5 Solar System Tests of General Relativity

5.1 Schwarzschild Metric

5.1.1 Spherically Symmetric and Static Solution

We will look for the simplest case, in which there exists a time-independent spherical source of mass M at the center. The metric tensor will be static and spherically symmetric. Due to time reversal symmetry, there is no cross term in the metric. Consider the metric in a spherical coordinate

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu =: -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \,, \tag{5.1}$$

and far away from the source we demand that it reduces to the Minkowski metric

$$\lim_{r\to\infty} \nu(r) = \lim_{r\to\infty} \lambda(r) = 0 \,. \tag{5.2}$$

Now we compute the non-vanishing Christoffel symbols:

$$\begin{gathered}\Gamma^t_{tr} = \frac{1}{2}\nu' \,, \qquad \Gamma^r_{tt} = \frac{1}{2}e^{\nu-\lambda}\nu' \,, \qquad \Gamma^r_{rr} = \frac{1}{2}\lambda' \,, \qquad \Gamma^r_{\theta\theta} = -re^{-\lambda} \,, \\ \Gamma^r_{\phi\phi} = -re^{-\lambda}\sin^2\theta \,, \qquad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \,, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \,, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta \,,\end{gathered} \tag{5.3}$$

where prime represents the derivative with respect to r. The non-vanishing Riemann tensors are then

$$\begin{gathered}R_{trtr} = \frac{1}{4}e^\nu\left(2\nu'' + \nu'^2 - \nu'\lambda'\right) \,, \qquad R_{t\theta t\theta} = \frac{1}{2}re^{\nu-\lambda}\nu' \,, \qquad R_{r\theta r\theta} = \frac{1}{2}r\lambda' \,, \\ R_{t\phi t\phi} = \frac{1}{2}re^{\nu-\lambda}\nu'\sin^2\theta \,, \qquad R_{r\phi r\phi} = \frac{1}{2}r\lambda'\sin^2\theta \,, \qquad R_{\theta\phi\theta\phi} = r^2\left(1-e^{-\lambda}\right)\sin^2\theta \,,\end{gathered} \tag{5.4}$$

and the non-vanishing Ricci tensors and Ricci scalar are

$$R_{tt} = \frac{1}{4}e^{\nu-\lambda}\left(2\nu'' - \nu'\lambda' + \nu'^2 + \frac{4}{r}\nu'\right) \,, \qquad R_{rr} = -\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\nu'\lambda' + \frac{\lambda'}{r} \,, \tag{5.5}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \,, \qquad R_{\theta\theta} = -\frac{1}{2}re^{-\lambda}\left(\nu' - \lambda'\right) - e^{-\lambda} + 1 \,, \tag{5.6}$$

$$R = -\frac{1}{2}e^{-\lambda}\left[2\nu'' + \nu'^2 - \nu'\lambda' + \frac{4}{r}\left(\nu' - \lambda' + \frac{1}{r}\right)\right] + \frac{2}{r^2} \,. \tag{5.7}$$

Finally, the Einstein tensors are

$$\begin{gathered}G^t_t = \frac{1}{r^2}e^{-\lambda}\left(1 - r\lambda'\right) - \frac{1}{r^2} \,, \qquad G^r_r = \frac{1}{r^2}e^{-\lambda}\left(r\nu' + 1\right) - \frac{1}{r^2} \,, \\ G^\theta_\theta = G^\phi_\phi = \frac{1}{4r}e^{-\lambda}\left(2\nu' - 2\lambda' + 2r\nu'' + r\nu'^2 - r\nu'\lambda'\right) \,.\end{gathered} \tag{5.8}$$

According to the Birkhoff theorem, any change in the mass distribution while keeping the spherical symmetry has no impact on the exterior, i.e., even radially pulsating source (spherically symmetric) maintains the spherically symmetric solution outside. While we adopted a time-independent metric as our ansatz, we could have started with time-dependence $\nu(t,r)$ and $\lambda(t,r)$ and derived from the vacuum field equation that they are indeed time-independent.

5.1.2 Schwarzschild Solution

Outside the spherical source $r \gg r_s$, there is no source of gravity, and the vacuum Einstein equation $G_{\mu\nu} = 0$ can be solved by the first two components as

$$\left(r\,e^{-\lambda}\right)' = 1 \,, \qquad 0 = \lambda' + \nu' \,, \tag{5.9}$$

and with the boundary condition we derive the solution

$$\therefore \quad \lambda = -\nu \,, \qquad\qquad e^{-\lambda} = e^{\nu} =: 1 - \frac{r_s}{r} \,, \tag{5.10}$$

where r_s here is just an integration constant. By taking the Newtonian limit:

$$g_{00} = \eta_{00} + h_{00} \,, \qquad\qquad h_{00} = -2\phi = \frac{r_s}{r} \,, \tag{5.11}$$

we can verify that the integration constant is indeed the *Schwarzschild radius*:

$$r_s := \frac{2GM}{c^2} = 3 \text{ km} \left(\frac{M}{M_\odot}\right) \,, \qquad\qquad \frac{r_s}{1 \text{ AU}} \approx 10^{-8} \,. \tag{5.12}$$

Hence the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{5.13}$$

and

$$\lambda' = -\nu' = -\frac{r_s}{r^2}\left(1 - \frac{r_s}{r}\right)^{-1} = -\frac{r_s/r}{r - r_s} \,. \tag{5.14}$$

The Schwarzschild solution has *four* singularities, two of which are just due to a spherical coordinate ($\theta = 0, \ \pi$) and hence removable. The singularity at $r = 0$ is physical:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \,, \tag{5.15}$$

while the singularity at $r = r_s$ is also removable by a change of coordinate. Consider a transformation to a new radial coordinate ρ

$$r =: \rho\left(1 + \frac{r_s}{4\rho}\right)^2 \,, \tag{5.16}$$

and the Schwarzschild metric becomes

$$ds^2 = -\frac{\left(1 - \frac{r_s}{4\rho}\right)^2}{\left(1 + \frac{r_s}{4\rho}\right)^2} dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right) \,. \tag{5.17}$$

The apparent singularity at $r = r_s$ is mapped into

$$\rho = \left(\frac{3}{4} \pm \frac{1}{\sqrt{2}}\right) r_s \,, \tag{5.18}$$

and the metric is regular at $r = r_s$.

• ***Killing vectors***.— A straightforward computation shows that there exist four Killing vectors in the Schwarzschild metric, i.e., one time-like

$$K^\mu \propto (1, 0) \,, \tag{5.19}$$

and three space-like

$$K^\mu \propto -(0, 0, \sin\phi, \cot\theta\cos\phi) \,, \qquad K^\mu \propto (0, 0, \cos\phi, -\cot\theta\sin\phi) \,, \qquad K^\mu \propto (0, 0, 0, 1) \,. \tag{5.20}$$

Three space-like Killing vectors represent SO(3) symmetry in 2D sphere.

5.1.3 Radial Geodesics in Schwarzschild Metric

• ***Radial null geodesic***.— Consider a null path x^μ_λ with affine parameter λ along the radial direction ($\dot\theta = \dot\phi = 0$), where dot here represents with respect affine parameter. The null condition imposes

$$\left(1-\frac{r_s}{r}\right)^2 \dot t^2 = \dot r^2 \,, \qquad \dot r := \frac{dr_\lambda}{d\lambda} \,, \qquad \dot t := \frac{dt_\lambda}{d\lambda} \,. \tag{5.21}$$

The geodesic equation can be solved for the time component

$$0 = \frac{d^2 t_\lambda}{d\lambda^2} + \Gamma^t_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \ddot t + 0 + 2\Gamma^t_{tr}\dot t\dot r + 0 = \ddot t + \nu'\dot t\dot r = e^{-\nu}\frac{d}{d\lambda}\left(e^\nu \dot t\right) \,, \tag{5.22}$$

and the solution can be readily obtained as

$$\left(1-\frac{r_s}{r}\right)\dot t =: \mathbb{C} \geq 0 \,, \tag{5.23}$$

where $\mathbb{C}$ is an integral constant and we chose λ with $\dot t > 0$. Mind that this constraint is valid for any geodesic in any motion. Finally, using the null condition, we obtain

$$\dot r^2 = \left(1-\frac{r_s}{r}\right)^2 \dot t^2 = \mathbb{C}^2 \,, \tag{5.24}$$

where the affine parameter for a null path is arbitrary, and hence $\mathbb{C}$ is undefined. For an incoming radial geodesic $\dot r < 0$, we derive

$$\frac{\dot t}{\dot r} = \frac{dt}{dr} = -\left(1-\frac{r_s}{r}\right)^{-1} = -\frac{r}{r-r_s} \qquad \rightarrow \qquad \therefore\; t = -\left(r + r_s\,\ln|r-r_s|\right) + \text{const.}\,. \tag{5.25}$$

The null geodesic makes 45° at far away $r \gg r_s$, but as it approaches r_s, the angle becomes steeper $|dt/dr| \gg 1$, such that it takes forever for an observer far away to see the light fall into black hole (infinite redshift), i.e., $r = r_s$ is the horizon. Note that at far away the metric becomes Minkowski, such that the coordinate can be identified as the observer rest frame at infinity. When a star collapses into a black hole, suppose an observer at the surface of the star keeps sending signals in a regular time interval. These signals arrive at another observer at infinity, but with increasingly longer interval. In fact, the intensity of these signals gets redshifted, such that the signals get dimmer as well.

In fact, the Schwarzschild solution we derived is also valid inside r_s, as long as $r > 0$. However, at $r < r_s$, a time-like and a space-like vectors switch their roles, as apparent in Eq. (5.13), i.e.,

$$T^\mu = (1,0,0,0) \,, \qquad ds^2 = g_{\mu\nu}T^\mu T^\mu < 0 \quad \text{at } r > r_s \,, \qquad ds^2 = g_{\mu\nu}T^\mu T^\mu > 0 \quad \text{at } r < r_s \,. \tag{5.26}$$

Hence, a particle trajectory inside r_s cannot be stable at a constant r (space-like trajectory), and it falls inevitably to the singularity at $r = 0$.

• ***Radial timelike geodesic***.— Now we consider an incoming particle on a time-like geodesic parametrized by the proper time τ:

$$-1 = -\left(1-\frac{r_s}{r}\right)\dot t^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot r^2 \,, \qquad \dot r := \frac{dr_\lambda}{d\tau} \,, \qquad \dot t := \frac{dt_\lambda}{d\tau} \,. \tag{5.27}$$

The time component of the geodesic equation is identical to that for a null geodesic, but at infinity we obtain $\mathbb{C} \equiv 1$, as the proper time runs exactly the same as the coordinate, under the assumption that the particle is at rest. Plugging this back to the normalization condition, we derive

$$\dot r^2 = \frac{r_s}{r} \qquad \rightarrow \qquad \tau - \tau_0 = \frac{2}{3\sqrt{r_s}}\left(r_0^{3/2} - r^{3/2}\right) \,, \qquad \dot r < 0 \,, \tag{5.28}$$

and the radial trajectory of a particle indicates that the falling particle will reach the singularity in a finite proper time, and of course no problem crossing the horizon at $r = r_s$.

5.1.4 Relativistic Binet Equation

Having understood the radial motion, we now study a general motion around a spherically symmetric mass, in which the symmetry of the system imposes that the particle motion should be confined in a plane ($\theta = \pi/2$) due to the angular momentum conservation. To verify this claim, we consider the geodesic equation for θ-component:

$$0 = \frac{d^2\theta_\lambda}{d\lambda^2} + \Gamma^\theta_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \ddot\theta + \frac{2}{r}\dot\theta\dot r - \sin\theta\cos\theta\,\dot\phi^2 \,, \qquad x^\mu_\lambda = (t, r, \theta, \phi)_\lambda \,, \tag{5.29}$$

and it admits a trivial solution:

$$\theta_\lambda \equiv \frac{\pi}{2} \,, \qquad \dot\theta_\lambda = 0 \,, \tag{5.30}$$

where λ is again the affine parameter. This is valid regardless of whether the geodesic is null or time-like. In the same way, the ϕ-component of the geodesic equation at $\theta = \pi/2$

$$0 = \frac{d^2\phi_\lambda}{d\lambda^2} + \Gamma^\phi_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \ddot\phi + \frac{2}{r}\dot r\dot\phi + 2\cot\theta\,\dot\theta\,\dot\phi = \frac{1}{r^2}\left(r^2\dot\phi\right)^{\cdot} + 0 \,, \tag{5.31}$$

yields the angular momentum conservation

$$r^2\frac{d\phi}{d\lambda} = \text{constant} =: L \,. \tag{5.32}$$

The only remaining component is the radial component of the geodesic equation:

$$0 = \frac{d^2 r_\lambda}{d\lambda^2} + \Gamma^r_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \ddot r + \frac{1}{2}e^{\nu-\lambda}\nu'\dot t^2 + \frac{1}{2}\lambda'\dot r^2 - re^{-\lambda}\dot\phi^2 = \frac{e^{-\lambda}}{2\dot r}\frac{d}{d\lambda}\left(e^\lambda\dot r^2 - e^{-\nu}\mathbb{C}^2 + \frac{L^2}{r^2}\right) \,, \tag{5.33}$$

where $\mathbb{C}$ from the time component of geodesic equation (5.23) replaces $\dot t$ and the first term in the round bracket accounts for the first and the third terms in the previous step. The solution to the radial component of geodesic equation is

$$\left(1 - \frac{r_s}{r}\right)^{-1}\left(\dot r^2 - \mathbb{C}^2\right) + \frac{L^2}{r^2} = \left\{\begin{array}{cc} 0 & \text{for } m = 0 \\ -1 & \text{for } m \neq 0 \end{array}\right\} \,, \tag{5.34}$$

which is indeed the same as the null condition or time-like condition. Since we consider a general motion, $\mathbb{C} \neq 1$ (particles can move at infinity).

We take a moment to look at the dynamical equation in Newtonian gravity (no horizon). For a mass spherically symmetric distribution, a test particle is subject to the same equation (5.32) for the angular momentum conservation in a planar motion. The dynamical equation is then

$$\ddot r - \frac{L^2}{r^3} = -\frac{r_s}{2r^2} \,, \tag{5.35}$$

where dot in this case is of course the derivative with respect to time and r_s simply represents $2GM$ in Newtonian gravity. It is customary to rewrite the equation in terms of $u := 1/r$ and express it as a function of ϕ:

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} \,. \tag{5.36}$$

This equation is called Binet's equation, and the solution is

$$u = \frac{r_s}{2L^2} + c_0\cos(\phi - \phi_0) \,, \qquad \text{or} \qquad \frac{R}{r} = 1 + e\,\cos(\phi - \phi_0) \,, \tag{5.37}$$

where c_0, ϕ_0 are integral constants and they are fixed in terms of semilatus rectum R and the eccentricity e:

$$R := \frac{2L^2}{r_s} \,, \qquad e := \frac{2c_0L^2}{r_s} \,. \tag{5.38}$$

The semilatus rectum is the distance from one focus perpendicular to the major axis, and the eccentricity is the ratio of focus to the semi-major axis (see the diagram). They are related to each other as

$$R = a|1 - e^2| \,. \tag{5.39}$$

In Newtoninan dynamics, the orbital parameters are further related to the energy E as

$$E := \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{r_s}{2r} = \frac{1}{2}L^2c_0^2 - \frac{r_s^2}{8L^2} , \qquad \therefore \quad e^2 = 1 + \frac{8EL^2}{r_s^2} . \tag{5.40}$$

For $E = 0$ or any motion with $L = 0$, the eccentricity is one (parabola), and the motion is plunge. For $E < 0$, the motion is an ellipse with $e = 0 \sim 1$, and for $E > 0$ the motion is a hyperbola with $e > 1$.

The constraint equation (5.34) for $\dot{r}$ in general relativity can also be arranged in terms of u:

$$L^2\left(\frac{du}{d\phi}\right)^2 - \mathbb{C}^2 + u^2L^2 - r_su^3L^2 = \left\{ \begin{array}{ll} 0 & \text{for } m = 0 \\ -1 + r_su & \text{for } m \neq 0 \end{array} \right\} , \tag{5.41}$$

where we used

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = -\frac{du}{d\phi}L , \qquad \frac{du}{d\phi} = \frac{\dot{u}}{\dot{\phi}} = \frac{\dot{u}}{Lu^2} . \tag{5.42}$$

By taking another derivative with respect to ϕ, we arrive at the relativistic version of Binet's equation:

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}r_su^2 = \left\{ \begin{array}{ll} 0 & \text{for } m = 0 \\ \frac{r_s}{2L^2} & \text{for } m \neq 0 \end{array} \right\} , \tag{5.43}$$

where the normalization $\mathbb{C}$ is removed. For a motion of test particles ($m \neq 0$), it is clear that there exists an extra relativistic contribution in proportion to r_su^2, which will become relevant only at small separation ($u \gg 1$). Any deviation from a Newtonian trajectory for a test particle can be computed by treating the relativistic contribution as a perturbation to the Newtonian orbit.

5.1.5 General Motion and Effective Potential

• ***Time-like motion***.— We analyze a particle motion ($m \neq 0$) in a Schwarzschild metric by re-arranging the constraint equation (5.34):

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left[\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - 1\right] = \frac{\mathbb{C}^2 - 1}{2} =: \mathcal{E} , \qquad \therefore \quad \frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \mathcal{E} , \tag{5.44}$$

where we defined the square bracket as the effective potential:

$$V_{\text{eff}} := \frac{1}{2}\left[\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - 1\right] = -\frac{r_s}{2r} + \frac{L^2}{2r^2} - \frac{r_sL^2}{2r^3} . \tag{5.45}$$

The constraint equation in relativistic dynamics is now cast in the usual energy equation in Newtonian dynamics, where $V_{\text{eff}} = 0$ at infinity and $\mathcal{E}$ is constant. As discussed, the effective potential has the ordinary Newtonian potential, the centrifugal force, and additional relativistic contribution.

At infinity the effective potential approaches Newtonian, and at $r = r_s$ it becomes $V_{\text{eff}} = -1/2$. For $L = 0$, the effective potential is simply Newtonian and attractive, and the radial motion was already discussed. For $L \neq 0$, the Newtonian potential is balanced by the centrifugal force at small separation, as L^2/r^2 becomes larger than r_s/r, and the barrier is infinite in Newtonian dynamics. However, the relativistic contribution r_sL^2/r^3 takes over at smaller separation, and the barrier is finite in relativistic dynamics.

First, we compute the minima of the effective potential:

$$\frac{dV_{\text{eff}}}{dr} = \frac{r_s}{2r^4}\left[\left(r - \frac{L^2}{r_s}\right)^2 + 3L^2 - \frac{L^4}{r_s^2}\right] , \tag{5.46}$$

such that if $L \leq \sqrt{3}r_s$, there exists no minimum, and the effective potential has no barrier (infall). In Newtonian dynamics, as long as $L > 0$ there exists a barrier with infinite height, and a particle turns around. With $L > \sqrt{3}r_s$, two minima exist

$$r_\pm := \frac{L^2}{r_s}\left[1 \pm \sqrt{1 - 3\left(\frac{r_s}{L}\right)^2}\right] , \tag{5.47}$$

the outer minimum r_+ is $r_+ \geq 3r_s$ and asymptotically increases with L^2, while the inner minimum is bounded by $r_-/r_s = 3/2 \sim 3$, decreasing to $r_- = \frac{3}{2}r_s$ with $L \to \infty$. As $L \to \sqrt{3}r_s$, both r_+ and r_- converge to $3r_s$, which is called the *inner most stable circular orbit* (ISCO). The effective potential at minima is

$$V_{\rm eff}(r_-) = \frac{(2r_s - r_-)L^2}{2r_-^3}\,, \qquad V_{\rm eff}(r_+) = -\frac{(r_+ - 2r_s)L^2}{2r_+^3} < 0\,. \tag{5.48}$$

The outer minimum is stable and always negative, while the inner minimum can be positive or negative, and $V_{\rm eff}(r_-) = 0$ at $L = 2r_s$.

With $\mathcal{E} < 0$, the orbits are bound, but not closed. For $V_{\rm eff}(r_-) > \mathcal{E} > 0$, the orbits are unbound, i.e., scattering. Those with $0 < V_{\rm eff}(r_-) < \mathcal{E}$ plunge into the center. The bound orbits are not closed, i.e., when a particle returns to $r_{\rm min}$ (or $r_{\rm max}$), the angular area swept by the orbit is larger than 2π, and the perihelion changes, known as the precession. Circular (bound) orbits ($\dot{r} = 0$) are possible at $r_\pm$, if $V_{\rm eff}(r_\pm) = \mathcal{E}$. If $\mathcal{E} > V_{\rm eff}$, $\dot{r} \neq 0$. So, the only possibilities are at $r_\pm$, but one at r_- is unstable. The smallest radius from the massive object for a circular orbit is therefore $r_+ = 3r_s$ (ISCO). In classical mechanics, any circular orbits of a test particle are allowed, as the orbital radius shrinks and the test particle moves faster, but in GR the particle can only move up to the speed of light, providing the minimum radius for the orbit. For Kerr black holes, the ISCO depends on the alignment of the particle's angular momentum and the black hole spin, but between $2r_s$ and $4.5r_s$. Hence, only BHs and neutrons stars have ISCO outside their surface.

• ***Null motion***.— For a trajectory of light, we again start by re-arranging the constraint equation (5.34):

$$\frac{1}{L^2}\,\dot{r}^2 + \frac{1}{r^2}\left(1 - \frac{r_s}{r}\right) = \frac{\mathbb{C}^2}{L^2} =: \mathcal{E}\,, \qquad \therefore\quad \frac{1}{L^2}\dot{r}^2 + W_{\rm eff} = \mathcal{E}\,, \tag{5.49}$$

where we defined $\mathcal{E}$ in a similar way and the effective potential is

$$W_{\rm eff}(r) := \frac{1}{r^2}\left(1 - \frac{r_s}{r}\right) > 0 \qquad \text{for } r > r_s\,. \tag{5.50}$$

Mind that dots now represent the derivative with respect to an affine parameter. While the affine parameter can be transformed, the products $Ld\lambda$ and $\mathbb{C}d\lambda$ remains unaffected, and hence $\mathcal{E}$ remains invariant under the affine transformation. In contrast to the time-like trajectory, the light path depends only on $\mathcal{E}$, not on L. A similar analysis can be performed for $W_{\rm eff}(r)$. The effective potential $W_{\rm eff} \to 0$ at infinity, and $W_{\rm eff}(r_s) = 0$. There exists one maximum:

$$\frac{dW_{\rm eff}}{dr} = \frac{3r_s - 2r}{r^4}\,, \qquad W_{\rm eff}\left(r = \frac{3}{2}r_s\right) = \frac{4}{27r_s^2} > 0\,, \tag{5.51}$$

such that if $\mathcal{E} < 4/27r_s^2$, the light is scattered away, while it plunges in the other case. Similar analysis can be made for light emitted between r_s and $3r_s/2$. A circular orbit is possible at $r = 3r_s/2$ with $dW_{\rm eff}/dr = 0$, but it is unstable.

The energy $\mathcal{E}$ for a null path is the inverse length squared, and the length scale corresponds to the impact parameter b at $r = \infty$:

$$\frac{1}{\sqrt{\mathcal{E}}} = \frac{L}{\mathbb{C}} = \frac{r^2\dot{\phi}}{(1 - \frac{r_s}{r})\dot{t}} \underset{r\to\infty}{\longrightarrow} r^2\frac{d\phi}{dt} = b\,, \qquad \because\ \lim_{r\to\infty} r\phi = b\,, \qquad \frac{dr}{dt} = -1\,. \tag{5.52}$$

At the closest approach $r_{\rm min}$ to the source, $\dot{r} = 0$, and hence the solution $r_{\rm min}$ can be found via

$$W_{\rm eff}(r_{\rm min}) = \frac{1}{b^2}\,. \tag{5.53}$$

5.2 Solar System Tests

5.2.1 Parametrized Post Newotnian (PPN)

Given the mass M and the gravitational constant G, the only dimensionfull quantity we can construct is the Schwarzschild radius in the metric in Eq. (5.1), and given $r_s/r \ll 1$, the dimensionless quantities ν and λ in the metric tensor can be expanded as

$$e^\nu =: 1 - \frac{r_s}{r} + \frac{1}{2}(\beta - \gamma)\left(\frac{r_s}{r}\right)^2 + \mathcal{O}(r_s/r)^3\,, \qquad e^\lambda =: 1 + \gamma\,\frac{r_s}{r} + \mathcal{O}(r_s/r)^2\,, \tag{5.54}$$

where the term linear in r_s is fixed due to the Newtonian limit. Both β and γ in the historic convention are two independent parameters, and they are part of the parametrized post-Newtonian (PPN) description. General relativity predicts

$$\beta = \gamma = 1 \, , \tag{5.55}$$

as in the Schwarzschild metric in Eq. (5.13), while Newtonian gravity $\beta = \gamma = 0$.

In terms of PPN parameters, several tests of gravity in the Solar system can be summarized as follows.

- The precession of the perihelion of Mercury per orbit yields

$$\delta\phi = \frac{2 + 2\gamma - \beta}{3} \frac{3\pi r_s}{a(1 - e^2)} \longrightarrow \frac{6\pi GM}{ac^2(1 - e^2)} \simeq 42.98''/100 \text{ yr} \, , \tag{5.56}$$

  where $a = 5.791 \times 10^{12}$ cm is the semi-major axis and $e = 0.2056$ is the eccentricity of Mercury, or semilatus rectum $R = a(1 - e^2)$. The Newtonian prediction is again smaller by $2/3$.

- The light deflection due to a mass M yields

$$\delta\phi = \left(\frac{1 + \gamma}{2}\right) \frac{2r_s}{b} \longrightarrow \frac{4GM}{bc^2} = 1.75'' \left(\frac{M}{1\, M_\odot}\right) \left(\frac{b}{1\, R_\odot}\right)^{-1} \, , \tag{5.57}$$

  where b is the impact parameter and the Newtonian prediction is a factor two smaller.

- Gravitational time delay of light signal is

$$\delta t = \left(\frac{1 + \gamma}{2}\right) 2r_s \left[\log\left(\frac{4r_e r_r}{r_c^2}\right) + 1\right] \longrightarrow \frac{4GM}{c^3} \left[\log\left(\frac{4r_e r_r}{r_c^2}\right) + 1\right] \, , \tag{5.58}$$

  where r_e, r_r are the distances of the emitter and the receiver ($r_c \ll r_e$, $r_c \ll r_r$), r_c is the distance to the sum at the closest approach, and the typical time delay is

$$\frac{4GM}{c^3} = 2 \times 10^{-5} \text{ sec} \left(\frac{M}{1\, M_\odot}\right) \, . \tag{5.59}$$

  The Newtonian prediction is a factor two smaller.

- Gravitational redshift has been measured with a great precision. However, it is in fact independent of the Einstein equation, and any metric theory of gravity will predict gravitational redshift.

5.2.2 Perihelion Precession of Mercury

Starting with the relativistic Binet equation (5.43), we treat the relativistic contribution as a perturbation to the Newtonian solution $\bar{u}$, and the perturbation equation is therefore,

$$\frac{d^2}{d\phi^2}\delta u + \delta u - \frac{3}{2}\, r_s \bar{u}^2 = 0 \, . \tag{5.60}$$

Given the Newtonian solution $\bar{u}$ in Eq. (5.37), the relativistic contribution is

$$\bar{u}^2 \propto 1 + \frac{1}{2}e^2 + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi \, , \tag{5.61}$$

and the solution is

$$\delta u = \frac{3r_s^3}{8L^4}\left(1 + \frac{1}{2}e^2 + e\phi\sin\phi - \frac{e^2}{6}\cos 2\phi\right) \, , \tag{5.62}$$

where we set $\phi_0 = 0$. Noting that the contribution in proportion to ϕ is the dominant one, the full solution is then

$$u = \bar{u} + \delta u \simeq \frac{r_s}{2L^2}\Big[1 + e\cos\phi(1 - \varepsilon)\Big] \, , \tag{5.63}$$

where we used $\cos\phi(1-\varepsilon) \simeq \cos\phi + \varepsilon\phi\sin\phi\cdots$ and defined

$$\varepsilon := \frac{3r_s^2}{4L^2} = \frac{3r_s}{2R} . \tag{5.64}$$

Therefore, the precession of the perihelion per orbit is then

$$\delta\phi = 2\pi\varepsilon = \frac{3\pi r_s}{R} = 0.104''/\text{orbit} , \qquad \delta\phi = 42.98''/100\text{ yr} , \tag{5.65}$$

where semilatus rectum is $R = 5.5\times 10^{12}$ cm and Mercury orbits 415 times per 100 years with $e = 0.2$.

The observed precession of Mercury is indeed

$$\delta\phi = 5599.74'' \pm 0.41''/\text{yr} , \tag{5.66}$$

about 100 times larger than the prediction in general relativity. In fact, there exist numerous Newtonian contributions that need to be taken into account. The observed value is from an Earth-based observatory, and the rotation axis of the Earth is precessing with respect to an inertial frame, the contribution of which is $5025.64'' \pm 0.50''/\text{yr}$ and is the largest among various Newtonian contributions. The next leading contributions arise from gravitational perturbations of other planet to the Sun's potential. The remaining value that is not explained by Newtonian gravity matches the prediction in general relativity.

Furthermore, the Sun is not exactly spherical, as it rotates, while our calculation assumes a perfect symmetry. Asymmetric deviation in gravitational potential is captured in terms of multipole expansion as

$$\Phi = -\frac{GM}{r} + J_2\frac{GM}{r}\left(\frac{R}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \cdots , \tag{5.67}$$

where J_2 is a dimensionless measure of the quadrupolar deviation. Given the $1/r^3$ dependence, it produces the same precession as the relativistic effect. A precise helioseismological measurement yields, however,

$$J_2 \sim 10^{-7} , \tag{5.68}$$

for the Sun, whose contribution to the precession is negligible.

5.2.3 Light Deflection: Gravitational Lensing of the Sun

Now we derive the trajectory of light propagation. Again, by treating the relativistic contribution as a perturbation,

$$\frac{d^2\bar{u}}{d\phi^2} + \bar{u} = 0 , \qquad \frac{d^2}{d\phi^2}\delta u + \delta u - \frac{3}{2}\, r_s\bar{u}^2 = 0 , \tag{5.69}$$

we first derive the background solution (i.e., no deflection)

$$\bar{u} = \frac{1}{b}\sin(\phi - \phi_0) , \qquad b = r\sin(\phi - \phi_0) , \tag{5.70}$$

which is simply a straight path with the incident angle ϕ_0, where ϕ is again the angle from the center and b is the impact parameter. For simplicity, we can set $\phi_0 = 0$. As the relativistic contribution in this case is $\bar{u}^2 \propto \sin^2\phi$, the general solution for the perturbation is

$$\delta u = \frac{r_s}{2b^2}\left(1 + c_1\cos\phi + c_2\sin\phi + \cos^2\phi\right) , \tag{5.71}$$

where c_1, c_2 are integral constants. The constants c_1 and c_2 can be fixed by imposing the boundary condition that the incoming light approaches the Sun from far away ($u\to 0$) with an angle $\phi_i = \pi + \varepsilon_i$ and it moves away from the Sun ($u \to 0$) with an an angle $\phi_o = -\varepsilon_o$:

$$0 = \frac{-\varepsilon_i}{b} + \frac{r_s}{2b^2}(2 - c_1) , \qquad 0 = \frac{-\varepsilon_o}{b} + \frac{r_s}{2b^2}(2 + c_1) , \tag{5.72}$$

where the contribution of c_2 is the second-order and hence ignored. The light deflection from a straight path is then

$$\delta\phi = \varepsilon_i + \varepsilon_o = \frac{2r_s}{b} = \frac{4GM}{b\,c^2} \,. \tag{5.73}$$

In Newtonian dynamics, massless particles like photons do not feel the gravity. However, a trick can be used to derive the light deflection angle in Newtonian dynamics. A massive particle moving with $v_{\rm rel}$ at infinity with impact parameter b gains a velocity kick from the source

$$\delta v = \frac{2GM}{b\,v_{\rm rel}} \,, \tag{5.74}$$

which is independent of the particle mass. Applying it to massless particles, the deflection angle in Newtonian dynamics is

$$\delta\phi = \frac{2GM}{b\,c^2} \,, \tag{5.75}$$

a factor two smaller than the prediction in general relativity.

5.2.4 Gravitational Time Delay

A spaceship Viking at r_V beyond the Sun sends a signal to the Earth at $r_\oplus$, and we send it back. Since the light path around the Sun is bent, the closed distance r_c of the light path to the Sun is not simply defined in terms of the triangular configuration. For the time delay, we start with

$$\frac{dt}{dr} = \pm\frac{1}{b}\left(1-\frac{r_s}{r}\right)^{-1}\left[\frac{1}{b^2} - W_{\rm eff}(r)\right]^{-1/2} \,, \qquad \mathcal{E} = \frac{\mathbb{C}^2}{L^2} = \frac{1}{b^2} \,, \tag{5.76}$$

where we used Eqs. (5.23) and (5.49) for $\dot t$ and $\dot r$. The $\pm$ sign arises, as r decreases towards the Sun and then increases past the Sun at the center. The total time elapsed for the light communication is

$$\Delta t_{\rm tot} = 2\int_{r_c}^{r_\oplus} dr\,\left|\frac{dt}{dr}\right| + 2\int_{r_c}^{r_V} dr\,\left|\frac{dt}{dr}\right| \,, \tag{5.77}$$

and the excess time delay for one round trip is $\Delta t_{\rm tot}$ minus the time it would take in Minkowski spacetime:

$$\delta t := \Delta t_{\rm tot} - 2\left(\sqrt{r_\oplus^2 - r_c^2} + \sqrt{r_V^2 - r_c^2}\right) \,. \tag{5.78}$$

The integral can be performed by using a perturbative method with expansion $r_c/r_\oplus \ll 1$ and $r_c/r_V \ll 1$ along the background trajectory, and we find

$$\delta t \simeq \frac{4GM}{c^3}\left[\log\left(\frac{4r_V r_\oplus}{r_c^2}\right) + 1\right] \,. \tag{5.79}$$

With two Viking missions from NASA in 1976, Irwin Shapiro performed a time-delay measurement in 1977. The landers at Mars send signals to the Earth and receive back. Depending on the positions of Mars and Earth, the light travel time changes, and furthermore the Sun's gravitational field increases the travel time (or time delay), depending on the position of the Sun in a triangular configuration. So, the goal is to measure the excess time delay at any given moment, rather than measure the total time elapsed. A typical round trip of signals takes 41 minutes, and the maximum time delay is 247 μsec$\sim 10^{-7}$. The gravitational time delay was confirmed at 1% level in 1977, and it is often called *Shapiro* time delay.

• ***Time Machines in relativity.***— Box 9.1 in Hartle (2003). Is time machine possible in real world? Yes, in a sense: *forward* time machine is possible. A twin paradox shows that if a twin accelerates enough, the twin comes back to the Earth, when the other twin at the Earth is old (or has already passed away). In a sense, the twin can experience the future of the other twin beyond age limits of ordinary human being.

As shown, the gravitational field causes time delay, and in the same sense a *forward* time machine. If an astronaut can circle around a black hole, the clock in the spaceship slows down significantly $(1 - r_s/r)^{-1/2}$, compared to the clock at the Earth at infinity. You can receive signals from the Earth beyond age limits of ordinary human being. The fuel cost to maintain the orbit would be, however, enormous. If you decided for a *no*-cost option, which is to jump into a black hole, you won't be able to return home, of course, and the duration of your forward time machine is as big as three hours in the largest black hole.

An interesting possibility is to come up with a shell of mass M, in which there is no gravity and you can build your home. However, outside the shell the gravitational field is exactly like a black hole, causing the time delay. The mass required to put in the shell is of course related to the Schwarzschild radius, and in fact there is no material that can stand the stress.

A *backward* time machine is in principle not allowed due to causality. However, in general relativity, there exist possibilities that a time-like curve can be connected to a distant past (closed spacetime geometry), or a warmhole connects one spacetime point to another. None at the moment is found in nature.