiv 0$, or $q_\mu \equiv 0$.

As noted, the fluid quantities are observer-dependent. Given the energy momentum tensor in terms of the fluid rest-frame quantities ($q_\mu = 0$), if the observer is moving with e^μ_t relative to the fluid velocity u^μ, the observer measures different fluid quantities from those defined in the rest frame:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu} + \pi_{\mu\nu} \,, \qquad \mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \,, \tag{4.62}$$

The energy-momentum tensor can be projected into the observer rest-frame as

$$\tilde{T}_{ab} = T_{\mu\nu}[e_a]^\mu [e_b]^\nu = (\rho + p) u_a u_b + p\, \eta_{ab} + \pi_{ab} \,, \qquad u_a := u_\mu [e_a]^\mu \,, \tag{4.63}$$

where we put tilde to emphasize that the components of $\tilde{T}_{ab}$ are observables. The fluid velocity and the anisotropic pressure satisfy

$$-1 = u^a u_a = -u_t^2 + u_i^2 \,, \qquad 0 = \pi^\mu_\mu \;\; \to \;\; \pi_{tt} = \pi_{ii} \,. \tag{4.64}$$

Furthermore, the anisotropic pressure is perpendicular to the fluid velocity:

$$0 = u^a \pi_{ab} = u^t \pi_{ta} + u^i \pi_{ia} \,, \qquad \pi_{ti} = \pi_{ij} \frac{u^j}{u^t} \,. \tag{4.65}$$

Therefore, the energy density and the pressure measured by the observer are

$$\tilde{\rho} := \tilde{T}_{tt} = (\rho{+}p)u_t^2 {-} p {+} \pi_{tt} \,, \qquad \tilde{p} := \frac{1}{3}\tilde{T}_{ii} = \frac{1}{3}(\rho{+}p)u_i u_i {+} p {+} \frac{1}{3}\pi_{ii} = p {+} \frac{1}{3}\left[(\rho + p)(u_t^2 - 1) + \pi_{tt}\right] \,, \tag{4.66}$$

and the anisotropic pressure is

$$\tilde{\pi}_{ij} = (\rho + p) u^i u^j + \pi_{ij} - \frac{1}{3}\,\delta_{ij}\left[(\rho + p)(u_t^2 - 1) + \pi_{tt}\right] \,, \qquad 0 = \tilde{\pi}_{ti} = \tilde{\pi}_{tt} \,. \tag{4.67}$$

Since the velocities of the fluid and the observer are different, the observer measures the non-vanishing spatial energy flux

$$\tilde{q}_i = \tilde{T}^t{}_i = (\rho + p) u^t u_i + \pi^t{}_i \,. \tag{4.68}$$

If the observer velocity is the fluid velocity, we obtain the consistency relation:

$$u_a = \eta_{ta} \,, \qquad \tilde{\rho} = \rho \,, \qquad \tilde{p} = p \,, \qquad \tilde{q}_i = 0 \,, \qquad \tilde{\pi}_{ij} = \pi_{ij} \,. \tag{4.69}$$

From Eq. (4.53), it is clear that at the linear order in perturbations the energy density ρ, pressure p, and anisotropic pressure $\Pi_{\alpha\beta}$ are the same as those measured by an observer, independent of the fluid or the observer velocity. However, the velocity of the fluid and the spatial flux measured by the observer are

$$u^a = e^a_\mu u^\mu = \left(1, U^i - U^i_{\rm obs}\right) + \mathcal{O}(2) \,, \qquad \tilde{q}_i = (\rho + p)(U^i - U^i_{\rm obs}) + \mathcal{O}(2) \,, \tag{4.70}$$

where the presence of the relative velocity between the fluid and the observer is apparent. In addition to the fluid quantities ρ, p, $\pi_{\mu\nu}$ (same for any observer), the energy-momentum tensor $T_{\mu\nu}$ can be written in terms of the fluid velocity without spatial flux $q_\mu \equiv 0$, and the off-diagonal part of the energy-momentum tensor is

$$T^0_\alpha = (\bar{\rho} + \bar{p})\left(U^f_\alpha - B_\alpha\right) + 0 \,. \tag{4.71}$$

One can also express $T_{\mu\nu}$ in terms of the observer velocity $u^\mu_{\rm obs}$ with non-vanishing spatial flux $\tilde{q}_i$ in Eq. (4.70), which defines

$$\tilde{q}_i =: [e_i]^\mu_{\rm obs} q^{\rm obs}_\mu = Q^{\rm obs}_\alpha \delta^\alpha_i + \mathcal{O}(2) \,, \tag{4.72}$$

and the off-diagonal part of the energy-momentum tensor is again identical:

$$T^0_\alpha = (\bar\rho+\bar p)\left(U^{\rm obs}_\alpha - B_\alpha\right) + Q^{\rm obs}_\alpha = (\bar\rho+\bar p)\left(U^f_\alpha - B_\alpha\right) \, . \tag{4.73}$$

In summary, the fluid quantities measured by the observer are

$$0=\pi_{tt}=\pi_{ti} \, , \qquad \tilde\rho=\rho \, , \qquad \tilde p = p \, , \qquad \tilde\pi_{ij}=\pi_{ij} \, , \qquad \tilde q_i = (\bar\rho+\bar p)\left(U^i - U^i_{\rm obs}\right) \, . \tag{4.74}$$

In other words, whoever the observer is, the fluid quantities the observer measures are identical to those at the fluid rest frame, except the spatial energy flux.

4.2.4 Distribution Function

In cosmology, photons and neutrinos are the most important radiation components, and they are not described by the fluid approximation. Their statistical properties are captured by the distribution function F:

$$F := \bar f + f \, , \tag{4.75}$$

where the background distribution $\bar f$ often follows the equilibrium distribution and the perturbation f describes the deviation from the equilibrium. The equilibrium distribution for massless particles is fully described by the physical momentum and the temperature, and it is independent of position and time.[2] In the rest-frame of an observer, the physical energy E and the momentum P^a can be measured, and the energy-momentum tensor can be re-constructed as

$$T^{ab} = g\int \frac{d^3P}{E} P^aP^bF \, , \tag{4.76}$$

where the four momentum satisfies the on-shell condition $-m^2 = P_aP^a$ and $E = P^t$ and g is the spin-degeneracy of the particle, equal to two for photons and one for left-handed neutrinos.

The fluid elements can be readily computed as

$$\rho = g\int d^3P\, EF \, , \qquad q^i = Q_\alpha\delta^\alpha_i = g\int d^3P\, P^iF \, , \qquad p\delta^{ij}+\pi^{ij} = g\int d^3P\,\frac{P^iP^j}{E}F \, . \tag{4.77}$$

For later convenience, we introduce an angular decomposition

$$f(P,\hat n) := \sum_{lm}(-i)^l\sqrt{\frac{4\pi}{2l+1}}f_{lm}(P)Y_{lm}(\hat n) \, , \qquad f_{lm}(P) \equiv i^l\sqrt{\frac{2l+1}{4\pi}}\int d^2\hat n\, Y^*_{lm}(\hat n)f(P,\hat n) \, , \tag{4.78}$$

where $P^i = Pn^i$ and $\hat n$ is the unit directional vector. The normalization convention may differ in literature. The perturbations in the fluid quantities are then related to the distribution function as

$$\delta\rho = 4\pi g\int_0^\infty dP\, P^2Ef_{00} \, , \qquad \delta p = \mathrm{Tr}\,\delta T^{ij} = \frac{4\pi g}{3}\int_0^\infty dP\,\frac{P^4}{E}f_{00} \, , \tag{4.79}$$

where we performed the angular integration. Higher moments of the fluid elements will be related to the higher-moments of the distribution function. At the linear order, the spatial energy flux from the distribution function is related to the relative velocity as

$$q_i = (\bar\rho+\bar p)\left(U^i_f - U^i_{\rm obs}\right) = Q^{\rm obs}_\alpha\delta^\alpha_i \, . \tag{4.80}$$

[2] In the background, the physical momentum and the temperature redshift in the same way.

4.2.5 Multiple Cosmological Fluids

The details are summarized in Hwang and Noh (2002). In a universe with multiple fluids $i = 1, \cdots, N$, each fluid component has different velocity $u^\mu_{(i)}$. Given the fluid components in a coordinate system, the energy-momentum tensor $T_{\mu\nu}$ is completely set (i.e., all the components of $T_{\mu\nu}$):

$$T^{\rm tot}_{\mu\nu} = \sum_i T^{(i)}_{\mu\nu}, \tag{4.81}$$

which defines (ignoring the super-script "total")

$$T^0_0 = -\rho + \mathcal{O}(2) , \qquad T^\alpha_\beta = p\,\delta^\alpha_\beta + \Pi^\alpha_\beta + \mathcal{O}(2) , \tag{4.82}$$

or defines

$$\rho_{\rm tot} = (\bar\rho + \delta\rho)_{\rm tot} = \sum_i \rho_{(i)} = -T^0_0 + \mathcal{O}(2) , \qquad p_{\rm tot} = \sum_i p_{(i)} , \qquad \Pi^{\rm tot}_{\alpha\beta} = \sum_i \Pi^{(i)}_{\alpha\beta} , \tag{4.83}$$

such that the energy density, pressure, anisotropic pressure are just the sum of individual fluids, regardless of fluid and observer velocities at the linear order. However, the velocity $u^\mu_{\rm tot}$ (and $q^{\rm tot}_\mu$) that would appear in the total energy momentum tensor is yet to be determined. In fact, we can define $u^\mu_{\rm tot}$ as one without spatial flux,[3] i.e.,

$$T^0_\alpha = (\bar\rho + \bar p)\,(U_\alpha - B_\alpha)_{\rm tot} + \mathcal{O}(2) := \sum_i (\bar\rho + \bar p)_{(i)} (U^{(i)}_\alpha - B_\alpha) + \mathcal{O}(2) . \tag{4.84}$$

For the case of multiple fluids, it is possible to have interactions between fluids, even in the background, such that the energy conservation law is

$$T^{(i)\mu}{}_{\nu;\mu} =: I^{(i)}_\nu , \qquad 0 = \sum_i I^{(i)}_\mu , \qquad T^{\rm tot}_{\mu\nu;\mu} = 0 . \tag{4.85}$$

In the background the conservation equation becomes

$$\dot{\bar\rho}_{(i)} + 3H\,(\bar\rho + \bar p)_{(i)} = \bar I_{(i)} , \qquad \frac{\dot{\bar\rho}_{(i)}}{\rho_{(i)}} = -3H(\bar\rho + \bar p)_{(i)}(1 - q_{(i)}) , \tag{4.86}$$

where we defined (ignoring the vector type)

$$I^{(i)}_0 =: -a\left[\bar I_{(i)}(1+\alpha) + \delta I_{(i)}\right] , \qquad I^{(i)}_\alpha =: J^{(i)}_{,\alpha} , \qquad \bar I_{(i)} =: 3H(\bar\rho + \bar p)_{(i)} q_{(i)} . \tag{4.87}$$

At the background level, the equation of state and the sound speed of the individual components are

$$w_{(i)} := \frac{\bar p_{(i)}}{\bar\rho_{(i)}} , \qquad c^2_{s(i)} := \frac{\dot{\bar p}_{(i)}}{\dot{\bar\rho}_{(i)}} = w_{(i)} + \frac{dw_{(i)}}{d\ln\bar\rho_{(i)}} , \qquad \frac{1}{1+w} = \sum_i \frac{x_{(i)}}{1+w_{(i)}} , \tag{4.88}$$

$$\dot w_{(i)} = -3H(c^2_{s(i)} - w_{(i)})(1 + w_{(i)})(1 - q_{(i)}) , \tag{4.89}$$

where we defined

$$x_{(i)} := \frac{\bar\rho_{(i)} + \bar p_{(i)}}{\bar\rho + \bar p} , \qquad \sum_i x_{(i)} = 1 . \tag{4.90}$$

At the perturbation level, we derive

$$\delta p = \sum_i \delta p_{(i)} = \sum_i c^2_{s(i)}\delta\rho_{(i)} + \sum_i e_{(i)} =: c^2_s\delta\rho + e , \tag{4.91}$$

$$e = \sum_i \left(c^2_{s(i)} - c^2_s\right)\delta\rho_{(i)} + \sum_i e_{(i)} =: e^{\rm rel} + e^{\rm int} , \tag{4.92}$$

[3] This is possible nonlinearly, as we trade three dof in spatial flux with three dof in velocity.

where we defined the intrinsic and the relative entropy perturbations

$$e^{\rm rel} := \sum_i \left(c^2_{s(i)} - c^2_s\right)\delta\rho_{(i)} \tag{4.93}$$

$$= \frac{1}{2}\sum_{i,j}\frac{(\bar\rho+\bar p)_{(i)}(\bar\rho+\bar p)_{(j)}}{\bar\rho+\bar p}(c^2_{(i)}-c^2_{(j)})S_{ij} + \sum_i \frac{(\bar\rho+\bar p)_{(i)}}{\bar\rho+\bar p}q_{(i)}c^2_{s(i)}\delta\rho\,, \tag{4.94}$$

where the relative fluctuation is

$$S_{ij} := \frac{\delta n_{(i)}}{n_{(i)}} - \frac{\delta n_{(j)}}{n_{(j)}} = \frac{\delta\rho_{(i)}}{\bar\rho_{(i)}+\bar p_{(i)}} - \frac{\delta\rho_{(j)}}{\bar\rho_{(j)}+\bar p_{(j)}}\,. \tag{4.95}$$

• ***Gauge-transformation properties***.—

$$\widetilde{\delta I}_{(i)} = \delta I_{(i)} - \bar I'_{(i)}T\,, \qquad \tilde J_{(i)} = J_{(i)} + a\bar I_{(i)}T\,, \qquad \tilde S_{ij} = S_{ij} - 3\mathcal{H}T(q_{(i)}-q_{(j)})\,, \tag{4.96}$$

where S_{ij} is gauge-invariant only when there is no energy transfer in the background $q_{(i)} \equiv 0$.

4.3 Einstein Equations

4.3.1 Christoffel Symbols

In the absence of the torsion, the Christoffel symbols are uniquely determined by the metric tensor as

$$\Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial\xi^\sigma}\frac{\partial^2\xi^\sigma}{\partial x^\nu\partial x^\rho} = \frac{1}{2}g^{\mu\sigma}(g_{\nu\sigma,\rho}+g_{\rho\sigma,\nu}-g_{\nu\rho,\sigma})\,, \qquad 0 = \frac{d^2\xi^\mu}{d\tau^2}\,, \qquad d\tau^2 = -\eta_{\mu\nu}d\xi^\mu d\xi^\nu\,, \tag{4.97}$$

where ξ^μ is a freely falling coordinate and $d\tau$ is the proper time. To the linear order in perturbations, we derive

$$\Gamma^0_{00} = \frac{a'}{a} + A' \;\to\; \mathcal{H}+\psi'\,, \qquad \Gamma^0_{0\alpha} = A_{,\alpha} - \frac{a'}{a}B_\alpha \;\to\; \psi_{,\alpha}\,, \tag{4.98}$$

$$\Gamma^\alpha_{00} = A^{|\alpha} - B^{\alpha\prime} - \frac{a'}{a}B^\alpha \;\to\; \psi^{,\alpha}\,, \tag{4.99}$$

$$\Gamma^0_{\alpha\beta} = \frac{a'}{a}\bar g_{\alpha\beta} - 2\frac{a'}{a}\bar g_{\alpha\beta}A + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta} \;\to\; \mathcal{H}\bar g_{\alpha\beta}(1-2\psi) + \left(\phi'+2\mathcal{H}\phi\right)\bar g_{\alpha\beta}\,, \tag{4.100}$$

$$\Gamma^\alpha_{0\beta} = \frac{a'}{a}\delta^\alpha_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right) + C^{\alpha\prime}_\beta \;\to\; \mathcal{H}\delta^\alpha_\beta + \phi'\delta^\alpha_\beta\,, \tag{4.101}$$

$$\Gamma^\alpha_{\beta\gamma} = \bar\Gamma^\alpha_{\beta\gamma} + \frac{a'}{a}\bar g_{\beta\gamma}B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\alpha} \;\to\; \bar\Gamma^\alpha_{\beta\gamma} + 2\phi_{,(\gamma}\delta^\alpha_{\beta)} - \phi^{,\alpha}\bar g_{\beta\gamma}\,, \tag{4.102}$$

where the conformal Hubble parameter is $\mathcal{H} = a'/a$ and $\bar\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols based on 3-metric $\bar g_{\alpha\beta}$.

4.3.2 Riemann Tensor

The Riemann tensor can then be constructed in terms of the Christoffel symbols as

$$R^\mu_{\nu\rho\sigma} := \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} - \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon} = \Gamma^\mu_{\nu\sigma;\rho} - \Gamma^\mu_{\nu\rho;\sigma} - \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} + \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon}\,, \tag{4.103}$$

and the Riemann tensor has all the information of the geometry, such that how any four vector changes locally is fully determined by the Riemann tensor

$$2u_{\mu;[\nu\rho]} = u_\sigma R^\sigma{}_{\mu\nu\rho}\,, \qquad u^\rho{}_{;[\nu\mu]} = \frac{1}{2}R^\rho{}_{\sigma\mu\nu}u^\sigma\,. \tag{4.104}$$

Out of the Riemann tensor, we can construct the Ricci tensor (and Ricci scalar) by contracting the Riemann tensor as

$$R_{\mu\nu} := R^\rho_{\mu\rho\nu}\,, \qquad R = R^\mu_\mu\,, \tag{4.105}$$

and construct the (conformal) Weyl tensor as

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) . \tag{4.106}$$

The Riemann tensor has the symmetry:

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu} , \qquad R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = R_{\mu[\nu\rho\sigma]} = 0 , \tag{4.107}$$

such that its 20 independent components can be separated into the Ricci tensor (10 components) and the (traceless) Weyl curvature tensor (also 10 components with all the symmetry of the Riemann tensor)

$$C^{\mu\nu}{}_{\mu\nu} = 0 , \qquad C_{\mu\nu\rho\sigma} = C_{[\mu\nu][\rho\sigma]} = C_{\rho\sigma\mu\nu} , \qquad C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = C_{\mu[\nu\rho\sigma]} = 0 . \tag{4.108}$$

The Ricci tensor is algebraically set by matter distribution through the Einstein equation, but the Weyl tensor is determined by differential equations with suitable boundary conditions.

To the background, we derive

$$R^0_{\alpha0\beta} = \mathcal{H}'\bar g_{\alpha\beta} , \qquad R^\alpha_{00\beta} = \mathcal{H}'\delta^\alpha_\beta , \qquad R^\alpha_{\beta\gamma\delta} = 2\left(K + \mathcal{H}^2\right)\delta^\alpha_{[\gamma}\bar g_{\delta]\beta} = \bar R^\alpha_{\beta\gamma\delta} + 2\mathcal{H}^2\delta^\alpha_{[\gamma}\bar g_{\delta]\beta} , \tag{4.109}$$

$$R_{00} = -3\mathcal{H}' , \qquad R_{\alpha\beta} = \left(2K + \mathcal{H}' + 2\mathcal{H}^2\right)\bar g_{\alpha\beta} = \bar R_{\alpha\beta} + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\bar g_{\alpha\beta} , \qquad R = \frac{6}{a^2}\left(K + \mathcal{H}' + \mathcal{H}^2\right) ,$$

where we defined the Riemann tensor for a 3-hypersurface based on 3-metric $\bar g_{\alpha\beta}$

$$\bar R^\alpha_{\beta\gamma\delta} = 2K\delta^\alpha_{[\gamma}\bar g_{\delta]\beta} , \qquad \bar R_{\alpha\beta} = 2K\bar g_{\alpha\beta} , \qquad \bar R = 6K . \tag{4.110}$$

and we used for any second-rank tensor $F_{\alpha\beta}$

$$F_{\alpha\beta|[\gamma\delta]} = K\left(\bar g_{\alpha[\delta}F_{\gamma]\beta} + \bar g_{\beta[\delta}F_{\gamma]\alpha}\right) . \tag{4.111}$$

To the linear order in perturbations, we derive the Riemann tensor:

$$R^\mu{}_{\nu00} = 0 , \qquad R^0{}_{00\alpha} = -\mathcal{H}'B_\alpha , \qquad R^0{}_{0\alpha\beta} = 0 , \tag{4.112}$$

$$R^0{}_{\alpha0\beta} = \mathcal{H}'\bar g_{\alpha\beta} - \left[\mathcal{H}A' + 2\mathcal{H}'A\right]\bar g_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + \mathcal{H}B_{(\alpha|\beta)} + C''_{\alpha\beta} + \mathcal{H}C'_{\alpha\beta} + 2\mathcal{H}'C_{\alpha\beta} , \tag{4.113}$$

$$R^0{}_{\alpha\beta\gamma} = 2\mathcal{H}\bar g_{\alpha[\beta}A_{,\gamma]} - B_{\alpha|[\beta\gamma]} + \frac{1}{2}(B_{\gamma|\alpha\beta} - B_{\beta|\alpha\gamma}) - 2C'_{\alpha[\beta|\gamma]} , \tag{4.114}$$

$$R^\alpha{}_{00\beta} = \mathcal{H}'\delta^\alpha_\beta - \mathcal{H}A'\delta^\alpha_\beta - A^{|\alpha}{}_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta}\right)' + \frac{1}{2}\mathcal{H}\left(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta}\right) + C^{\alpha\prime\prime}_\beta + \mathcal{H}C^{\alpha\prime}_\beta , \tag{4.115}$$

$$R^\alpha{}_{0\beta\gamma} = 2\mathcal{H}\delta^\alpha_{[\beta}A_{,\gamma]} - B_{[\beta}{}^{|\alpha}{}_{\gamma]} + B^\alpha{}_{|[\beta\gamma]} - 2\mathcal{H}^2\delta^\alpha_{[\beta}B_{\gamma]} - 2C^{\alpha\prime}_{[\beta|\gamma]} , \tag{4.116}$$

$$R^\alpha{}_{\beta0\gamma} = \mathcal{H}\left(\bar g_{\beta\gamma}A^{,\alpha} - \delta^\alpha_\gamma A_{,\beta}\right) + \mathcal{H}'\bar g_{\beta\gamma}B^\alpha - \mathcal{H}^2\left(\bar g_{\beta\gamma}B^\alpha - \delta^\alpha_\gamma B_\beta\right) - \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right)_{|\gamma} + C^{\alpha\prime}_{\gamma|\beta} - C'_{\beta\gamma}{}^{|\alpha} , \tag{4.117}$$

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} = {} & \bar R^\alpha_{\beta\gamma\delta} + \mathcal{H}^2\left(\delta^\alpha_\gamma\bar g_{\beta\delta} - \delta^\alpha_\delta\bar g_{\beta\gamma}\right)(1 - 2A) \\ & + \frac{1}{2}\mathcal{H}\left[\bar g_{\beta\delta}\left(B_\gamma{}^{|\alpha} + B^\alpha{}_{|\gamma}\right) - \bar g_{\beta\gamma}\left(B_\delta{}^{|\alpha} + B^\alpha{}_{|\delta}\right) + 2\delta^\alpha_\gamma B_{(\beta|\delta)} - 2\delta^\alpha_\delta B_{(\beta|\gamma)}\right] \\ & + \mathcal{H}\left[\bar g_{\beta\delta}C^{\alpha\prime}_\gamma - \bar g_{\beta\gamma}C^{\alpha\prime}_\delta + \delta^\alpha_\gamma C'_{\beta\delta} - \delta^\alpha_\delta C'_{\beta\gamma} + 2\mathcal{H}\left(\delta^\alpha_\gamma C_{\beta\delta} - \delta^\alpha_\delta C_{\beta\gamma}\right)\right] \\ & + 2C^\alpha_{(\beta|\delta)\gamma} - 2C^\alpha_{(\beta|\gamma)\delta} + C_{\beta\gamma}{}^{|\alpha}{}_\delta - C_{\beta\delta}{}^{|\alpha}{}_\gamma , \end{aligned} \tag{4.118}$$

and by contracting the Riemann tensor we derive the Ricci tensor and the Ricci scalar:

$$R_{00} = -3\mathcal{H}' + 3\mathcal{H}A' + \Delta A - B^{\alpha\prime}{}_{|\alpha} - \mathcal{H}B^\alpha{}_{|\alpha} - C^{\alpha\prime\prime}_\alpha - \mathcal{H}C^{\alpha\prime}_\alpha , \tag{4.119}$$

$$R_{0\alpha} = 2\mathcal{H}A_{,\alpha} - \mathcal{H}'B_\alpha - 2\mathcal{H}^2B_\alpha + \frac{1}{2}\Delta B_\alpha - \frac{1}{2}B^\beta{}_{|\alpha\beta} - C^{\beta\prime}_{\beta|\alpha} + C'_{\alpha\beta}{}^{|\beta} , \tag{4.120}$$

$$\begin{aligned} R_{\alpha\beta} = {} & 2K\bar g_{\alpha\beta} + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\bar g_{\alpha\beta}(1 - 2A) - \mathcal{H}A'\bar g_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + 2\mathcal{H}B_{(\alpha|\beta)} + \mathcal{H}\bar g_{\alpha\beta}B^\gamma{}_{|\gamma} \\ & + C''_{\alpha\beta} + 2\frac{a'}{a}C'_{\alpha\beta} + 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)C_{\alpha\beta} + \mathcal{H}\bar g_{\alpha\beta}C^{\gamma\prime}_\gamma + 2C^\gamma_{(\alpha|\beta)\gamma} - C^\gamma_{\gamma|\alpha\beta} - \Delta C_{\alpha\beta} , \end{aligned} \tag{4.121}$$

$$\begin{aligned} R = {} & \frac{1}{a^2}\left[6\left(\mathcal{H}' + \mathcal{H}^2 + K\right) - 6\mathcal{H}A' - 12\left(\mathcal{H}' + \mathcal{H}^2\right)A - 2\Delta A\right. \\ & \left. + 2B^{\alpha\prime}{}_{|\alpha} + 6\mathcal{H}B^\alpha{}_{|\alpha} + 2C^{\alpha\prime\prime}_\alpha + 6\mathcal{H}C^{\alpha\prime}_\alpha - 4KC^\alpha_\alpha - 2\Delta C^\alpha_\alpha + 2C^{\alpha\beta}{}_{|\alpha\beta}\right] . \end{aligned} \tag{4.122}$$

• HW: derive the Riemann tensor and the Ricci tensor in the conformal Newtonian gauge

4.3.3 Einstein Equation and Background Equation

The Einstein equation is that the Einstein tensor $G_{\mu\nu}$ is proportional to the energy-momentum tensor $T_{\mu\nu}$:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda \, g_{\mu\nu} = 8\pi G \, T_{\mu\nu} \,, \tag{4.123}$$

where G is the Newton's constant and Λ is the cosmological constant. The cosmological constant can be put in the right-hand side as a part of the energy-momentum tensor:

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} \,. \tag{4.124}$$

The trace of the Einstein equation gives

$$T = -\rho + 3p \,, \qquad R = 8\pi G(\rho - 3p) + 4\Lambda \,, \tag{4.125}$$

and the Ricci tensor is completely set by the trace of the energy-momentum tensor. To the background, the Einstein equation yields the Friedmann equation give

$$H^2 = \frac{8\pi G}{3} \bar\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \,, \qquad H^2 + \dot H = \frac{\ddot a}{a} = -\frac{4\pi G}{3} \left(\bar\rho + 3\bar p\right) + \frac{\Lambda}{3} \,, \tag{4.126}$$

and the energy-momentum conservation yields

$$\dot{\bar\rho} + 3H \left(\bar\rho + \bar p\right) = 0 \,. \tag{4.127}$$

In terms of the conformal time, the Friedmann equation becomes

$$\mathcal{H}' = a^2(H^2 + \dot H) = -\frac{4\pi G}{3} a^2 (\bar\rho + 3\bar p) \,, \qquad \mathcal{H}^2 + \mathcal{H}' = \frac{a''}{a} = \frac{4\pi G}{3} a^2 (\bar\rho - 3\bar p) - K \,. \tag{4.128}$$

In a flat Universe ($K = 0$) dominated by an energy component $\bar\rho \propto a^{-n}$, we can derive the analytic solutions to the Friedmann equation:

$$a \propto \eta^{\frac{2}{n-2}} \propto t^{2/n} \,, \qquad t \propto \eta^{\frac{n}{n-2}} \,, \qquad H = H_o \left(\frac{t_o}{t}\right) = \frac{2}{nt} \,, \qquad \mathcal{H} = \mathcal{H}_o \left(\frac{\eta_o}{\eta}\right) = \frac{2}{n-2} \frac{1}{\eta} \,, \tag{4.129}$$

or in terms of equation of state $\bar\rho \propto a^{-3(1+w)}$,

$$a \propto \eta^{\frac{2}{1+3w}} \propto t^{2/3(1+w)} \,, \qquad t \propto \eta^{\frac{3(1+w)}{1+3w}} \,, \qquad H = \frac{2}{3(1+w)t} \,, \qquad \mathcal{H} = \frac{2}{1+3w} \frac{1}{\eta} \,, \tag{4.130}$$

where we used $n = 3(1+w)$.

• ***Einstein-de Sitter Universe***.— This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations are

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{\eta_0}\right)^2 \,, \qquad \frac{t}{t_0} = \left(\frac{\eta}{\eta_0}\right)^3 \,, \qquad \eta_0 = 3t_0 \,, \tag{4.131}$$

$$H = \frac{2}{3t} \,, \qquad \mathcal{H} = \frac{2}{\eta} \,, \qquad \rho_m = \frac{1}{6\pi G t^2} \,, \qquad r = \eta_0 - \eta = \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right) \,, \tag{4.132}$$

where the reference point t_0 satisfies $a(t_0) = 1$, but it can be any time $t_0 \in (0, \infty)$.

4.3.4 Linear-Order Einstein Equation

The Einstein equation can be expanded up to the linear order in perturbations, and decomposed into the evolution equations describing the scalar, the vector, and the tensor perturbations. At the linear order, they do not mix.

• ***Scalar perturbations***.—

$$G^0_0 \quad : \quad H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\rho \,, \tag{4.133}$$

$$G^0_\alpha \quad : \quad \kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\bar\rho + \bar p)av \,, \tag{4.134}$$

$$G^\alpha_\alpha - G^0_0 \quad : \quad \dot\kappa + 2H\kappa + \left(3\dot H + \frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\rho + 3\delta p\right) \,, \tag{4.135}$$

$$G^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta G^\gamma_\gamma \quad : \quad \dot\chi + H\chi - \varphi - \alpha = 8\pi G\Pi \,, \tag{4.136}$$

where we defined

$$\kappa := 3H\alpha - 3\dot\varphi - \frac{\Delta}{a^2}\chi \,, \qquad \chi := a\beta + a\gamma' \,, \qquad \Pi_{\alpha\beta} := \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}\bar g_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta} \,. \tag{4.137}$$

The energy-momentum conservation yields

$$T^\nu_{0;\nu} \quad : \quad \delta\dot\rho + 3H\left(\delta\rho + \delta p\right) - (\bar\rho + \bar p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right) = 0 \,, \tag{4.138}$$

$$T^\nu_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v]^\cdot}{a^4(\bar\rho + \bar p)} - \frac{1}{a}\alpha - \frac{1}{a(\bar\rho + \bar p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right) = 0 \,. \tag{4.139}$$

• ***Vector perturbations***.—

$$T^\nu_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v^{(v)}_\alpha]^\cdot}{a^4(\bar\rho + \bar p)} + \frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\bar\rho + \bar p} = 0 \,. \tag{4.140}$$

• ***Tensor perturbations***.—

$$G^\alpha_\beta \quad : \quad \ddot C^{(t)\alpha}_{\ \ \beta} + 3H\dot C^{(t)\alpha}_{\ \ \beta} - \frac{\Delta - 2K}{a^2}C^{(t)\alpha}_{\ \ \beta} = 8\pi G\Pi^{(t)\alpha}_{\ \ \beta} \,. \tag{4.141}$$

• ***Multiple fluids***.— In the presence of interactions in Eq. (4.87), the individual conservation becomes

$$T^{(i)\nu}_{0;\nu} \quad : \quad \delta\dot\rho_{(i)} + 3H\left(\delta\rho + \delta p\right)_{(i)} - \dot{\bar\rho}_{(i)}\alpha - (\bar\rho + \bar p)_{(i)}\left(\kappa + \frac{1}{a}\Delta v_{(i)}\right) = \delta I_{(i)} \,, \tag{4.142}$$

$$T^{(i)\nu}_{\alpha;\nu} \quad : \quad \frac{[a^4(\bar\rho + \bar p)v]^\cdot_{(i)}}{a^4(\bar\rho + \bar p)_{(i)}} - \frac{1}{a}\alpha - \frac{1}{a(\bar\rho + \bar p)_{(i)}}\left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi_{(i)} - J_{(i)}\right) = 0 \,. \tag{4.143}$$

The first conservation equation can be expressed as

$$\dot\delta_{(i)} + 3H(c_s^2 - w)_{(i)}\delta_{(i)} + 3H(1 + w_{(i)})q_{(i)}\delta_{(i)} = (1 + w_{(i)})\left[\kappa - 3H(1 - q_{(i)})\alpha + \frac{\Delta}{a}v_{(i)}\right] + \frac{1}{\bar\rho_{(i)}}(-3He + \delta I)_{(i)} \,. \tag{4.144}$$

4.4 Linear-Order Cosmological Solutions

4.4.1 Super-Horizon Solution

On super-horizon scales $k \ll \mathcal{H}$, many simplifications can be made to derive useful relations. First, the energy-momentum conservation in Eq. (4.138) can be re-arranged as

$$3(1+w)\dot{\varphi}_\delta + \frac{3H}{\bar{\rho}}\left(\delta p - \bar{p}\,\delta\right) + (1+w)\frac{\Delta}{a^2}(\chi - av) = 0\,, \qquad \varphi_\delta := \varphi + \frac{\delta}{3(1+w)}\,, \tag{4.145}$$

implies that in the super horizon limit we have a conservation law for a medium with adiabatic condition $e = \delta p - c_s^2\delta\rho \equiv 0$:

$$\dot{\varphi}_\delta = 0\,, \tag{4.146}$$

where we assumed the equation of state is constant. The uniform-density gauge curvature is often denoted as $\zeta = \varphi_\delta$. Note that in the conservation equation we assumed no energy transfers between any fluids, and this conservation law holds for individual adiabatic fluids.

The other important conservation law deals with the comoving-gauge curvature (often denoted as $\mathcal{R}$):

$$\varphi_v := \varphi - \mathcal{H}v\,. \tag{4.147}$$

As we derive in Section 5, the comoving-gauge curvature perturbation φ_v in a flat universe $K = 0$ is conserved on large scales throughout the evolution:

$$\dot{\varphi}_v = 0\,, \tag{4.148}$$

if the total matter content of the Universe is adiabatic, which is the case in the standard model. So, it is convenient to derive its relation to the conformal Newtonian gauge quantities by using the Einstein equation (4.134) to replace v in favor of κ

$$\varphi_v = \varphi - \frac{\mathcal{H}\left(3H\alpha - 3\dot{\varphi}\right)}{12\pi G a(\bar{\rho}+\bar{p})} = \varphi - \frac{2}{3}\frac{\alpha - \dot{\varphi}/H}{1+w}\,. \tag{4.149}$$

In the absence of anisotropic pressure $\Pi = 0$, the Einstein equation implies

$$\alpha_\chi = -\varphi_\chi\,, \tag{4.150}$$

and on super-horizon scales

$$\varphi_v = \frac{5+3w}{3(1+w)}\,\varphi_\chi\,, \tag{4.151}$$

where we chose the conformal Newtonian gauge and ignored $\dot{\varphi}_\chi/H = 0$, as obvious in the above equation. While φ_v is conserved on super-horizon scales throughout the whole evolution, the Newtonian gauge potential evolves, as the Universe transitions from the radiation dominated to the matter dominated eras:

$$\varphi_\chi^{\mathrm{RDE}} = \frac{2}{3}\varphi_v\,, \qquad \varphi_\chi^{\mathrm{MDE}} = \frac{3}{5}\varphi_v\,. \tag{4.152}$$

On super-horizon scales, the first two Einstein equations imply

$$H\kappa = -4\pi G\delta\rho = 12\pi G(\bar{\rho}+\bar{p})\mathcal{H}v\,, \tag{4.153}$$

which guarantees the equivalence

$$0 = \varphi_\delta - \varphi_v = \frac{\delta}{3(1+w)} + \mathcal{H}v\,, \qquad \varphi_v = \varphi_\delta\,, \tag{4.154}$$

independent of adiabatic conditions.

4.4.2 Einstein Equation in the conformal Newtonian Gauge

In the conformal Newtonian gauge we have

$$\kappa = 3H\psi - 3\dot{\phi} \,, \qquad \chi = 0 \,, \qquad U = v_\chi \,, \qquad \boldsymbol{v} = -\nabla U \,. \tag{4.155}$$

By substituting into the Einstein equation, we derive

$$H\kappa + \frac{\Delta + 3K}{a^2}\phi = -4\pi G\delta\rho \,, \qquad \phi + \psi = -8\pi G\Pi \,, \tag{4.156}$$

$$\kappa = 12\pi G(\bar{\rho} + \bar{p})av \,, \qquad \dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\psi = 4\pi G\left(\delta\rho + 3\,\delta p\right) \,. \tag{4.157}$$

To remove κ in favor of the other variables, we use Eq. (4.157) to arrive at

$$\dot{\phi} + H\phi = -4\pi G(\bar{\rho} + \bar{p})av - 8\pi GH\Pi \,, \tag{4.158}$$

and Eq. (4.156) can be further manipulated as

$$-(\Delta + 3K)\phi = 4\pi Ga^2\bar{\rho}\,\delta + a\mathcal{H}\kappa = 4\pi Ga^2\bar{\rho}\,[\delta + 3\mathcal{H}v(1+w)] \equiv 4\pi Ga^2\bar{\rho}\,\delta_v \,, \tag{4.159}$$

where δ_v is the density fluctuation in the comoving gauge:

$$\delta_v := \delta - \frac{\dot{\bar{\rho}}}{\bar{\rho}}av = \delta + 3\mathcal{H}v \,. \tag{4.160}$$

Finally, the equation for the velocity can be obtained from Eq. (4.157) as

$$v' + \mathcal{H}v = -\phi + \frac{\delta p_v}{\bar{\rho} + \bar{p}} - 8\pi G\Pi + \frac{2}{3}\frac{\Delta + 3K}{a^2}\frac{\Pi}{\bar{\rho} + \bar{p}} \,, \tag{4.161}$$

where the pressure fluctuation in the comoving gauge is

$$\delta p_v := \delta p - \dot{\bar{p}}\,av \,. \tag{4.162}$$

Assuming a flat Universe ($K = 0$) and a pressureless medium ($p = \delta p = 0$), we can further simplify the equation as

$$\phi + \psi = 0 \,, \qquad \kappa = \frac{3}{a}\left(a\psi\right)^{\cdot} = 12\pi G\bar{\rho}av \,, \qquad \Delta\phi = -4\pi Ga^2\bar{\rho}\,\delta_v \,, \qquad v' + \mathcal{H}v = \psi \,. \tag{4.163}$$

4.4.3 Newtonian Correspondence

As apparent, the relativistic equations appear quite similar or identical to those in the Newtonian dynamics. Here we identify such correspondence made available in a particular choice of gauge. However, keep in mind that the relativistic dynamics is intrinsically different from the Newtonian, and such correspondence is only identified in a limited case (e.g., linear order for pressureless media).

With $\beta \equiv 0$ in the conformal Newtonian gauge, we find the velocity potential in the standard Newtonian perturbation theory

$$U = v \,, \qquad \mathbf{v} = -\nabla v \,, \qquad \theta := -\frac{1}{a}\nabla\cdot\mathbf{v} = \frac{1}{a}\Delta v \,, \tag{4.164}$$

and by taking the divergence of v in Eq. (4.163), we obtain the governing equation

$$\Delta\psi = \frac{1}{a}\Delta(av)' = \left(a^2\theta\right)^{\cdot} \,, \qquad \therefore\;\; \dot{\theta} + 2H\theta = \frac{1}{a^2}\Delta\psi = 4\pi G\bar{\rho}\,\delta_v \,. \tag{4.165}$$

The last remaining equation in the SPT can be obtained by taking the time derivative of δ_v in Eq. (4.163):

$$\dot{\delta}_v = -\frac{\Delta(a\phi)^{\cdot}}{4\pi Ga^3\bar{\rho}} = \theta \,. \tag{4.166}$$

where we assumed in this case $\bar\rho \propto 1/a^3$. With a proper identification of gauge-invariant variables to the standard Newtonian perturbation theory

$$v_\chi \rightarrow U \,, \qquad \alpha_\chi = -\varphi_\chi \rightarrow \delta\Phi \,, \qquad \delta_v \rightarrow \delta_m \,, \tag{4.167}$$

we find the governing equation in SPT is fully relativistic at the linear order.

Manipulating the Newtonian gauge equations, we find that the density fluctuation then follows the same evolution equation as in the standard Newtonian perturbation theory

$$\ddot\delta_v + 2H\dot\delta_v - 4\pi G\bar\rho_m\delta_v = 0 \,. \tag{4.168}$$

With a mathematical identity

$$\frac{1}{a^2H}\left[a^2H^2\left(\frac{\delta}{H}\right)^{\cdot}\right]^{\cdot} = \ddot\delta + 2H\dot\delta - \delta\left(\frac{\ddot H}{H} + 2\dot H\right) \,, \tag{4.169}$$

and by using the Friedmann equation with $\bar p = w\bar\rho$ (valid for any K)

$$\left(\frac{\ddot H}{H} + 2\dot H\right) = 4\pi G\bar\rho(1+w)(1+3w) \,, \tag{4.170}$$

we can derive a formal solution for the differential equation for δ_v in case $w = 0$:

$$\delta_v(\mathbf{k},t) = c_1(\mathbf{k})H(t)\int\frac{dt}{a^2H^2} + c_2(\mathbf{k})H(t) \,, \tag{4.171}$$

where the first term is the growing solution and the second term is the decaying solution.

4.4.4 General Solutions

In fact, the most general evolution equation for the density fluctuation: was derived already in Bardeen (1980); Hwang and Noh (1999) by solving the Einstein equations with full generality:

$$\begin{aligned}
&\ddot\delta_v + (2+3c_s^2-6w)H\dot\delta_v + \left[c_s^2\frac{k^2}{a^2} - 4\pi G\bar\rho(1-6c_s^2+8w-3w^2) + 12(w-c_s^2)\frac{K}{a^2} + (3c_s^2-5w)\Lambda\right]\delta_v \\
&\equiv \frac{1+w}{a^2H}\left[\frac{H^2}{a(\bar\rho+\bar p)}\left(\frac{a^3\bar\rho}{H}\delta_v\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\delta_v = -\frac{k^2-3K}{a^2}\frac{1}{\bar\rho}\left\{e + 2H\dot\Pi + 2\left[-\frac{1}{3}\frac{k^2}{a^2} + 2\dot H + 3(1+c_s^2)H^2\right]\Pi\right\} \,,
\end{aligned} \tag{4.172}$$

where the equation of state, the sound speed c_s^2, and the entropy perturbation are defined as

$$w := \frac{\bar p}{\bar\rho} \,, \qquad c_s^2 := \frac{\dot{\bar p}}{\dot{\bar\rho}} \,, \qquad \delta p := c_s^2\delta\rho + e \,. \tag{4.173}$$

This equation is fully general for any Λ and K. In the same way, the most general evolution equation for the potential fluctuation was also derived in Hwang and Noh (1999)

$$\begin{aligned}
&\ddot\varphi_\chi + (4+3c_s^2)H\dot\varphi_\chi + \left[c_s^2\frac{k^2}{a^2} + 8\pi G\bar\rho(c_s^2-w) - 2(1+3c_s^2)\frac{K}{a^2} + (1+c_s^2)\Lambda\right]\varphi_\chi \\
&\equiv \frac{\bar\rho+\bar p}{H}\left[\frac{H^2}{a(\bar\rho+\bar p)}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\varphi_\chi = \mathcal{F}(e,\Pi) \,,
\end{aligned} \tag{4.174}$$

where the source term $\mathcal{F}$ is some function of e and Π. These equations are greatly simplified in the absence of anisotropic pressure Π and entropy perturbation e. The equations in terms of conformal time can be obtained by using

$$\dot f = \frac{1}{a}f' \,, \qquad \ddot f = \frac{1}{a^2}f'' - \frac{H}{a}f' \,. \tag{4.175}$$

In a universe with constant equation of state and $K = \Lambda = e = \Pi = 0$, the differential equation becomes

$$\varphi_\chi'' + 3(1+w)\mathcal{H}\varphi_\chi' + wk^2\varphi_\chi = 0 \,, \qquad \mathcal{H} = \frac{2}{1+3w}\frac{1}{\eta} \,, \tag{4.176}$$

and the solutions are the Bessel functions of order α:

$$\varphi_\chi = y^{-\alpha}\left[c_1(k)J_\alpha(y) + c_2(k)Y_\alpha(y)\right] \,, \qquad y := \sqrt{w}k\eta \,, \qquad \alpha := \frac{1}{2}\left(\frac{5+3w}{1+3w}\right) \,. \tag{4.177}$$

In RDE ($w = 1/3$) and MDE ($w = 0$), the solutions are

$$\varphi_\chi = \frac{1}{y^2}\left[c_1(k)\left(\frac{\sin y}{y} - \cos y\right) + c_2(k)\left(\frac{\cos y}{y} + \sin y\right)\right] \,, \qquad w = \frac{1}{3} \,, \qquad \alpha = \frac{3}{2} \,, \tag{4.178}$$

$$\varphi_\chi = c_1(k) + \frac{c_2(k)}{y^5} \,, \qquad w = 0 \,, \qquad \alpha = \frac{5}{2} \,, \tag{4.179}$$

where c_2 is the decaying mode. The growing mode of the gravitational potential is constant at all k in MDE and also outside the horizon in RDE, while it decays inside the horizon in RDE. Ignoring the oscillatory part insider the horizon in RDE, the gravitational potential can be well approximated as

$$\varphi_\chi = \frac{2}{3}\frac{1}{1+(k\eta)^2} \qquad \text{for } \eta \le \eta_{\rm eq} \,, \quad \forall k \,, \tag{4.180}$$

where the normalization is set at the super-horizon scale, compared to φ_v. The solution for the density fluctuation is from the Einstein equation

$$\delta_v = \frac{k^2 - 3K}{4\pi G\bar\rho a^2}\,\varphi_\chi \,. \tag{4.181}$$

4.4.5 ΛCDM Universe

The Universe today is best described by a flat universe ($K = 0$) with a cosmological constant and cold dark matter. Here we study analytical solutions in ΛCDM universe.

• ***Background equations***.— A flat homogeneous universe with pressureless matter and a cosmological constant is governed by

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} \,, \qquad H^2 + \dot H = \frac{\ddot a}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3} \,, \qquad \dot H = -4\pi G\rho_m = -\frac{3}{2}H^2\Omega_m(z) \,,$$
$$\mathcal{H}' = -\frac{4\pi G}{3}a^2\rho_m - \frac{\Lambda}{3}a^2 \,, \qquad \Omega_m(z) := \frac{\rho_m}{\rho_c} = \frac{8\pi G\rho_m}{3H^2} \,, \qquad H^2\Omega_m(z) = \frac{8\pi G}{3}\rho_m \,. \tag{4.182}$$

The Einstein-de Sitter Universe is obtained by setting $\Lambda = 0$ and $\Omega_m(z) = 1$. In this section, we will use $\Omega_m \equiv \Omega_m(z)$.

• ***Perturbations***.— We will derive the solutions first in the comoving gauge and derive the relation to the conformal Newtonian gauge. The energy conservation and the energy constraint equations give

$$0 = \dot\delta_v - \kappa_v \,, \qquad H\kappa_v + 4\pi G\rho_m\delta_v = -\frac{1}{a^2}\Delta\varphi_v \equiv \delta R^{(h)} \quad \to \quad H\dot\delta_v + \frac{3}{2}H^2\Omega_m\delta_v = -\frac{1}{a^2}\Delta\varphi_v \,. \tag{4.183}$$

Using the background solution for H, the homogeneous solution for δ_v (where the RHS is set zero) can be readily derived as $\delta_v^{(h)} \propto H$. The homogeneous solution is the decaying mode, and the particular solution (or the growing mode) can be derived as

$$\delta_v^{(p)} = \delta_v^{(h)}\int\frac{dt}{\delta_v^{(h)}}\left(-\frac{\Delta\varphi_v}{a^2H}\right) := -D\Delta\varphi_v \,, \qquad D := H\int\frac{dt}{\mathcal{H}^2} \,, \qquad \dot\varphi_v = 0 \,, \tag{4.184}$$

where the momentum constraint equation gives the conservation of the comoving gauge curvature and we defined the growth function D for the density.[4] This solution is identical to that in the Newtonian dynamics despite the presence of Λ, and it is indeed consistent with the general equation (4.172) with $w = c_s^2 = 0$.

If we define the logarithmic growth rate f,

$$f := \frac{d\ln D}{d\ln a} = \frac{1}{\mathcal{H}}\frac{d}{d\eta}\ln D = \frac{1}{H}\frac{d}{dt}\ln D \quad \rightarrow \quad \dot{D} = HfD \,, \tag{4.185}$$

the energy constraint equation can be re-arranged as

$$H^2 fD + \frac{3}{2}H^2\Omega_m D = \frac{1}{a^2} \quad \rightarrow \quad \therefore\ D = \frac{1}{\mathcal{H}^2 f\Sigma} \,, \qquad \Sigma := 1 + \frac{3}{2}\frac{\Omega_m}{f} \xrightarrow[\Omega_m=1]{} \frac{5}{2} \,, \qquad \delta_v = -\frac{\Delta\varphi_v}{\mathcal{H}^2 f\Sigma} \,. \tag{4.186}$$

The remaining perturbations are

$$\chi_v := a\beta_v \,, \qquad \kappa_v \equiv -\frac{\Delta}{a^2}\chi_v \equiv \dot{\delta}_v \equiv -\frac{\Delta\varphi_v}{a^2 H\Sigma} \,, \qquad \chi_v = \frac{\varphi_v}{H\Sigma} \,, \tag{4.187}$$

where we used $\dot{D} = HfD$.

• ***Newtonian correspondence***.— The velocity and the gravitational potential in the Newtonian dynamics correspond to the conformal Newtonian gauge quantities: $U^i = -v_\chi^i$ and $\alpha_\chi = -\varphi_\chi$. The simplest way to derive the relations is the gauge transformation from the comoving gauge to the conformal Newtonian gauge:

$$\gamma_v = \gamma_\chi \equiv 0 \quad \rightarrow \quad L = 0 \,, \qquad v_\chi = 0 - T \,, \qquad \varphi_\chi = \varphi_v - \mathcal{H}T \,, \tag{4.188}$$

$$0 = \beta_\chi = \beta_v - T \qquad \rightarrow \qquad T = \beta_v = \frac{1}{a}\chi_v = \frac{\varphi_v}{\mathcal{H}\Sigma} \,, \tag{4.189}$$

such that we derive

$$v_\chi = -\beta_v = -\frac{1}{a}\chi_v = -\frac{\varphi_v}{\mathcal{H}\Sigma} \,, \qquad \varphi_\chi = \varphi_v + \mathcal{H}v_\chi = \left(1 - \frac{1}{\Sigma}\right)\varphi_v = \dot{\chi}_v \,, \tag{4.190}$$

where we used a useful relation in ΛCDM

$$1 \equiv \frac{1}{a}\left(\frac{a}{H\Sigma}\right)^{\cdot} = \frac{1}{\Sigma} + \left(\frac{1}{H\Sigma}\right)^{\cdot} \quad \rightarrow \quad 1 - \frac{1}{\Sigma} = \left(\frac{1}{H\Sigma}\right)^{\cdot} \,. \tag{4.191}$$

The remaining relations are then

$$\delta_v = \delta_\chi + 3\mathcal{H}v_\chi \,, \qquad \Delta\varphi_\chi = -\frac{3}{2}\mathcal{H}^2\Omega_m\delta_v \,, \qquad \Delta v_\chi = \delta_v' \,, \qquad \kappa_v = \dot{\delta}_v = \theta \,. \tag{4.192}$$

4.4.6 Cosmological Gravitational Waves

In the absence of anisotropic pressure $\Pi_{\alpha\beta}^{(t)}$ in the tensor component, the cosmological gravitational waves $C_{\alpha\beta}^{(t)}$ propagate freely in an expanding universe. By decomposing the transverse traceless tensor $C_{\alpha\beta}^{(t)}$ in terms of two polarization basis $e_{\alpha\beta}^s(\mathbf{k})$, with $s = +, \times$, the propagation equation (4.141) becomes

$$\ddot{h}_\mathbf{k}^s + 3H\dot{h}_\mathbf{k}^s - \frac{1}{a^2}\Delta h_\mathbf{k}^s = 0 \,, \qquad h_\mathbf{k}^{s\prime\prime} + 2\mathcal{H}h_\mathbf{k}^{s\prime} - \Delta h_\mathbf{k}^s = 0 \,, \tag{4.193}$$

where we assumed $K = 0$ and used

$$h_{\alpha\beta}^{(t)} = 2C_{\alpha\beta}^{(t)}(\eta,\mathbf{k}) \equiv e_{\alpha\beta}^+(\mathbf{k})h^+(\eta,\mathbf{k}) + e_{\alpha\beta}^\times(\mathbf{k})h^\times(\eta,\mathbf{k}) \,, \qquad e_{\alpha\beta}^s(\mathbf{k})e^{s'\alpha\beta}(\mathbf{k}) = 2\delta^{ss'} \,. \tag{4.194}$$

[4]Note that the solution D is unique with the boundary condition $D = 0$ at $t = 0$. So, the usual growth function $\hat{D}$ that is normalized today is then $\hat{D} := D(t)/D(t_0)$ and the density is $\delta_v(t) = \hat{D}(t)\delta_v(t_0)$.

Mind the normalization convention for the polarization basis with $s = +, \times$. By change of variable $v^s_{\mathbf{k}} := ah^s_{\mathbf{k}}$, the propagation equation becomes

$$(v^s_{\mathbf{k}})'' + \left(k^2 - \frac{a''}{a}\right) v^s_{\mathbf{k}} = 0 \,, \qquad \mathcal{H}^2 + \mathcal{H}' = \frac{a''}{a} \,. \tag{4.195}$$

On large scales ($k^2 \ll a''/a$), we can readily find the solution

$$\frac{1}{a} v^s_{\mathbf{k}} = h^s_{\mathbf{k}} = c^s_1(\mathbf{k}) + c^s_2(\mathbf{k}) \int \frac{dt}{a^3} \,, \tag{4.196}$$

where the first one is the growing mode and is constant on large scales. Furthermore, by assuming $\Lambda = 0$ and a constant equation of state, the exact solution can be obtained in terms of Bessel functions as

$$h^s_{\mathbf{k}} = c_1 \frac{J_\nu(k\eta)}{(k\eta)^\nu} + c_2 \frac{Y_\nu(k\eta)}{(k\eta)^\nu} \,, \qquad \nu := \frac{3(1-w)}{2(1+3w)} \,. \tag{4.197}$$

In the absence of parity violating process, two different polarization states behave statistically in the same way, and we can omit the superscript s. In RDE ($a \propto \eta$, $\nu = 1/2$), two solutions are

$$(v^s_{\mathbf{k}})'' + k^2 v^s_{\mathbf{k}} = 0 \,, \qquad v^s_{\mathbf{k}} \propto \sin(k\eta) \,, \quad \cos(k\eta) \,, \qquad h^s_{\mathbf{k}} \propto \frac{1}{\eta} \sin(k\eta) \,, \quad \frac{1}{\eta} \cos(k\eta) \,. \tag{4.198}$$

In MDE ($a \propto \eta^2$; $\nu = 3/2$), two solutions are the spherical Bessel functions

$$(v^s_{\mathbf{k}})'' + \left(k^2 - \frac{2}{\eta^2}\right) v^s_{\mathbf{k}} = 0 \,, \qquad v^s_{\mathbf{k}} \propto \eta j_1(k\eta) \,, \quad \eta y_1(k\eta) \,, \qquad h^s_{\mathbf{k}} \propto \frac{1}{\eta} j_1(k\eta) \,, \quad \frac{1}{\eta} y_1(k\eta) \,. \tag{4.199}$$

The second solution blows up at $k \to 0$. Given the normalization convention, the total power spectrum of the cosmological gravitational waves is

$$P_T = 2(P_{h^+} + P_{h^\times}) = P_{h^{+2}} + P_{h^{-2}} \,, \tag{4.200}$$

where $P_{h^{\pm 2}}$ is the power in the helicity basis.