Energy - momentum tensor

RHS of Einstein eq.

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$ ⟷ perfect fluid (only ρ & p) — observer dependent!

in a homogeneous & isotropic univ. $U^\mu = \frac{1}{a}(1, 0, 0, 0)$ (not moving) $\quad U^\mu U_\mu = -1$ timelike

$\therefore T_{\mu\nu} = a^2 \begin{pmatrix} \rho & & & 0 \\ & p g_{11} & & \\ & & p g_{22} & \\ 0 & & & p g_{33} \end{pmatrix}$, $T^\mu{}_\nu = g^{\mu\rho} T_{\rho\nu} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ $\quad T^{tot}_{\mu\nu} = \sum_{i=1}^{N} T^{(i)}_{\mu\nu}$ each component

$T = T^\mu{}_\mu = -\rho + 3p$ $\quad$ with conformal time $\quad dt = a\, d\eta$

* Energy-momentum conservation $\quad T_{\mu\nu;\mu} = 0$ $\quad$ covariant derivative

$0 = T^\mu{}_{0;\mu} = \dot{\rho} + 3H(\rho + p)$ $\quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$

$\Gamma^0_{00} = \mathcal{H}$. $\Gamma^0_{\alpha\beta} = \mathcal{H} \tilde{g}_{ij}$. $\Gamma^\alpha_{\beta 0} = \mathcal{H} \delta^\alpha_\beta$

$\therefore \rho \propto a^{-3(1+\omega)}$ $\quad p \equiv \omega \cdot \rho$ $\quad$ Equation of state for perfect fluids

* Components

$\omega = 0$ dust, matter (pressureless) mde $\quad \rho \propto a^{-3}$ (energy density mc^2/V), $p = 0$ $\quad$ also baryons

$\omega = 1/3$ radiation, relativistic rde $\quad \rho \propto a^{-4}$, $p = 1/3\, \rho$ $\quad n_\gamma \sim 1/a^3$. $E_\gamma \sim \frac{1}{a}$ $\therefore \rho \sim 1/a^4$

$\omega = -1$ $\quad p = -\rho$ (negative pressure) vacuum energy (not empty) $\quad \rho$ = constant

$dQ = 0$ $\quad \therefore dU = \rho\, dV = -p\, dV$

Λ : Einstein's biggest blunder
static & infinite vs collapse
1917 before Hubble
$\Lambda \to p<0$ balance $\Rightarrow$ unstable

Friedmann - Lemaître equation

solution

* Einstein eq. : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, $G_{\mu\nu} \equi R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

↑ ~ integral constant, satisfying the Bianchi identity $g_{\mu\nu;\lambda} = 0$

metricity, length preservation

* Friedman equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$G^0_0 = 8\pi G T^0_0$ $\qquad$ $G^\alpha_\beta = 8\pi G T^\alpha_\beta$ & 1st. Fried.

$R_{00} = -3(\dot H + H^2)$ $\quad$ $R_{\alpha\beta} = a^2 g_{\alpha\beta}\left(\dot H + 3H^2 + \frac{2K}{a^2}\right)$

$R = 6\left(\dot H + 2H^2 + \frac{K}{a^2}\right)$ $\qquad$ $\frac{\ddot a}{a} = H^2 + \dot H$

* how fast the expansion? acceleration ($w < -1/3$) or deceleration ($w > -1/3$)

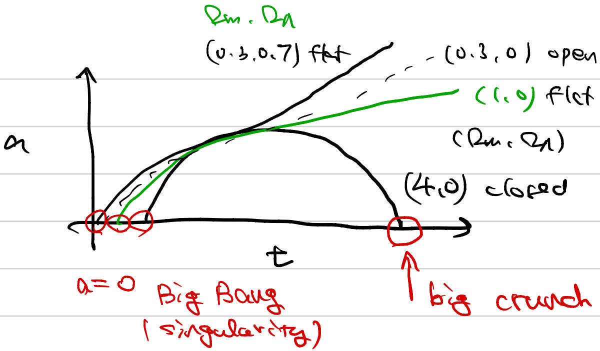
for ordinary compo. $\rho \geq 0$, $p \geq 0$ $\xrightarrow{K=0}$ $\ddot a < 0$. decelerating expansion
↑ gravity ↑ initial cond.

cosmological constant $w = -1$ → $\ddot a > 0$

* density parameter $\Omega_i \equiv \frac{8\pi G}{3H^2}\rho_i \equiv \rho_i / \rho_{crit}$ $\qquad \rho_{crit}^{-1} = \frac{8\pi G}{3H^2}$ $\qquad H_0 = 100\, h$ km/s/Mpc

$H^2 = H^2 \Omega_{tot} - \frac{K}{a^2} = H^2(\Omega_{tot} + \Omega_K)$, $\Omega_K \equiv -\frac{K}{a^2H^2}$, $\Omega_{tot} + \Omega_K = 1$.

$\Omega_{tot} \ni \Omega_{dm}, \Omega_b, \Omega_{rad}, \Omega_\Lambda, \cdots$ $\qquad$ ↖ cannot be put in $T_{\mu\nu}$



$\Omega_{tot} + \Omega_K = 1$, $H^2 = H^2 \Omega_{tot} - K/a^2$ $\quad \Omega_K \sim 0$

today : $\Omega_m \sim 0.3$. $\Omega_\Lambda \sim 0.7 \rightarrow$ acceleration (dark energy) $\quad \Omega_r \sim 10^{-6}$

20 years ago : $\Omega_m \sim 0.3 - 1.0$. $\Omega_r \sim 10^{-6}$. $\Omega_K \sim \Omega_\Lambda \sim 0$

deceleration parameter

$K > 0$ closed $\rightarrow$ $\dot{a} = 0$ possible

$\therefore \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-n}$, $H = H_0 \left(\frac{t_0}{t}\right) = \frac{2}{nt}$

Solutions to Friedman equation

$K = 0$, $\rho \propto a^{-n}$, $n = 3(1+w)$, $H = \frac{\dot{a}}{a} = \frac{d\ln a}{dt}$ $\quad \therefore \sqrt{\frac{8\pi G}{3}\rho_0}\, dt = \left(\frac{a}{a_0}\right)^{n/2} d\ln a$

$\therefore a \sim t^{2/n} \sim \eta^{2/(2-n)}$, $H = H_0 (t_0/t) = \frac{2}{nt}$, $H = 2/3t$ for mde $\quad H = 1/2t$ for rde

de Sitter solution (empty univ.)

$\Lambda > 0$ $\quad K = 0$, $a \sim \exp\left[\pm \sqrt{\frac{\Lambda}{3}}\, t\right]$

how can we tell which one describes our univ?