3 Tensor Algebra in a Curved Spacetime

Here we first briefly review technique of tensors and differential geometry. Needless to say that there exist a lot more than presented here about the subjects. Then we discuss the metric tensor and the Riemann curvature tensor in a curved spacetime.

3.1 Tensor Analysis & Curved Geometry

3.1.1 Basics in Tensor Calculus

• ***Manifold***.— is a geometry which locally looks like a bit of n-dimensional Euclidean space $\mathbb{R}^n$. For example, a 2d sphere is a manifold and is locally like $\mathbb{R}^2$. However, they are globally different, i.e., you need many (at least two) patches of $\mathbb{R}^2$ to cover the sphere. This set of coordinate patches is called an *atlas* or a *chart*. So, a manifold is in a simple term a set of open neighborhoods that look like $\mathbb{R}^n$.

• ***Congruence of curves***.— is the family of curves $x^\mu(u)$ that only one curve goes through each point in the manifold. For a given vector field X^μ, the congruence of curves whose tangent vector is X^μ is called *orbits* or *trajectories* of X^μ:

$$\frac{dx^\mu}{du} = X^\mu \,. \tag{3.1}$$

• ***Contravariant and covariant tensors***.— Given a coordinate transformation to cover the manifold,

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu) \,, \qquad J := \left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right| \,, \qquad x^\mu = x^\mu(\tilde{x}^\nu) \,, \qquad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \frac{1}{J} \,, \tag{3.2}$$

contravariant tensors transform like a differential dx^μ as

$$\tilde{X}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} X^\nu(x) \,, \qquad \tilde{X}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma} X^{\rho\sigma}(x) \,, \qquad \cdots \,, \tag{3.3}$$

while covariant tensors transform like a gradient $\partial_\mu \phi$ as

$$\tilde{X}_\mu(\tilde{x}) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} X_\nu(x) \,, \qquad \tilde{X}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} X_{\rho\sigma}(x) \,, \qquad \cdots \,. \tag{3.4}$$

Of course a tensor of rank zero, or scalar, is invariant:

$$\tilde{\phi}(\tilde{x}) = \phi(x) \,. \tag{3.5}$$

General tensors are of rank (p,q), and one can create one $Z = XY$ of (p_1+p_2, q_1+q_2) by a direct product of X and Y with each type. Contraction in contrast reduces the rank to $(p-1, q-1)$ by summing over one upper and one lower indicies together.

• ***Tangent space and co-tangent space***.— Tensors are geometric objects, i.e., independent of coordinate systems. For this interpretation, consider basis vectors $\boldsymbol{e}_{(\mu)}$, i.e., four basis vectors in 4D, and a simple choice is $\boldsymbol{e}_{(\mu)} \propto \partial_\mu$. A vector can be expressed in terms of basis vectors

$$V = V^\mu \boldsymbol{e}_{(\mu)} \,, \tag{3.6}$$

where V^μ is the components. In this expression, it is made obvious that the vector is independent of coordinate systems, and we derive that the basis vectors transform with the inverse as

$$V = V^\mu \boldsymbol{e}_{(\mu)} = \tilde{V}^\mu \tilde{\boldsymbol{e}}_{(\mu)} \,, \qquad \therefore \;\; \tilde{\boldsymbol{e}}_{(\mu)} = \boldsymbol{e}_{(\nu)} \left(\Lambda^{-1}\right)^\nu{}_\mu = \Lambda_\mu{}^\nu \boldsymbol{e}_{(\nu)} \,, \tag{3.7}$$

where we used the Lorentz transformation notation to make the structure clear, but Λ can be replaced with a general transformation $\partial \tilde{x}^\mu / \partial x^\nu$ and its inverse. In general, at a given point p, a set of all vectors make up a tangent vector space T_p, and we can define its dual space T_p^* or a co-tangent vector space, where a dual vector $w \in T_p^*$ is a linear map of a vector $V \in T_p$ to a real number: $w(V) \in \mathbb{R}$. For example, consider dual basis vectors $\boldsymbol{t}^{(\mu)}$, satisfying the orthonormality condition, then any dual vector can also be decomposed in terms of its basis as

$$\boldsymbol{t}^{(\mu)}(\boldsymbol{e}_{(\nu)}) := \delta^\mu_\nu , \qquad w = w_\mu \boldsymbol{t}^{(\mu)} , \qquad w(V) = w_\mu V^\mu . \tag{3.8}$$

In this way, we can deal with vectors and dual vectors with their components only. Mind that a tangenet space T_p itself is a different manifold with the only exceptions of Euclidean and Minkowski manifolds). All the tangent spaces T_p and co-tangent spaces T^*_p for all p in the manifold are referred to as the tangent and co-tangent bundles. One can generalize this concept to general tensor of rank (p,q), which takes p dual vectors and q vectors, and a tensor of (p,q) can be written in terms of tensor components

$$T = T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q} \boldsymbol{e}_{(\mu_1)} \otimes \cdots \otimes \boldsymbol{e}_{(\mu_p)} \otimes \boldsymbol{t}^{(\nu_1)} \otimes \cdots \otimes \boldsymbol{t}^{(\nu_q)} , \tag{3.9}$$

• ***Lie derivative***.— In a curved spacetime, *no two quantities at two different points can be meaningfully compared.* This implies that a differentiation in a curved manifold needs to be re-considered, and a simple coordinate differentiation of any tensor does not transform tensorially. The *Lie derivative* $\pounds_X$ defines a differentiation of any tensor along a tangent vector X^μ. Now consider a coordinate transformation

$$\tilde{x}^\mu = x^\mu + \varepsilon\, X^\mu , \qquad |\varepsilon| \ll 1 , \tag{3.10}$$

and interpret it in an *active* way, i.e., we move the point x^μ_p to another point $x^\mu_q := \tilde{x}^\mu$ along X^μ by an infinitesimal amount ε, rather than the same physical point in a *passive* interpretation. A tensor at x_q can be approximated as

$$T^{\mu\nu}(x_q) = T^{\mu\nu}(x_p) + \varepsilon\, X^\rho \partial_\rho T^{\mu\nu}(x_p) + \cdots , \tag{3.11}$$

in the limit $\varepsilon \to 0$. With

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \varepsilon\, \partial_\nu X^\mu , \tag{3.12}$$

we shift the tensor at the point p to the point q (active transformation):

$$\tilde{T}^{\mu\nu}_{\text{shift}}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma} T^{\rho\sigma}(x) = T^{\mu\nu}(x) + \varepsilon \Big[\partial_\rho X^\mu\, T^{\rho\nu}(x) + \partial_\sigma X^\nu\, T^{\mu\sigma}(x) \Big] + \mathcal{O}(\varepsilon^2) , \tag{3.13}$$

Now we define the *Lie derivative* as

$$\pounds_X T^{\mu\nu} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left[T^{\mu\nu}(x_q) - T^{\mu\nu}_{\text{shift}} \right] = X^\rho \partial_\rho T^{\mu\nu} - T^{\mu\rho}\partial_\rho X^\nu - T^{\rho\nu}\partial_\rho X^\mu , \tag{3.14}$$

and a few more examples are

$$\pounds_X A^\mu = A^\mu{}_{,\nu} X^\nu - X^\mu{}_{,\nu} A^\nu , \qquad \pounds_X T_{\mu\nu} = T_{\mu\nu,\rho} X^\rho + T_{\rho\nu} X^\rho{}_{,\mu} + T_{\mu\rho} X^\rho{}_{,\nu} . \tag{3.15}$$

The Lie derivative is independent of coordinate systems and preserves the tensor rank. It reduces to ordinary differentiation with respect to x^ν, in case $X^\mu = \delta^\mu_\nu$. Furthermore, it is linear in the tensor field and satisfies the Leibniz chain rule. All the partial derivatives in the Lie derivative can be replaced in terms of covariant derivatives defined below.

• ***Covariant derivatives***.— Similarly, a vector field V^μ at a point q can be approximated

$$V^\mu(x_q) = V^\mu(x_p) + \delta x^\nu \partial_\nu V^\mu(x_p) + \cdots , \qquad x^\mu_q := x^\mu_p + \delta x^\mu , \tag{3.16}$$

if two points are close to each other. However, this vector cannot be directly compared to the vector at a point p, so we shift the vector at p to the point q. With infinitesimal displacement, the shifted vector at q can be written as (or again actively transform)

$$V^\mu_{\text{shift}}(x_q) = V^\mu(x_p) - \Gamma^\mu_{\rho\nu}(x_p) V^\rho(x_p) \delta x^\nu + \cdots , \qquad \delta x^\mu{}_{,\rho} V^\rho =: -\Gamma^\mu_{\rho\nu} V^\rho \delta x^\nu , \tag{3.17}$$

where the affine connection Γ is defined in terms of a proportionality constant. Now we define the covariant derivative as

$$V^\mu{}_{;\nu} := \nabla_\nu V^\mu = \lim_{\delta x\to 0}\frac{1}{\delta x^\nu}\left[V^\mu(x_q) - V^\mu_{\text{shift}}\right] = \partial_\nu V^\mu + \Gamma^\mu_{\rho\nu}V^\rho \,. \tag{3.18}$$

Note that $\pounds_X T$ and $X^\mu\nabla_\mu T$ are similar, but different, in that the latter is a covariant derivative of a tensor T at a given point followed by an inner product with X^μ, while the former is a comparison of T separated by X^μ. Given the definition, we can readily derive

$$\nabla_\mu\phi = \partial_\mu\phi \,, \qquad X_{\mu;\nu} = X_{\mu,\nu} - \Gamma^\rho_{\mu\nu}X_\rho \,, \qquad T^\mu_{\nu;\rho} = T^\mu_{\nu,\rho} + \Gamma^\mu_{\rho\sigma}T^\sigma_\nu - \Gamma^\sigma_{\nu\rho}T^\mu_\sigma \,. \tag{3.19}$$

Contraction over two indicies commutes with the covariant derivative:

$$\nabla_\mu\left(T^{a_1\cdots a_i=c\cdots a_n}_{b_1\cdots b_i c\cdots b_m}\right) = \left(\nabla_\mu T^{a_1\cdots a_n}_{b_1\cdots c_m}\right)\Big|^{a_i=c}_{b_i=c} \,. \tag{3.20}$$

• ***Geodesic and Affine connection***.— Geodesic is in a simple term a *straightest possible curve* in a curved manifold. For example, in a sphere, geodesic provides a shorted path (a great circle on a sphere) between two points, and a parallel direction at a one point is no longer parallel along the geodesic. Note that any parallel direction remains parallel in Euclidean geometries, when shifted to any points.

If we demand that the covariant derivative of a contravariant tensor transforms as $(1,1)$ tensor, we derive that the affine connection transforms as

$$\tilde\Gamma^\lambda_{\mu\nu} = \frac{\partial\tilde x^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial\tilde x^\mu}\frac{\partial x^\sigma}{\partial\tilde x^\nu}\,\Gamma^\rho_{\tau\sigma} + \frac{\partial\tilde x^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial\tilde x^\mu\partial\tilde x^\nu} \,. \tag{3.21}$$

Any quantity which transforms as above is called an *affine connection*, and a manifold with affine connection is an *affine manifold*. A *torsion* tensor can be defined as

$$T^\mu_{\rho\sigma} := \Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho} \,, \tag{3.22}$$

and indeed it transforms tensorially, as the non-tensorial part in $\Gamma^\lambda_{\mu\nu}$ is removed. If the torsion tensor vanishes identically, the affine connection is symmetric over the lower indicies.

• ***Parallel transport and geodesic equation***.— Given a curve $x^\mu(u)$ and its tangent vector X^μ, we define a *total derivative* and a *parallel transport* along the curve:

$$\frac{D}{du}T := \nabla_X T = X^\mu\nabla_\mu T \,, \qquad \frac{D}{du}T \equiv 0 \,. \tag{3.23}$$

Furthermore, if a tangent vector X^μ is parallel transported along its curve $x^\mu(u)$ to be parallel to the tangent vector, this curve is called an *affine geodesic*:

$$\nabla_X X^\mu = \lambda X^\mu \,, \qquad \frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{du}\frac{dx^\sigma}{du} = \lambda(u)X^\mu \,, \tag{3.24}$$

where λ is a constant. Alternatively, this can be derived by shifting the tangent vector at p to q and using $\delta x^\mu = X^\mu\delta u$. When $\lambda \equiv 0$, or the parallel transported tangent vector is exactly the tangent vector, this parameter is called an *affine parameter* s, and the geodesic is invariant under re-parametrization $s \to c_1 s + c_2$ with arbitrary constants c_1 and c_2.

• ***Tensor density of weight W***.— transforms tensorially, but with extra power of the Jacobian J:

$$\tilde T^{\mu\cdots}_{\nu\cdots} = J^W\frac{\partial\tilde x^\mu}{\partial x^\kappa}\cdots\frac{\partial x^\rho}{\partial\tilde x^\nu}T^{\kappa\cdots}_{\nu\cdots} \,. \tag{3.25}$$

For instance, we will always deal with the metric tensor $g_{\mu\nu}$, and its determinant is denoted as g. Then, $\sqrt{-g}$ is a scalar density of weight -1 (other literature might use different sign convention, or weight +1):

$$\tilde g_{\mu\nu}(\tilde x) = \frac{\partial x^\rho}{\partial\tilde x^\mu}\frac{\partial x^\sigma}{\partial\tilde x^\nu}g_{\rho\sigma}(x) \,, \qquad \sqrt{-\tilde g} = J^{-1}\sqrt{-g} \,. \tag{3.26}$$

The *volume factor* d^4x is of weight $+1$, so that $\sqrt{-g}\, d^4x$ is an ordinary scalar. In the same way, for any vector density $V^\mu_{(W)}$ (or tensors) of weight W can be made an ordinary vector:

$$V^\mu := \left(\sqrt{-g}\right)^W V^\mu_{(W)} , \qquad \nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\nu} V^\nu . \tag{3.27}$$

Under the assumption that the covariant derivative of the metric tensor vanishes (which is true, if the connection is the Christoffel symbol), we can show that the covariant derivative of a vector density $V^\mu_{(W)}$ of W is

$$\nabla_\rho V^\mu_{(W)} = \partial_\rho V^\mu_{(W)} + \Gamma^\mu_{\rho\nu} V^\nu_{(W)} + W \Gamma^\nu_{\nu\rho} V^\mu_{(W)} , \tag{3.28}$$

with one extra term, which is also true for any rank tensor density $T^{\mu\nu\cdots}_{\rho\sigma\cdots\,(W)}$. In fact, this formula is valid, even if the connection is *not* Christoffel symbol (use the transformation rule). As a special case, we obtain

$$\nabla_\mu V^\mu_{(-1)} = \partial_\mu V^\mu_{(-1)} , \tag{3.29}$$

and hence we derive

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g}\, V^\mu) . \tag{3.30}$$

This can be directly verified by an explicit calculation of the covariant derivative with

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} , \tag{3.31}$$

from Eq. (2.11).

3.1.2 Levi-Civita Tensor and Matrix Determinant

The Levi-Civita tensor is defined always in any coordinates as

$$\varepsilon^{\mu\nu\rho\sigma} = +1 \quad \text{for even of } txyz , \qquad \varepsilon^{\mu\nu\rho\sigma} = -1 \quad \text{for odd} , \tag{3.32}$$

where the coordinates run in sequence.[1] Given this definition of the Levi-Civita tensor and any 4-by-4 matrix $\mathbf{M} = M_{\mu\nu}$, the determinant of $\mathbf{M}$ can be derived as

$$\det \mathbf{M} := \varepsilon^{\mu\nu\rho\sigma} M_{1\mu} M_{2\nu} M_{3\rho} M_{4\sigma} = \frac{1}{4!} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} M_{\alpha\mu} M_{\beta\nu} M_{\gamma\rho} M_{\delta\sigma} . \tag{3.33}$$

To demonstrate that the Levi-Civita symbol is in fact a tensor density of weight $+1$, we consider

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \frac{\partial \tilde{x}^\gamma}{\partial x^\rho} \frac{\partial \tilde{x}^\delta}{\partial x^\sigma} \varepsilon^{\mu\nu\rho\sigma} = J \varepsilon^{\alpha\beta\gamma\delta} . \tag{3.34}$$

Given the nature of index structure in LHS, the right-hand side should be proportional to $\varepsilon^{\alpha\beta\gamma\delta}$, and the proportionality constant can be computed by considering the reference sequence ($txyz$ for $\alpha\beta\gamma\delta$). Furthermore, we infer another useful relation

$$\det \mathbf{M} \cdot \varepsilon^{\mu\nu\rho\sigma} = \varepsilon^{\alpha\beta\gamma\delta} M^\mu{}_\alpha M^\nu{}_\beta M^\rho{}_\gamma M^\sigma{}_\delta , \tag{3.35}$$

for any matrix representation $\mathbf{M} = M^\mu{}_\nu$. The covariant Levi-Civita, defined by lowering the indicies with $g_{\mu\nu}$ as

$$\varepsilon_{\alpha\beta\gamma\delta} := g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma}\, \varepsilon^{\mu\nu\rho\sigma} = g\, \varepsilon^{\alpha\beta\gamma\delta} , \tag{3.36}$$

is then a tensor density of weight -1, as g is a density of weight -2 and the weights add up for a product of two tensor densities. Note that the indicies are down (co-variant), and the equation simply indicates that the value of the LHS is identical to the RHS given the index values. The second equality can be derived by the same inference and using the reference sequence.

[1] For example, coordinates can be t, x, y, z or t, r, θ, ϕ. However, mind that $\varepsilon^{txyz} = -\varepsilon^{xyzt}$, so the coordinate sequence should be checked. The normalization convention can also be different, for example, $\varepsilon^{0123} \equiv 1$, or $\varepsilon^{1230} \equiv 1 = -\varepsilon^{0123}$. In some cases, the convention $\varepsilon_{0123} = 1$ is adopted.

Now consider a matrix representation $\mathbf{M} = M_{\mu\nu}$ and its inverse $\mathbf{M}^{-1} = M^{\mu\nu}$, there exists a useful identity

$$M^{\mu\nu}\frac{\partial}{\partial x^\alpha}M_{\nu\mu} \equiv \mathrm{Tr}\Big[\mathbf{M}^{-1}\partial_\alpha\mathbf{M}\Big] = \frac{\partial}{\partial x^\alpha}\ln|\det\mathbf{M}| \,, \tag{3.37}$$

and we can readily show that

$$\partial_\rho g = g\, g^{\mu\nu}\,\partial_\rho g_{\mu\nu} = 2g\,\Gamma^\mu_{\mu\rho} \,, \qquad \therefore\ \Gamma^\mu_{\mu\rho} = \frac{1}{2g}\frac{\partial}{\partial x^\rho}g = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\rho}\sqrt{-g} \,, \tag{3.38}$$

where we used the relation between $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ in Eq. (2.11). Since g is a scalar of weight -2, this relation leads to

$$\nabla_\mu g = 0 \,, \qquad \nabla_\rho g_{\mu\nu} = 0 \,. \tag{3.39}$$

Some useful identities are

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = g\ 4! \,, \qquad \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\kappa\nu\rho\sigma} = g\ 3!\delta^\mu_\kappa \,, \tag{3.40}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\kappa\epsilon\rho\sigma} = 2g\,\delta^{\mu\nu}_{\kappa\epsilon} := 2g\begin{vmatrix}\delta^\mu_\kappa & \delta^\nu_\kappa \\ \delta^\mu_\epsilon & \delta^\nu_\epsilon\end{vmatrix} = \begin{cases} +1 & \mu\neq\nu\,,\ \ \mu=\kappa\,,\ \ \nu=\epsilon \\ -1 & \mu\neq\nu\,,\ \ \mu=\epsilon\,,\ \ \nu=\kappa \\ 0 & \text{otherwise}\end{cases} \,, \tag{3.41}$$

where we assumed four spacetime dimension. $g=-1$ in the Minkowski and $g=1$ in the Euclidean.

3.1.3 Differential Forms

A differential p-form is a tensor of rank $(0,p)$ with completely anti-symmetric indicies. A scalar is $(0,0)$, and hence a 0-form. A co-vector (dual vector) is $(0,1)$, and a 1-form. The E&M field strength tensor $F_{\mu\nu}$ is $(0,2)$ and anti-symmetric over its indicies, and hence a 2-form. There is no p-form in d-dimension, if $p>d$. In n-dimension, there exist only

$$\begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{p!(n-p)!} \,, \tag{3.42}$$

number of independent p-forms. A wedge product of p-form A and q-form B is a simple product, but their indicies should be anti-symmetrized:

$$A\wedge B = \frac{(p+q)!}{p!q!}\,A_{[\mu_1\cdots\mu_p}B_{\mu_1\cdots\mu_q]} = (-1)^{pq}B\wedge A \,. \tag{3.43}$$

For instance, consider ϕ, A_μ, B_μ, and we obtain

$$(\phi\wedge A)_\mu = \phi A_\mu \,, \qquad (A\wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu \,. \tag{3.44}$$

With a coordinate independent notation, a differential p-form ω can be expressed as

$$\omega = \omega_{\mu_1\cdots\mu_p}(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}) \,. \tag{3.45}$$

An exterior derivative d increases the rank of a p-form A by one:

$$(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]} \,, \tag{3.46}$$

and the simplest examples are the gradient and the curl

$$(d\phi)_\mu = \partial_\mu\phi \,, \qquad (dA)_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.47}$$

Due to the anti-symmetric nature, a repeat application of the exterior derivative is always zero:

$$0 = d(dA) \,. \tag{3.48}$$

A p-form F is called *closed* if $dF=0$, and is called *exact* if there exists a $(p-1)$-form A with $F=d(A)$. All exact forms are always closed, but the converse is true, provided that the manifold is simply connected. For example, the E&M field strength tensor $F_{\mu\nu}$ is a 2-form, which can be expressed in terms of a 1-form A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu = (dA)_{\mu\nu} \,, \qquad dF = 0 \,. \tag{3.49}$$

As noted in E&M, however, A_μ is *not* uniquely determined, as

$$F = d(A) = d(\tilde{A}) \, , \qquad\qquad \tilde{A} = A + du \, , \tag{3.50}$$

where we introduced a $(p-2)$-form u. In E&M, $du \to \partial_\mu \phi$, corresponding to the gauge transformation.

• ***Stokes' theorem***.— While it makes little sense to compare or manipulate quantities at two different points, it is fine if the quantities are scalar. For example, a summation of a scalar field at two points remains invariant under a coordinate transformation:

$$\tilde{\phi}(\tilde{x}_1) + \tilde{\phi}(\tilde{x}_2) = \phi(x_1) + \phi(x_1) \, . \tag{3.51}$$

Drawing on this concept, we can construct an integral over some region in the manifold. As discussed before, however, the coordinate volume factor d^4x is of weight $+1$, such that we need to include a scalar of weight -1 in the integration to compensate for it. Consider the volume element of m-dimensional subspace in n-dimensional manifold

$$dV^{\mu_1\cdots\mu_m} := \delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \frac{\partial x^\nu}{\partial u_1} \cdots \frac{\partial x^\nu}{\partial u_m} \, du^1 \cdots du^m \, , \qquad\qquad m \le n \, , \tag{3.52}$$

where the curves are parametrized as $x^\mu(u)$. This volume element is a tensor of rank m (and density 0) and invariant under re-parametrization of the curves with u. A covariant tensor $X_{\mu_1\cdots\mu_m}$ of rank m can be used to construct a scalar and form a volume integral over the m-dimensional subspace Ω:

$$\int_\Omega X_{\mu_1\cdots\mu_m} dV^{\mu_1\cdots\mu_m} \, . \tag{3.53}$$

For example, consider a 2D curve $x^\mu = (x, y)$ parametrized in terms of $u^\mu = (r, \theta)$, and the volume element is then

$$dV^{xy} = \delta^{xy}_{\mu\nu} \frac{\partial x^\mu}{\partial u_1} \frac{\partial x^\nu}{\partial u_2} \, du^1 du^2 = \left(\frac{\partial x^x}{\partial u_1} \frac{\partial x^y}{\partial u_2} - \frac{\partial x^y}{\partial u_1} \frac{\partial x^x}{\partial u_2} \right) du^1 du^2 = r \, dr \, d\theta = -dV^{yx} \, . \tag{3.54}$$

In 4D, the ordinary volume element and the surface element are related as

$$dV = \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma} dV^{\mu\nu\rho\sigma} = g \, dt dx dy dz \, , \qquad\qquad dS_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} dV^{\nu\rho\sigma} = g \, (dx dy dz, \; dt dy dz, \; dt dx dz, \; dt dx dy) \, . \tag{3.55}$$

The Stoke's theorem states that in a m-dimensional sub-space the integral of a covariant tensor of rank $(m-1)$ over the boundary $\partial\Omega$ is equal to the integral of a derivative over the volume Ω:

$$\int_{\partial\Omega} X_{\mu_1\cdots\mu_{m-1}} dV^{\mu_1\cdots\mu_{m-1}} = \int_\Omega \partial_{\mu_m} X_{\mu_1\cdots\mu_{m-1}} dV^{\mu_1\cdots\mu_m} \, . \tag{3.56}$$

Special classes of this general Stoke's theorem are

$$\int \nabla \cdot \boldsymbol{F} \, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S} \, , \qquad\qquad \int_S \nabla \times \boldsymbol{F} \cdot d\boldsymbol{S} = \int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{\Gamma} \, , \tag{3.57}$$

the Gauss divergence theorem and (also) the Stoke's theorem in three-dimensional space.

3.2 Metric and Riemann Curvature Tensors

With the equivalence principle, one can always find a coordinate, in which the metric is locally Minkowski $g_{\mu\nu} = \eta_{\mu\nu}$ and hence the Christoffel symbol vanishes $\Gamma^\rho_{\mu\nu} = 0$. Hence, anything that distinguishes from the Minkowski spacetime should come from at least the second derivatives of $g_{\mu\nu}$. This also implies that the metric alone is not enough to prove that the spacetime is curved, i.e., the metric in a spherical coordinate is not Minkowski, but the spacetime is in fact not curved.

3.2.1 Metric Tensor

Any symmetric covariant tensor $g_{\mu\nu}$ of rank two can be used to define distances of any vectors. In particular, the *line element* or infinitesimal distance between two neighboring points is

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \,, \tag{3.58}$$

where ds^2 can be either positive or negative despite the notation. Our convention is *mostly plus*, such that the interval is called *time-like* for $ds^2 < 0$, *space-like* for $ds^2 > 0$, and *null* for $ds^2 = 0$. The metric tensor is often called the *first fundamental form*. A manifold with a metric is called a *Riemannian* manifold. Given the metric tensor, its inverse $g^{\mu\nu}$ and the determinant g are

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu \,, \qquad\qquad g := \det\ g_{\mu\nu} \,. \tag{3.59}$$

• ***Metric geodesic***.— Consider a time-like curve $x^\mu(u)$ and the distance between two points is

$$s = \int_p^q ds = \int du \, \frac{ds}{du} = \int du \sqrt{-g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}} \,. \tag{3.60}$$

By using the Euler-Lagrange equation with respect to dx^μ/du, we find that the path x^μ minimizing the distance between two points should satisfy

$$g_{\mu\nu}\frac{d^2x^\nu}{du^2} + \{\nu\rho,\mu\}\frac{dx^\nu}{du}\frac{dx^\rho}{du} = -\frac{d^2s}{du^2}\left(\frac{ds}{du}\right)^{-1} g_{\mu\nu}\frac{dx^\nu}{du} \,, \tag{3.61}$$

or by removing $g_{\mu\nu}$

$$\frac{d^2x^\mu}{du^2} + \left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\} \frac{dx^\nu}{du}\frac{dx^\rho}{du} = -\frac{d^2s}{du^2}\left(\frac{ds}{du}\right)^{-1} \frac{dx^\mu}{du} \,, \tag{3.62}$$

where we defined the Christoffel symbols of the first kind and the second kind:

$$\{\nu\rho,\kappa\} := \frac{1}{2}\left(\partial_\rho g_{\nu\kappa} + \partial_\nu g_{\rho\kappa} - \partial_\kappa g_{\nu\rho}\right) \,, \tag{3.63}$$

$$\left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\} := g^{\mu\kappa}\{\nu\rho,\kappa\} = \frac{1}{2}g^{\mu\kappa}\left(\partial_\rho g_{\nu\kappa} + \partial_\nu g_{\rho\kappa} - \partial_\kappa g_{\nu\rho}\right) \,. \tag{3.64}$$

If we parametrize the curve in terms of s, the geodesic equation becomes

$$\frac{d^2x^\mu}{ds^2} + \left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\} \frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \,, \qquad\qquad -1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \,. \tag{3.65}$$

It is now apparent that the affine and the metric geodesics coincide, if we equate the affine connection to the Christoffel symbol of the second kind:

$$\Gamma^\mu_{\nu\rho} := \left\{ \begin{array}{c} \mu \\ \nu\rho \end{array} \right\} = \frac{1}{2}g^{\mu\kappa}\left(\partial_\rho g_{\nu\kappa} + \partial_\nu g_{\rho\kappa} - \partial_\kappa g_{\nu\rho}\right) \,. \tag{3.66}$$

This automatically leads to the vanishing torsion tensor and the metricity condition

$$T^\mu_{\nu\rho} = 0 \,, \qquad\qquad 0 = \nabla_\rho g_{\mu\nu} = g_{\mu\nu,\rho} - \Gamma^\epsilon_{\rho\mu}\, g_{\epsilon\nu} - \Gamma^\epsilon_{\rho\nu}\, g_{\mu\epsilon} \,. \tag{3.67}$$

For the latter, one can go to a free-falling frame to see the triviality. Alternatively, if we assume the metricity condition, we can show that the affine connection is the metric connection.

In short, if we take the Christoffel symbol as our affine connection, the metricity relation holds, and the converse is also true (see § 3.1 of Wald (1984) for the uniqueness). The metricity relation can also be derived, if we demand that the inner product of two vectors remains unchanged, when both vectors are parallel transported.

3.2.2 Riemann Curvature Tensor

Unlike coordinate derivatives, covariant differentiation does not commute. A straightforward computation yields

$$X^\mu{}_{;\rho\sigma} = \partial_\sigma\left(\partial_\rho X^\mu + \Gamma^\mu_{\nu\rho}X^\nu\right) + \Gamma^\mu_{\kappa\sigma}\left(\partial_\rho X^\kappa + \Gamma^\kappa_{\nu\rho}X^\nu\right) - \Gamma^\kappa_{\rho\sigma}\left(\partial_\kappa X^\mu + \Gamma^\mu_{\nu\kappa}X^\nu\right) , \tag{3.68}$$

$$X^\mu{}_{;\sigma\rho} = \partial_\rho\left(\partial_\sigma X^\mu + \Gamma^\mu_{\nu\sigma}X^\nu\right) + \Gamma^\mu_{\kappa\rho}\left(\partial_\sigma X^\kappa + \Gamma^\kappa_{\nu\sigma}X^\nu\right) - \Gamma^\kappa_{\sigma\rho}\left(\partial_\kappa X^\mu + \Gamma^\mu_{\nu\kappa}X^\nu\right) , \tag{3.69}$$

and we obtain

$$\nabla_\sigma\nabla_\rho X^\mu - \nabla_\rho\nabla_\sigma X^\mu = 2X^\mu{}_{;[\rho\sigma]} = R^\mu{}_{\nu\sigma\rho}X^\nu - T^\kappa_{\rho\sigma}\nabla_\kappa X^\mu , \tag{3.70}$$

where $T^\kappa_{\rho\sigma}$ is the torsion tensor and we defined the Riemann tensor in terms of the Christoffel symbols only:

$$R^\mu_{\nu\rho\sigma} := \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} - \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon} = \Gamma^\mu_{\nu\sigma;\rho} - \Gamma^\mu_{\nu\rho;\sigma} - \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} + \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon} . \tag{3.71}$$

Despite the appearance, the Riemann tensor is indeed a tensor, transforming tensorially. We will consider only the connections with vanishing torsion $T^\kappa_{\rho\sigma} \equiv 0$: The Riemann tensor has all the information of the geometry, such that how any four vector changes locally is fully determined by the Riemann tensor

$$2u_{\mu;[\nu\rho]} = u_\sigma R^\sigma{}_{\mu\nu\rho} , \qquad u^\rho{}_{;[\nu\mu]} = \frac{1}{2}R^\rho{}_{\sigma\mu\nu}u^\sigma . \tag{3.72}$$

Consider a small closed path (or a round trip) and parallel transport a vector along the path. Only when the Riemann vanishes identically, the transported vector (or any tensors) will come back to the original configuration, regardless of paths taken (see my note).

• ***Symmetries in Riemann tensor***.— With the metric tensor, we define the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R:

$$R_{\mu\nu} := R^\kappa{}_{\mu\kappa\nu} , \qquad R := R^\mu_\mu = g^{\mu\nu}R_{\nu\mu} , \qquad R_{\mu\nu\rho\sigma} = g_{\mu\kappa}R^\kappa{}_{\nu\rho\sigma} . \tag{3.73}$$

Given the definition, the Riemann tensor has the symmetry in its indices

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu} . \tag{3.74}$$

Furthermore, due to the symmetry in the connection, the Riemann tensor satisfies

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho} = R^\mu{}_{[\nu\rho\sigma]} = 0 . \tag{3.75}$$

It can be shown that the Riemann tensor satisfies the Bianchi identities: (use the normal coordinate in Section 3.3.1)

$$R_{\mu\nu\rho\sigma;\kappa} + R_{\mu\nu\kappa\rho;\sigma} + R_{\mu\nu\sigma\kappa;\rho} = 0 . \tag{3.76}$$

• ***2D Sphere as an example***.— Given a metric tensor in a 2D sphere of radius r

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \tag{3.77}$$

we first compute the Christoffel symbols

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\ \cos\theta , \qquad \Gamma^\phi_{\theta\phi} = \cot\theta , \tag{3.78}$$

and the Riemann tensor

$$R^\theta{}_{\phi\theta\phi} = \sin^2\theta , \qquad R_{\phi\phi} = \sin^2\theta , \qquad R_{\theta\theta} = 1 , \qquad R = \frac{2}{r^2} . \tag{3.79}$$

3.2.3 Weyl Conformal Tensor

The symmetries in the Riemann tensor reduce the degree of freedom from n^4 to

$$R^\mu{}_{\nu\rho\sigma} \ni \frac{n^2(n^2-1)}{12} \quad \text{dof} \, . \tag{3.80}$$

For the case of four-dimensional spacetime, the degree of freedom is reduced to 20 from $4^4 = 256$:

- $n = 1$: the Riemann tensor vanishes identically $R^\mu{}_{\nu\rho\sigma} = 0$, i.e., dof is zero,
- $n = 2$: only one component of the Riemann tensor is independent, i.e., dof is one, essentially the Ricci scalar R,
- $n = 3$: six dofs are essentially captured by the Ricci tensor $R_{\mu\nu}$,
- $n = 4$: twenty dofs exist. Ten of them is represented by the Ricci tensor $R_{\mu\nu}$ and the remaining ten by the Weyl tensor $C^\mu{}_{\nu\rho\sigma}$.

The (traceless) Weyl curvature tensor or the conformal tensor is defined as

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \, , \tag{3.81}$$

where the factors 2 and 6 in the denominator are indeed $(n-2)$ and $(n-1)(n-2)$ in n-dimension.

Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic (see Section 3.3.2). The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force. The Ricci curvature, or trace component of the Riemann tensor contains precisely the information about how volumes change in the presence of tidal forces, so the Weyl tensor is the traceless component of the Riemann tensor. In other words, the Ricci tensor is algebraically set by matter distribution through the Einstein equation, but the Weyl tensor is determined by differential equations with suitable boundary conditions.

The Weyl tensor has the same symmetries as in the Riemnann tensor, in addition to the traceless condition (metric contraction on any pair of indices yields zero)

$$C^\mu{}_{\nu\mu\sigma} = 0 \, . \tag{3.82}$$

Two metric tensors are *conformally related* if

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \, , \tag{3.83}$$

for a non-vanishing differentiable function Ω in the manifold. Two metrics describe two different geometries, but the causal structure (or null geodesic) is identical, and the Weyl tensors are also identical:

$$\hat{C}^\mu{}_{\nu\rho\sigma} = C^\mu{}_{\nu\rho\sigma} \, . \tag{3.84}$$

If the Weyl tensor vanishes identically, the metric is conformally flat:

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \, , \tag{3.85}$$

and the converse is also true. For instance, the background FRW metric is conformally flat.

3.2.4 Killing Vector and Maximally Symmetric Spacetime

• ***Killing vectors***.— A metric $g_{\mu\nu}(x)$ is called *form-invariant*, if

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) \, , \tag{3.86}$$

under a coordinate transformation for all x, and this transformation is called *isometry*. Note that this condition is different from the condition for a scalar $\tilde{\phi}(\tilde{x}) = \phi(x)$. Given the transformation law for a metric tensor, the condition for a form-invariant metric puts a very complicated restriction on the transformation. For an infinitesimal transformation, this can be expressed as

$$0 = \xi_{\mu;\nu} + \xi_{\nu;\mu} = \mathcal{L}_\xi g \, , \qquad \tilde{x}^\mu = x^\mu + \xi^\mu(x) \, . \tag{3.87}$$

Any four vector that satisfies this equation is called a *Killing* vector. The vanishing Lie derivative states that the metric tensor remains unchanged along ξ^μ. Given the structure of the Killing vector, the full functional form of $\xi^\mu(x)$ in the manifold is fully determined by ξ^μ and its derivative $\xi^\mu{}_{;\nu}$ at some point p (see Weinberg (1972) § 13 for more details).

For two Killing vectors ξ_1^μ and ξ_2^μ, we can readily show that

$$[\xi_1, \xi_2]^\mu := \xi_2^\nu \nabla_\nu \xi_1^\mu - \xi_1^\nu \nabla_\nu \xi_2^\mu \,, \tag{3.88}$$

is another Killing vector

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}]\, g_{\mu\nu} = \mathcal{L}_{[\xi_1,\xi_2]} g_{\mu\nu} = 0 \,. \tag{3.89}$$

For a geodesic x^μ_λ and its tangent vector k^μ, the inner product $\xi_\mu k^\mu$ with any Killing vector ξ^μ is conserved along the geodesic:

$$\frac{D}{d\lambda}\left(\xi_\mu k^\mu\right) := k^\nu \nabla_\nu \left(\xi_\mu k^\mu\right) = k^\mu k^\nu \nabla_\nu \xi_\mu \equiv \frac{1}{2} k^\mu k^\nu \left(\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu\right) = 0 \,. \tag{3.90}$$

In the same way, a current $J^\mu := T^{\mu\nu}\xi_\nu$ of an energy-momentum tensor $T_{\mu\nu}$ and a given Killing vector ξ_ν is also conserved:

$$\nabla_\mu J^\mu = T^{\mu\nu}\nabla_\mu \xi_\nu = \frac{1}{2} T^{\mu\nu}\left(\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu\right) = 0 \,. \tag{3.91}$$

• ***Stationary metric***.— The spacetime metric is said to be *stationary*, if there exists a *special coordinate*, in which the metric tensor is independent of a time coordinate (or *stationary*),

$$0 = \frac{\partial}{\partial t}\, g_{\mu\nu} \,, \tag{3.92}$$

which means the metric tensor can depend on x^0 in other coordinates (which is t in this special coordinate). This observation can be generalized by considering a time-like vector ξ^μ and a special coordinate, in which

$$\xi^\mu := \delta^\mu_t \,, \qquad 0 = \pounds_\xi g_{\mu\nu} = g_{\mu\rho}\Gamma^\rho_{\nu t} + g_{\nu\rho}\Gamma^\rho_{\mu t} \,, \qquad \therefore \;\; \frac{\partial}{\partial t}\, g_{\mu\nu} = 0 \,, \tag{3.93}$$

where the vanishing derivative is obtained by using $g_{\mu\nu;t} = 0$. Hence, a spacetime metric is stationary, only if a time-like Killing vector exists.

• ***Static metric***.— In the same way, if the metric admits a time-like Killing vector orthogonal to a coordinate hypersurface, the spacetime is called *static*. Since the Killing vector is time-like, the spacetime is stationary, and with the orthogonality to the hypersurface as an extra condition, the space-time cross-term cannot exist:

$$dx^\mu_\Sigma := (0, dx^i) \,, \qquad 0 = g_{\mu\nu}\xi^\mu dx^\nu_\Sigma \,, \qquad \therefore \;\; 0 = g_{ti} \,. \tag{3.94}$$

This also implies that *stationary* metric can still have space-time cross terms. Note also that only the space-time cross terms in metric changes sign under time-reversal, so that *static* metric is invariant under time-reversal. For example, a rotating system can have stationary fields, but it is not invariant under time-reversal, because of the cross terms or directionality of the rotating system. So, *static* metric is *irrotational* and *stationary*.

This condition can also be generalized by considering a hypersurface Σ and a function on the surface:

$$f(x^\mu_\Sigma) = \text{constant} \,, \qquad \therefore \;\; \xi^\mu \propto \nabla^\mu f \quad \text{for } x^\mu \in \Sigma \,. \tag{3.95}$$

Defining $\xi^\mu := \lambda(x)\nabla^\mu f$, one can show that the static condition is satisfied, only if

$$\xi_{[\mu}\nabla_\nu \xi_{\rho]} = 0 \,, \tag{3.96}$$

where the three indicies are totally anti-symmetrized.

• ***Spherically symmetric metric***.— Furthermore, a spacetime is said to be *spherically symmetric*, if three linearly independent spacelike Killing vectors $\mathbf{X}_i$ exist and satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3 \,, \qquad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_1 \,, \qquad [\mathbf{X}_3, \mathbf{X}_1] = \mathbf{X}_2 \,. \tag{3.97}$$

These three Killing vectors represent the rotational axes in three sphere.

• ***Maximally Symmetric Space***.— is defined as a space that possesses the largest possible number of Killing vectors, and it is homogeneous and isotropic. On an n-dimensional manifold this number is $n(n+1)/2$, and it is easy to understand. Consider an Euclidean space R^n, where the isometries are translations and rotations — there are n translations, one for each direction we can move, and $n(n-1)/2$ rotations, i.e., $(n-1)$ rotation axes for each direction we want to rotate. This applies to spacetime with non-Euclidean signature. For $n \geq 2$, there are more Killing vectors than the number of dimension. Strange at first, because more than n vectors in n-dimension cannot be independent at any point. However, Killing vectors here mean actually Killing vector fields, and a linear combination of Killing vector fields with constant coefficients is still a Killing vector field.

• ***Maximally symmetric spacetime***.— is described by a constant curvature (Ricci scalar R) at every space and time. There exist three cases: Minkowski ($R = 0$), de Sitter ($R > 0$), and Anti-de Sitter ($R < 0$). D-dimensional de Sitter spacetime is a sphere S_D ($D \geq 3$) embedded in a $(D+1)$-dimensional flat spacetime. D-dimensional AdS is a hyperboloid in a $(D+1)$-dimensional flat spacetime. The Riemann tensor and the curvature scalar of a maximally symmetric D-dimensional spacetime are

$$R_{\mu\nu\rho\sigma} = \hat{K}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) , \qquad R = D(D-1)\hat{K} , \tag{3.98}$$

where $\hat{K}$ and R are both constant. The normalization convention is consistent with our FLRW notation, i.e., $R^{(3)} = 6\hat{K}$.

• ***Birkhoff's theorem***.— The metric outside a spherically symmetric source is static in general relativity ($R_{\mu\nu} = 0$). In Newtonian gravity, time-dependence and spherical symmetry are independent. For example, if a pulsating mass distribution maintains spherical symmetry, the metric outside the source is static, i.e., no gravitational waves.

3.3 More on the Curvature

3.3.1 Riemann Normal Coordinate

General covariance allows a coordinate system, where the metric is locally Minkowski and the first derivative of the metric vanishes (so do the Christoffel symbols)

$$g_{\mu\nu} = \eta_{\mu\nu} , \qquad g_{\mu\nu,\rho} = 0 , \qquad \Gamma^{\rho}_{\mu\nu} = 0 . \tag{3.99}$$

Riemann normal coordinates realize such coordinates by using a set of four geodesics (one time-like and three space-like) starting from the fixed point P. Fermi normal coordinates are a specific extension of Riemann coordinates such that holds for every point along a fixed time-like geodesic.

Consider a coordinate transformation between $\tilde{x}^\mu$ and x^a, in which x^a represents a local normal coordinate. The metric tensor transforms as

$$g_{ab} = \frac{\partial\tilde{x}^\mu}{\partial x^a}\frac{\partial\tilde{x}^\nu}{\partial x^b}\,\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}[e_a]^\mu[e_b]^\nu , \tag{3.100}$$

and with 16 dof in the transformation at P (tetrad vectors)

$$[e_a]^\mu_P := \left.\frac{\partial\tilde{x}^\mu}{\partial x^a}\right|_P , \tag{3.101}$$

we can make the local metric Minkowski $g_{ab} = \eta_{ab}$ at P (only the symmetric part of 16 dofs participates). We want to show that the first derivative of the metric in the normal coordinate vanishes at P. Now consider three geodesic paths emanating from P along the spatial tetrad directions, and we parametrize them in terms of proper distance $d\lambda = ds$. Using these geodesics and assigning the proper interval to the local coordinate x^i, we can construct a coordinate system in the small neighborhood of a point P

$$\tilde{x}^\mu = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n\tilde{x}^\mu}{d\lambda^n}\right|_P\lambda^n = \tilde{x}^\mu_P + [e_i]^\mu_P x^i - \frac{1}{2}\left.\Gamma^\mu_{\rho\sigma}\right|_P[e_i]^\rho_P[e_j]^\sigma_P x^i x^j + \mathcal{O}\left(x^3\right) , \qquad x^a_P \equiv 0 , \tag{3.102}$$

where we used
$$\frac{dx^i}{d\lambda}=1\,,\qquad 0=\frac{d^2\tilde{x}^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma}\frac{d\tilde{x}^\rho}{d\lambda}\frac{d\tilde{x}^\sigma}{d\lambda}=\frac{d^2\tilde{x}^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma}[e_i]^\rho[e_j]^\sigma\,. \tag{3.103}$$
To compute the metric in the normal coordinate, we need to compute the derivatives of the coordinate transformation:
$$\begin{aligned}\frac{\partial\tilde{x}^\mu}{\partial x^0} &= \frac{\partial\tilde{x}^\mu_P}{\partial x^0}+\frac{\partial}{\partial x^0}\left([e_i]^\mu_P\right)x^i+\mathcal{O}(x^2)=[e_t]^\mu_P-\Gamma^\mu_{\rho\sigma}\Big|_P[e_i]^\rho_P[e_t]^\sigma_P x^i+\mathcal{O}(x^2)\,, &(3.104)\\ \frac{\partial\tilde{x}^\mu}{\partial x^k} &= 0+[e_k]^\mu_P-\Gamma^\mu_{\rho\sigma}\Big|_P[e_i]^\rho_P[e_k]^\sigma_P x^i+\mathcal{O}(x^2)\,, &(3.105)\end{aligned}$$

where the tetrad vectors are constructed along a time-like path parametrized by x^0
$$0=\frac{D}{dx^0}[e_i]^\mu=\frac{\partial[e_i]^\mu}{\partial x^0}+\Gamma^\mu_{\rho\sigma}[e_i]^\rho[e_t]^\nu\,, \tag{3.106}$$

and they are independent of x^i.

Therefore, the metric tensor in the normal coordinate around P is to the zeroth order and the linear order in x^i:
$$\begin{aligned}g_{ab} &= \frac{\partial\tilde{x}^\mu}{\partial x^a}\frac{\partial\tilde{x}^\nu}{\partial x^b}\,\tilde{g}_{\mu\nu}=\left([e_a]^\mu_P-\Gamma^\mu_{\rho\sigma}\Big|_P[e_i]^\rho_P[e_a]^\sigma_P x^i\right)\left([e_b]^\nu_P-\Gamma^\nu_{\rho\sigma}\Big|_P[e_j]^\rho[e_b]^\sigma_P x^j\right)\left(\tilde{g}^P_{\mu\nu}+\tilde{g}^P_{\mu\nu,\rho}[e_k]^\rho_P x^k\right)\\ &= [e_a]^\mu_P[e_b]^\nu_P\tilde{g}^P_{\mu\nu}+\left(g_{\mu\nu,\rho}-g_{\sigma\nu}\Gamma^\sigma_{\rho\mu}-g_{\mu\sigma}\Gamma^\sigma_{\rho\nu}\right)_P[e_a]^\mu_P[e_b]^\nu_P[e_i]^\rho_P x^i+\mathcal{O}(x^2)\\ &= \eta_{ab}+0+\mathcal{O}(x^2)\,, \end{aligned}\tag{3.107}$$

where the round bracket at P vanishes, as $\nabla_\rho g_{\mu\nu}=0$. Indeed, the local metric tensor at P ($x^i=0$) is the Minkowski and the first derivative of the metric at P vanishes.

3.3.2 Geodesic Deviation Equation

Due to the equivalence principle, one cannot measure the gravity in a free fall, as the test particle and the laboratory are moving together. However, two test particles separated by ξ^μ will follow different geodesic, and the time evolution of its separation can be measured. That is described by the geodesic deviation equation in general relativity. In the Newtonian gravity, it is described by the tidal force (or differential gravity):
$$\frac{d^2\xi^i}{dt^2}=-\delta^{ij}\frac{\partial^2\phi}{\partial x^j\partial x^k}\,\xi^k\,. \tag{3.108}$$

We will derive the geodesic equation in a curved spacetime. Consider a geodesic path x^μ_λ in terms of an affine parameter λ, and the geodesic equation is
$$0=\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}=\frac{dt^\mu}{d\lambda}+\Gamma^\mu_{\rho\sigma}t^\rho t^\sigma=t^\nu\nabla_\nu t^\mu\,,\qquad t^\mu:=\frac{dx^\mu}{d\lambda}\,, \tag{3.109}$$

where we defined the tangent direction t^μ. Again consider another geodesic path $\tilde{x}^\mu_\lambda$ parametrized by the same affine parameter, and the separation between two geodesics at λ is
$$\xi^\mu_\lambda:=\tilde{x}^\mu_\lambda-x^\mu_\lambda\,. \tag{3.110}$$

Assuming two geodesics are nearby $|\xi^\mu_\lambda|\ll 1$, we can expand the geodesic equation for $\tilde{x}^\mu$ and derive the equation for ξ^μ to the linear order as
$$0=\frac{d^2\xi^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma,\nu}\xi^\nu\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}+2\Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\,. \tag{3.111}$$
Now we compute how the separation vector is transported along the geodesic, or the change in ξ^μ along the flow:
$$\frac{D}{d\lambda}\xi^\mu=t^\nu\nabla_\nu\xi^\mu=\frac{d\xi^\mu}{d\lambda}+\Gamma^\mu_{\rho\sigma}\xi^\rho t^\sigma\,, \tag{3.112}$$

and the rate of change in ξ^μ along the flow (or the acceleration) is

$$\begin{aligned}\frac{D^2}{d\lambda^2}\xi^\mu &= \frac{d}{d\lambda}\left(\frac{d\xi^\mu}{d\lambda}+\Gamma^\mu_{\rho\sigma}\xi^\rho t^\sigma\right)+\Gamma^\mu_{\epsilon\kappa}\left(\frac{d\xi^\epsilon}{d\lambda}+\Gamma^\epsilon_{\rho\sigma}\xi^\rho t^\sigma\right)t^\kappa \\ &= \frac{d^2\xi^\mu}{d\lambda^2}+t^\kappa\frac{\partial}{\partial x^\kappa}\left(\Gamma^\mu_{\rho\sigma}\xi^\rho t^\sigma\right)+\Gamma^\mu_{\epsilon\kappa}\frac{d\xi^\epsilon}{d\lambda}t^\kappa+\Gamma^\epsilon_{\rho\sigma}\Gamma^\mu_{\kappa\epsilon}\xi^\rho t^\sigma t^\kappa \\ &= 0+\xi^\rho t^\kappa t^\sigma\left(\Gamma^\mu_{\rho\sigma,\kappa}-\Gamma^\mu_{\kappa\sigma,\rho}+\Gamma^\epsilon_{\rho\sigma}\Gamma^\mu_{\kappa\epsilon}-\Gamma^\epsilon_{\sigma\kappa}\Gamma^\mu_{\rho\epsilon}\right)=R^\mu{}_{\sigma\kappa\rho}t^\sigma t^\kappa\xi^\rho\ ,\end{aligned} \tag{3.113}$$

called the *geodesic deviation equation*, where we manipulated in the third line

$$t^\kappa\frac{\partial}{\partial x^\kappa}\left(\Gamma^\mu_{\rho\sigma}\xi^\rho t^\sigma\right)=\Gamma^\mu_{\rho\sigma,\kappa}\xi^\rho t^\kappa t^\sigma+\Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{d\lambda}t^\sigma+\Gamma^\mu_{\rho\sigma}\xi^\rho\left(-\Gamma^\sigma_{\kappa\epsilon}t^\kappa t^\epsilon\right)\ . \tag{3.114}$$

In the Newtonian gravity, the vacuum equation is the Laplace equation:

$$\nabla^2\phi=0\ , \tag{3.115}$$

i.e., the tidal tensor is traceless. In the same way, the vacuum equation in a curved space should correspond to

$$0=R^\mu{}_{\sigma\kappa\mu}t^\sigma t^\kappa\ , \tag{3.116}$$

for any given direction t^μ, implying that the Laplace equation corresponds to

$$\therefore\quad R_{\mu\nu}=0\ . \tag{3.117}$$

3.3.3 Palatini Equation

In the Riemann normal coordinate, the Christoffel symbol vanishes $\Gamma^\rho_{\mu\nu}=0$, while its derivatives are non-vanishing $\Gamma^\rho_{\mu\nu,\sigma}\neq 0$. Hence the Riemann tensor in this coordinate is simply

$$R^\mu{}_{\nu\rho\sigma}=\partial_\rho\Gamma^\mu_{\nu\sigma}-\partial_\sigma\Gamma^\mu_{\nu\rho}\ . \tag{3.118}$$

Now we change a coordinate, and all quantities will transform accordingly. The variation of the Riemann tensor in the normal coordinate is then

$$\delta R^\mu{}_{\nu\rho\sigma}=\partial_\rho(\delta\Gamma^\mu_{\nu\sigma})-\partial_\sigma(\delta\Gamma^\mu_{\nu\rho})\ , \tag{3.119}$$

where we used the commutation relation between partial derivatives ∂_μ and variation δ. Since the variation in the Christoffel symbol

$$\delta\Gamma^\rho_{\mu\nu}=\tilde\Gamma^\rho_{\mu\nu}-\Gamma^\rho_{\mu\nu}\ , \tag{3.120}$$

is the difference in Christoffel symbols, it transforms tensorially (one can verify by an explicit computation), and hence the expression for the variation of the Riemann in any coordinates should be

$$\delta R^\mu{}_{\nu\rho\sigma}=\nabla_\rho(\delta\Gamma^\mu_{\nu\sigma})-\nabla_\sigma(\delta\Gamma^\mu_{\nu\rho})\ . \tag{3.121}$$

This is called the *Palatini equation*. Contracting μ and ρ, we then obtain the variation of the Ricci tensor as

$$\delta R_{\mu\nu}=\nabla_\rho\left(\delta\Gamma^\rho_{\mu\nu}\right)-\nabla_\nu\left(\delta\Gamma^\rho_{\mu\rho}\right)\ . \tag{3.122}$$

3.3.4 Tetrad Formalism (TBD)

• **good discussion in Carroll p.88, concise in Wald 3.4b, Weinberg 12.5**