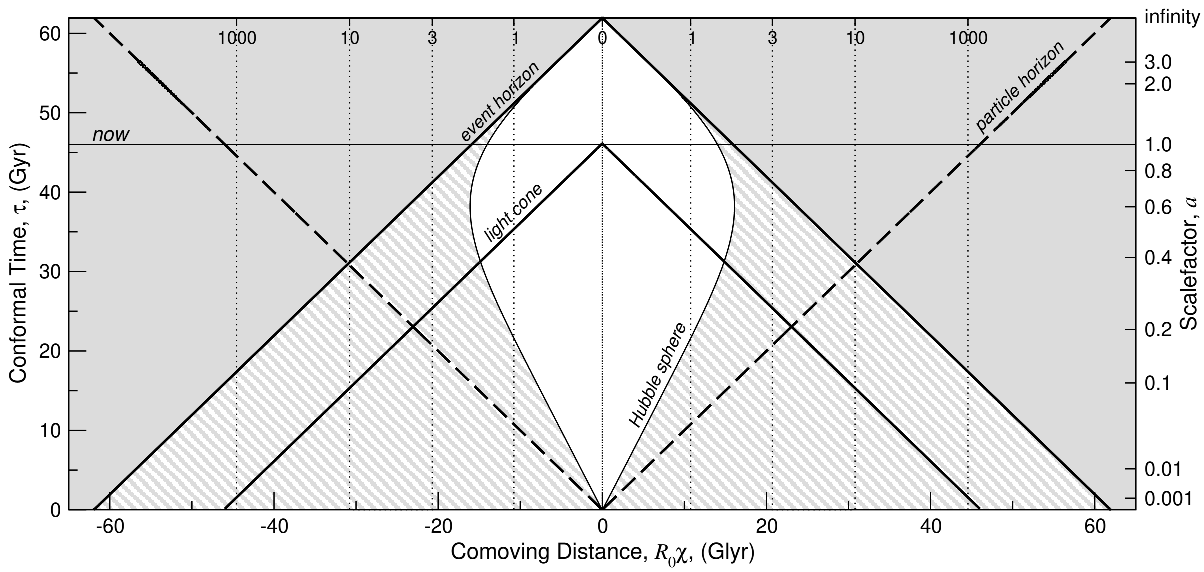

Figure 5.1: Horizons in terms of comoving distance and conformal time. The event and the particle horizon are shown as solid and dashed curves for the observer on the curve at any given time. Figure is taken from Davis and Lineweaver (2004).

5 Standard Inflationary Models

Standard single field inflationary models provide a mechanism for the inflationary expansion (horizon problem) and the perturbation generation (initial condition) by a single scalar field, called inflaton. The scalar field Lagrangian has the canonical kinetic term, but various single field models differ in the scalar field potential, according to which the inflaton rolls over. In most cases, the slow-roll condition is adopted, such that the scalar field dynamics is insensitive to the details of the scalar field potential.

The outcome of the standard model predictions is as follows: The curvature fluctuations are scale-invariant ($n_s \simeq 1$) and highly Gaussian. The tensor fluctuations are also scale-invariant, but its amplitude is very small compared to the scalar fluctuations. The running of the indices is very small. Recent observations confirm these predictions and constrain the parameters with high precision. However, beyond these basic features/constraints, we do not have a solid model for inflation. Note that the energy scale of inflation is beyond the validity of the standard model physics, and most inflationary models have many theoretical issues, when quantum corrections are considered.

5.1 Problems in the Standard Big Bang Model

5.1.1 Horizon Problem

The comoving radial distance between two points on the light cone can be computed by using $ds^2 = 0$ as

$$\chi_f - \chi_i = \eta_f - \eta_i = \int_{t_i}^{t_f} \frac{dt}{a} = \int_{a_i}^{a_f} \frac{da}{a^2 H} = \int_{z_f}^{z_i} \frac{dz}{H} , \tag{5.1}$$

where the subscripts i and f represent the initial and the final points for light emission and reception. When $a_i = t_i = 0$ or $z_i = \infty$ (beginning of the Universe), the comoving distance $\chi_f - \chi_i$ represents the particle horizon at $t = t_f$ for an observer (or the receiver) at χ_f, or the maximum comoving distance any one can receive signals from the beginning of the Universe up to t_f. For the infinite future ($t_f \to \infty$), the comoving distance $\chi_f - \chi_i$ represents the event horizon for an observer (or the sender) at t_i to communicate with an observer at χ_f and $t_f = \infty$, or the maximum comoving distance we can send signals at any given time until the infinite future.

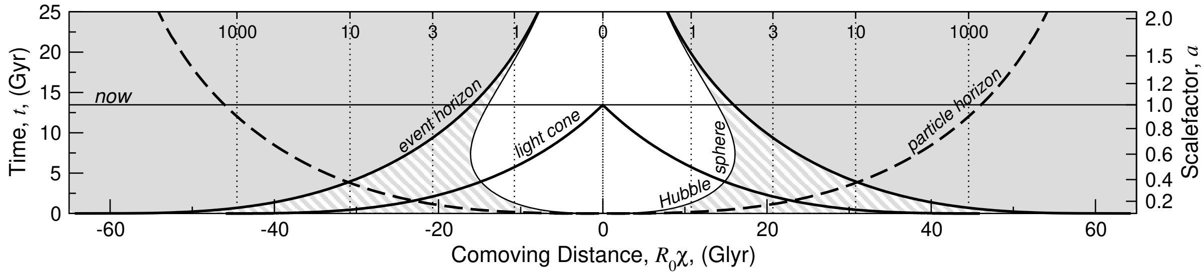

Figure 5.2: Same as in Fig. 5.1, but in terms of proper time. Note that the proper time is *not* bounded, such that the plot was simply cut at 25 Gyr.

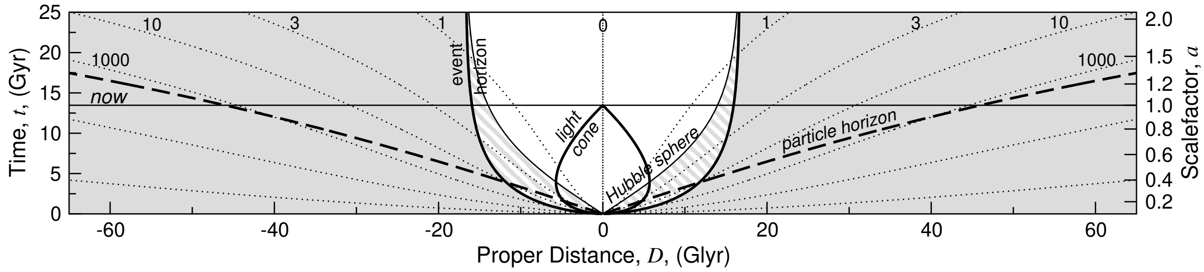

Figure 5.3: Same as in Fig. 5.1, but in terms of proper time and proper distance. Note that the proper time is *not* bounded, such that the plot was simply cut at 25 Gyr.

For an energy component

$$\bar{\rho} \propto a^{-n} , \qquad n = 3(1+w) , \qquad \frac{1}{a^2 H} \propto a^{-\frac{1}{2}+\frac{3}{2}w} , \tag{5.2}$$

the existence of the particle horizon is related to the convergence around $a = 0$ and the finite age of the Universe, which requires

$$\lim_{a_i \to 0} \frac{da}{a^2 H} = \text{finite} , \qquad w > -\frac{1}{3} . \tag{5.3}$$

Hence, in the standard phase, e.g., MDE ($w = 0$), RDE ($w = 1/3$), the particle horizon exists. Similarly, the existence of the event horizon is related to the convergence around $a = \infty$ (or $t = \infty$), which requires the opposite condition $w < -1/3$. Hence, there exist no event horizons for MDE or RDE. In contrast, for an exponential expansion $a = \exp(Ht)$ ($w = -1$) in Eq. (5.35), the situation is opposite to the ordinary phase with a power-law expansion:

$$\chi_f - \chi_i = -\frac{1}{H}\left(e^{-Ht_f} - e^{-Ht_i}\right) = \frac{1}{a_f H}\left(e^{H\Delta t} - 1\right) , \qquad t \in (-\infty, \infty) , \tag{5.4}$$

where H is a constant and a is normalized at $t = 0$. There exists an event horizon (convergence at $t_f = \infty$), but there is no particle horizon (no convergence at $a_i = 0$, $t_i = -\infty$). Note that in this de Sitter phase, the conformal time $\eta \in (-\infty, 0)$ is unbounded in the past, while bounded in the future.

Here we assume a fiducial cosmological ΛCDM model with $(\Omega_m, \Omega_\Lambda, h) = (0.3, 0.7, 0.7)$ to present spacetime diagrams from Davis and Lineweaver (2004). Figure 5.1 shows the light cone and the horizons in terms of comoving distance and conformal time. Note that in a Λ-dominated universe, the conformal time is finite in the infinite future:

$$dt = \frac{d\ln a}{H} , \qquad d\eta = \frac{d\ln a}{aH} , \qquad \lim_{a\to\infty} a^2 H = a^2 H_0 \sqrt{\Omega_\Lambda} . \tag{5.5}$$

The value $\eta \simeq 62$ Gyr on the y-axis in Figure 5.1 is the maximum conformal time, representing the infinite future, which is in turn related to the event horizon. Note that the light cone in this plot is 45°, as $a d\chi = dt = a d\eta$. The dashed line in Figure 5.1 shows the particle horizon for the observer at $\chi \simeq 46$ Gly, which corresponds to the light path emitted at

$\eta = \chi = 0$. The event horizon today is shorter $\chi_e \simeq 18\ h^{-1}$Mpc, who can send the signal today, arriving at $\chi = 0$ in the infinite future ($\eta \simeq 62$ Gyr).

The vertical dotted lines show the comoving observers at given comoving coordinates, and the labels on top shows the redshift the observer today at $\chi = 0$ can see. For example, the first vertical dotted line at $\chi \simeq 10\ h^{-1}$Mpc shows that this observer can be seen at $z = 1$. Today's event horizon is $\chi_e \simeq 18\ h\text{Mpc}^{-1}$, and the observer at this distance can be seen around $z \simeq 2$, but today we cannot send the signal to this observer in the future. Thin solid curves in Figure 5.1 show the Hubble sphere, or the physical distance, but in terms of the comoving coordinate r_H:

$$d_H := \frac{c}{H(z)} =: a r_H \ , \tag{5.6}$$

so that any objects (in the same-time hypersurface) beyond this Hubble sphere move away at speed faster than the speed of light, according to the Hubble law. The Hubble parameter was larger at the beginning of the Universe, decelerating in time, but after Λ-domination, the Hubble parameter is constant, and r_H is decreasing.

In the standard model, the analytic solution was derived for the scale factor $a(t) \propto t^{2/n}$ with $n = 3(1+w)$ in Eq. (4.149), and the integral in Eq. (5.1) becomes

$$\chi_f - \chi_i = \left.\frac{3(1+w)t}{(1+3w)a}\right|_i^f \ , \qquad t \in (0, \infty) \ . \tag{5.7}$$

At the last scattering of CMB photons ($z \simeq 1100$, or $t \simeq 300,000$ yrs), the particle horizon and the comoving angular diameter distance are

$$\chi_p \simeq \frac{3 \cdot 300,000 \text{ yrs}}{1/1100} \simeq 330 \text{ Mpc} \ , \qquad \chi(z = 1100) \simeq 3 \cdot 13.8 \text{ Gyr} \ , \tag{5.8}$$

where we used the age of the Universe is around 13.8 Gyr and the equation of state is $w = 0$. Hence, the horizon at the time of last scattering is subtended today by

$$\theta \simeq \frac{330 \text{ Mpc}}{3 \cdot 13.8 \text{ Gyr}} \approx 0.024 \text{ rad} = 1.37 \text{ deg} \ . \tag{5.9}$$

A more precise calculation yields about 1.8 degrees in the sky. So, any patches that are separated by about two degrees (four times the angular size of the Moon) were outside the causal contact by the time of the last scattering. Nevertheless, CMB sky is incredibly uniform in temperature with only 10^{-5} level fluctuations.

If there was a period (inflationary period), in which the (comoving) horizon $1/\mathcal{H}$ decreases in time, instead of increasing in the standard model:

$$d_H \sim \frac{1}{H} \ , \qquad \lambda_H \sim \frac{1}{\mathcal{H}} \ , \qquad \frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \ , \tag{5.10}$$

and if such period was sufficiently long, the scales we observe today would have been in the causal contact (or inside the horizon) before the standard expansion phase began. Note that this condition is not satisfied in MDE or RDE. However, during the inflationary period of an exponential expansion in Eq. (5.4), the Hubble parameter is constant, and this condition is certainly satisfied. To solve the horizon problem, we demand

$$\frac{e^N}{a_f H} \gg 330 \text{ Mpc} \ , \qquad \therefore \quad N = \ln\left(\frac{a_f}{a_i}\right) = H\Delta t \gg \ln\left(330 \text{ Mpc} \times a_f H\right) \simeq 54 \ , \tag{5.11}$$

where we ignored unity in Eq. (5.4) and used the entropy conservation ($a^3 g_s T^3$) to obtain

$$aH = 1.7 \times 10^{21} \text{ Mpc}^{-1} \left(\frac{g_{*,s}}{106.75}\right)^{-1/3} \left(\frac{g_*}{106.75}\right)^{1/2} \left(\frac{T}{10^{14} \text{ GeV}}\right) \ . \tag{5.12}$$

The Hubble during the inflation was assumed to be $T = 10^{14}$ GeV. Angular separation of 180 degrees would add $\ln(180) = 5.19$, and the total number of e-folding should be (much) larger than 60:

$$N \geq 60 \ , \qquad \frac{a_f}{a_i} \geq e^{60} = 1.14 \times 10^{26} \ . \tag{5.13}$$

The duration for $N = 60$ e-folding at this energy scale is

$$H = 1.4 \times 10^{10} \text{ GeV} \left(\frac{g_*}{106.75}\right)^{1/2} \left(\frac{T}{10^{14} \text{ GeV}}\right)^2 \ , \qquad \Delta t = \frac{60}{H} = 2.5 \times 10^{-32} \text{ sec} \ . \tag{5.14}$$

5.1.2 Flatness Problem

The total energy density today is close to the critical density

$$\rho_c := \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-29}\ h^2\ \mathrm{g\ cm^{-3}} = 1.123 \times 10^{-5}\ h^2\ m_p\ \mathrm{cm^{-3}}\ , \tag{5.15}$$

i.e., the curvature radius $R_K^2 = 1/K$ is currently constrained to be much larger than the horizon. Let's assume that the spatial curvature is non-zero, and we show that this current value requires an extreme fine-tuning in the standard model. The Friedmann equation can be re-arranged as

$$H^2 = H^2\Omega_t - \frac{K}{a^2}\ , \qquad\qquad \Omega_t^{-1} = 1 - \frac{3K}{8\pi G\rho a^2}\ , \tag{5.16}$$

where we divided both hand sides by $H^2\Omega_t = 8\pi G\rho/3$. Note that the total density parameter Ω_t does not include the spatial curvature component. Using this relation, we obtain the ratio in terms of scale factor

$$\frac{(\Omega_t^{-1}-1)_i}{(\Omega_t^{-1}-1)_0} = \frac{(\rho a^2)_0}{(\rho a^2)_i} = \frac{(\rho a^2)_0}{(\rho a^2)_{\mathrm{EQ}}}\frac{(\rho a^2)_{\mathrm{EQ}}}{(\rho a^2)_i} \simeq \frac{a_{\mathrm{EQ}}}{a_0}\left(\frac{a_i}{a_{\mathrm{EQ}}}\right)^2\ , \tag{5.17}$$

where we split the period into MDE and RDE and approximated an instantaneous transition. The equality epoch is $z_{\mathrm{EQ}} = 3400$, and $T_{\mathrm{EQ}} = T_\gamma(1+z_{\mathrm{EQ}}) = 0.8$ eV. Hence the ratio is

$$\frac{(\Omega_t^{-1}-1)_i}{(\Omega_t^{-1}-1)_0} = 2.1\times 10^{-51}\left(\frac{g_{*,s}}{106.75}\right)^{-2/3}\left(\frac{T}{10^{14}\ \mathrm{GeV}}\right)^{-2}\ . \tag{5.18}$$

The density parameter today is about $\Omega_t \simeq 1.0 \pm 0.02$, such that $|\Omega_t^{-1} - 1| \leq 0.1$. To reach the current value of Ω_t, its value at the initial condition has to be extremely close to unity, but not unity, which requires 50-digit fine-tuning (60 digits if $T_i \sim M_{\mathrm{pl}}$). Put it differently, if Ω_t at the initial epoch a_i was 0.99 (no fine tuning), the energy density would have been dominated by curvature already at $a = 10\ a_i$, which would ruin the standard RDE and MDE evolution.

An inflationary expansion can mitigate this fine-tuning issue, because

$$\frac{(\Omega_t^{-1}-1)_i}{(\Omega_t^{-1}-1)_0} = \frac{(\rho a^2)_0}{(\rho a^2)_{\mathrm{EQ}}}\frac{(\rho a^2)_{\mathrm{EQ}}}{(\rho a^2)_f}\frac{(\rho a^2)_f}{(\rho a^2)_i} \simeq \frac{a_{\mathrm{EQ}}}{a_0}\left(\frac{a_f}{a_{\mathrm{EQ}}}\right)^2 \times \left(\frac{a_f}{a_i}\right)^2\ , \tag{5.19}$$

the extra factor $(a_f/a_i)^2 = e^{2N} \geq 10^{52}$ from the inflationary period eliminates the need for fine tuning, where the subscript f represents the end of inflation and the beginning of RDE. If the spatial curvature is zero ($K = 0$), it is zero all the time. This also requires an extreme fine-tuning, because the amount of matter at a given epoch has to be exactly the critical density. So, some non-zero curvature in an inflationary model is most natural.

Given the energy density, the solution to the Friedmann equation needs boundary conditions, exactly the same way that given $T_{\mu\nu}$, the differential equation $G_{\mu\nu}$ needs to be solved with the boundary conditions. The boundary conditions for a homogeneous and isotropic universe can be the Hubble parameter $H_\star$ at some epoch $a_\star$ or the spatial curvature K. Our presentation of the flatness problem assumes that the boundary condition is given by $H_\star$ or H_0 today. Instead, the value of K can be given, and the Hubble parameter is determined by the amount of matter. In this case, if $K \neq 0$, it has to have the value consistent with the current observation, which requires an extremely fine-tuned initial condition. So, either $K = 0$ or $K \neq 0$, we are faced with the question of why the initial condition was given like that. An inflationary period can mitigate this challenge.

5.1.3 Primordial Monopole Problem

According to most grand unification theory (GUT), phase transitions that took place in the early Universe produce magnetic monopole that are much heavier than protons by $\approx 10^{16}$ with abundances that are at least one per horizon at the time. Its energy density today can then be estimated as

$$\rho_{\mathrm{mono}} \sim \left(\frac{m}{L^3}\right)_{\mathrm{mono}}\left(\frac{a_{\mathrm{mono}}}{a_0}\right)^3 \simeq \left(\frac{T_{\mathrm{GUT}}}{10^{11}\ \mathrm{GeV}}\right)^4 \rho_\gamma\ , \tag{5.20}$$

such that $\Omega_{\rm mono} \gg 1$ if $T_{\rm GUT} \gg 10^{12}$ GeV, where we used $L \sim t \sim 1/H \sim 1/T^2$ and $T_{\rm GUT}$ is the temperature at the monopole formation. Those monopoles need to be removed to be consistent with the current observations.

An inflationary period after the monopole formation would dilute the number density of the monopoles by

$$\left(\frac{a_i}{a_f}\right)^3 = e^{-3N} \leq 6.7 \times 10^{-79} \,. \tag{5.21}$$

Note that the monopole formation has to be before the inflationary period.

5.2 Single Scalar Field

5.2.1 Scalar Field Action

In addition to the Einstein-Hilbert action for gravity, we consider the action for a scalar field with canonical kinetic term and the potential V:

$$S = \int \sqrt{-g}\, d^4x \left[\frac{c^4}{16\pi G}\, R - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi)\right] \,, \tag{5.22}$$

where the kinetic term in the Minkowski spacetime reduces to the standard form

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi = \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2\right] \,. \tag{5.23}$$

The Euler-Lagrange equation yields the equation of motion for the scalar field

$$\Box\phi - V_{,\phi} = 0 \,, \qquad \Box := g^{\mu\nu}\nabla_\mu\nabla_\nu \,, \tag{5.24}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\,\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\,\phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu} \,. \tag{5.25}$$

The equation of motion in the Minkowski spacetime reduces to the usual:

$$\ddot{\phi} - \nabla^2\phi + V_{,\phi} = 0 \,. \tag{5.26}$$

It is often in literature that the Planck unit is adopted, and there exist two different conventions:

$$M_{\rm pl}^2 := \frac{1}{8\pi G} = \left(2.44 \cdot 10^{18}\ {\rm GeV}\right)^2 \,, \qquad m_{\rm pl}^2 := \frac{1}{G} = \left(1.22 \cdot 10^{19}\ {\rm GeV}\right)^2 \,. \tag{5.27}$$

The recent analysis of the Planck CMB mission yields that the scalar fluctuation amplitude $A_s \simeq 2.1 \times 10^{-9}$ and hence the energy scale of the inflation is

$$A_s = \frac{H^2}{8\pi^2\varepsilon M_{\rm pl}^2} = 2.1 \times 10^{-9} \,, \qquad \therefore\ \ H = 4.1 \times 10^{-4}\sqrt{\varepsilon}M_{\rm pl} = 1.0 \times 10^{14}\ {\rm GeV}\sqrt{\frac{\varepsilon}{0.01}} \,. \tag{5.28}$$

5.2.2 Background Relation and Evolution Equations

In the background, the non-vanishing fluid quantities for a scalar field are the energy density and the pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \,, \tag{5.29}$$

and accounting for the covariant derivative the equation of motion can be obtained as

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \,, \qquad \phi'' + 2\mathcal{H}\phi' + a^2V_{,\phi} = 0 \,. \tag{5.30}$$

The conservation equation of a scalar field

$$0=\dot\rho_\phi+3H(\rho_\phi+p_\phi)=\dot\rho_\phi+3H\dot\phi^2\,,\qquad\qquad \therefore\quad \frac{d\ln\rho_\phi}{d\ln a}=-3\,\frac{\dot\phi^2}{\rho_\phi}=-6\left(1-\frac{V}{\rho_\phi}\right)\,,\tag{5.31}$$

and it can be integrated to yield

$$\rho_\phi(a)=\rho_\phi(a_0)\ \exp\left\{-\int_{a_0}^a\frac{da}{a}\,6\left[1-\frac{V(\phi)}{\rho_\phi(a)}\right]\right\}\,.\tag{5.32}$$

With the dependence on ρ_ϕ in the integral, it is not an exact solution, but it is evident that $\rho_\phi\propto a^{-n}$ with $0\leq n\leq 6$, and we obtain

$$\dot\phi^2=\frac{n}{3}\,\rho_\phi=\rho_\phi+p_\phi\,,\qquad\qquad p_\phi=\left(\frac{1}{3}n-1\right)\rho_\phi\,.\tag{5.33}$$

The Friedmann equation for a scalar field is

$$H^2=\frac{\rho_\phi}{3M_{\rm pl}^2}\,,\qquad \dot H=-\frac{\rho_\phi+p_\phi}{2M_{\rm pl}^2}=-\frac{\dot\phi^2}{2M_{\rm pl}^2}\,,\qquad H^2+\dot H=\frac{\ddot a}{a}=\frac{1}{3M_{\rm pl}^2}\left(V-\dot\phi^2\right)\,,\tag{5.34}$$

where we assumed a flat universe and no cosmological constant. If the potential energy of the scalar field is the dominant energy component of the Universe or the kinetic energy is smaller than the potential energy (slow-roll), the expansion of the Universe is accelerating $\ddot a>0$. Various inflationary models with slow-roll condition state that the potential is sufficiently flat, such that $V(\phi)$ is nearly constant during the inflationary period and ϕ slowly evolves (rolls over V).

5.2.3 de-Sitter Spacetime

The de-Sitter universe is a highly symmetric spacetime, defined as a background FRW universe with no matter and constant Hubble parameter. A constant Hubble parameter leads to an exponential expansion, and we parametrize the de-Sitter solution as

$$H^2:=\frac{\Lambda}{3}\,,\qquad a(t)=e^{Ht}=-\frac{1}{H\eta}\,,\qquad a=(0,\infty)\,,\qquad t=(-\infty,\infty)\,,\qquad \eta=(-\infty,0)\,,\tag{5.35}$$

where the scale factor is normalized at $t=0$. The slow-roll parameter for the de-Sitter spacetime is

$$\varepsilon:=-\frac{\dot H}{H^2}=\frac{d}{dt}\left(\frac{1}{H}\right)=0\,.\tag{5.36}$$

5.2.4 Slow-Roll Parameters

In general, inflationary models slightly deviate from the de-Sitter phase ($\varepsilon\neq 0$), and its deviation is captured by the slow-roll parameter:

$$\varepsilon=\frac{d}{dt}\left(\frac{1}{H}\right)=-\frac{\dot H}{H^2}\,,\qquad \dot H=-H^2\varepsilon\,,\qquad \frac{\ddot a}{a}=H^2+\dot H=H^2(1-\varepsilon)\,,\tag{5.37}$$

To solve the horizon problem, we know that the comoving horizon has to decrease in time

$$0>\frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right)=-\frac{\ddot a}{a^2H^2}=-\frac{1-\varepsilon}{a}\,.\tag{5.38}$$

The background evolution of a scalar field can be re-phrased in terms of the slow-roll parameters as

$$\varepsilon=\frac{1}{2}\frac{\dot\phi^2}{H^2M_{\rm pl}^2}=\frac{3}{2}(1+w)\,,\qquad\qquad \dot\phi^2=\rho_\phi+p_\phi=2\varepsilon H^2M_{\rm pl}^2\,.\tag{5.39}$$

Splitting $\rho_\phi=K+V$, the ratio of kinetic to potential energy is obtained as

$$H^2=\frac{\rho_\phi}{3M_{\rm pl}^2}=\frac{K+V}{3M_{\rm pl}^2}\,,\qquad\qquad \frac{K}{V}:=\frac{\dot\phi^2}{2V}=\frac{\varepsilon H^2M_{\rm pl}^2}{V}=\varepsilon\,\frac{K+V}{V}\,,\tag{5.40}$$

such that we obtain

$$\therefore \quad \frac{K}{V} = \frac{\varepsilon}{3-\varepsilon} \,, \qquad H^2 = \frac{V}{(3-\varepsilon)M_{\rm pl}^2} \,. \tag{5.41}$$

When the inflation ends ($\varepsilon \simeq 1$), the ratio becomes 1/2. If we ignore the second derivative of the field ($\ddot\phi \simeq 0$) in the equation of motion,

$$3H\dot\phi \simeq -V_{,\phi} \,, \qquad \rho_\phi + p_\phi \simeq \left(\frac{V_{,\phi}}{3H}\right)^2 \,, \tag{5.42}$$

the slow-roll parameters are then further related to the slow-roll parameters defined in terms of the derivatives of the potential also used below)

$$\varepsilon_V := \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \simeq \varepsilon \,, \qquad \eta_V := M_{\rm pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) \simeq \varepsilon + \eta \,, \qquad \xi_V := \frac{M_{\rm pl}^4 V_{,\phi}V_{,\phi\phi\phi}}{V^2} \,, \tag{5.43}$$

where we used the second slow-roll parameter

$$\eta := -\frac{\ddot\phi}{H\dot\phi} \,. \tag{5.44}$$

In fact, one can show the exact relation

$$\varepsilon = \varepsilon_V\left(1 - \frac{4}{3}\varepsilon_V + \frac{2}{3}\eta_V\right) \,, \qquad \varepsilon_V = \varepsilon_1\left(\frac{3-\eta}{3-\varepsilon_1}\right)^2 \,. \tag{5.45}$$

In literature, different convention for slow-roll parameters are often used, in particular, in terms of Hubble flow as

$$\varepsilon_1 := \varepsilon \,, \qquad \varepsilon_2 := \frac{1}{H}\frac{d\ln\varepsilon}{dt} = -\frac{2\dot H}{H^2} + \frac{\ddot H}{H\dot H} = 2(\varepsilon - \eta) \,, \qquad \varepsilon_{i+1} := \frac{1}{H}\frac{d\ln\varepsilon_i}{dt} = \frac{\dot\varepsilon_i}{\varepsilon_i H} \,. \tag{5.46}$$

The last equation is the definition (sometimes $d\ln|\varepsilon_i|/Hdt$ is used, but it is essentially $\dot\varepsilon_i/\varepsilon_i H$). Furthermore, the inflation has to last for some time, such that the modes we measure in CMB have to expand at least by $40-60$ e-folds. So it is convenient to define the number of e-folding for a given mode as the number of e-folds the mode k expanded from the horizon crossing until the end of inflation,[1]

$$N(\phi_k) := \ln\frac{a_{\rm end}}{a(\phi_k)} = \int_{t_k}^{t_{\rm end}} H\,dt \,, \qquad k = aH \,, \tag{5.47}$$

where t_k is the time the k-mode crosses the horizon. Using the e-folding number, we can express the slow-roll parameters as

$$dN = -Hdt = -d\ln a \,, \qquad \varepsilon = \frac{d\ln H}{dN} \,, \qquad \varepsilon_{i+1} = -\frac{d\ln\varepsilon_i}{dN} \,. \tag{5.48}$$

Note that the other definition of e-folds is

$$\hat N(\phi_k) := \ln\frac{a(\phi_k)}{a(\phi_i)} = \int_{t_i}^{t_k} H\,dt \,, \qquad d\hat N = Hdt = -dN \,, \qquad \varepsilon = -\frac{d\ln H}{d\hat N} \,, \qquad \varepsilon_{i+1} = \frac{d\ln\varepsilon_i}{d\hat N} \tag{5.49}$$

where i stands for the initial time.

5.2.5 Linear-Order Evolution

Given the energy momentum tensor, we can derive the fluid quantities for a scalar field:

$$\delta\rho_\phi = \dot\phi\,\delta\dot\phi - \dot\phi^2\alpha + V_{,\phi}\delta\phi = \delta\rho_v - 3H\dot\phi\,\delta\phi \,, \qquad \delta\rho_v := \delta\rho - \rho' v \,, \tag{5.50}$$

$$\delta p_\phi = \dot\phi\,\delta\dot\phi - \dot\phi^2\alpha - V_{,\phi}\delta\phi = \delta\rho_v - 3c_s^2 H\dot\phi\,\delta\phi \,, \qquad v_\phi = \frac{\delta\phi}{\phi'} \,, \tag{5.51}$$

$$e := \delta p - c_s^2\,\delta\rho = (1-c_s^2)\delta\rho_v \,, \qquad \pi^\phi_{\alpha\beta} = q^\phi_\alpha = 0 \,, \tag{5.52}$$

[1] The end of inflation is a bit ill-defined, as we do not have a concrete model. However, in terms of N we can safely use the condition that the slow-roll parameter becomes order unity $\varepsilon \simeq 1$.

where we used the following relation (first from the conservation equation) and the sound speed is defined as

$$\dot\rho_\phi = \dot\phi(\ddot\phi + V_{,\phi}) = -3H\dot\phi^2 \,, \qquad \dot p_\phi = \dot\phi(\ddot\phi - V_{,\phi}) = \dot\phi(2\ddot\phi + 3H\dot\phi) \,, \qquad c_s^2 := \frac{\dot p_\phi}{\dot\rho_\phi} = -1 - \frac{2\ddot\phi}{3H\dot\phi} \,. \tag{5.53}$$

Note tht the sound speed defined above is negative, and in particular $c_s^2 \simeq -1$ for slow-roll inflation. The entropy perturbation is gauge-invariant, and it is non-negligible for scalar field ($e \simeq 2\delta\rho_v$ for slow roll models). Therefore, the comoving gauge corresponds to the uniform field gauge for the single-field models:

$$\varphi_v = \varphi - \mathcal{H}v = \varphi - H\frac{\delta\phi}{\dot\phi} = \varphi_{\delta\phi} \,. \tag{5.54}$$

Other useful gauge-invariant variables are

$$\widetilde{\delta\phi} = \delta\phi - \phi' T \,, \qquad \delta\phi_\varphi := \delta\phi - \frac{\dot\phi}{H}\varphi \,, \qquad \delta\phi_\chi := \delta\phi - \dot\phi\chi \,. \tag{5.55}$$

The equation of motion for a scalar field is then

$$\delta\ddot\phi + 3H\delta\dot\phi + \left(V_{,\phi\phi} + \frac{k^2}{a^2}\right)\delta\phi = \dot\phi(\dot\alpha + \kappa) + (2\ddot\phi + 3H\dot\phi)\alpha \,. \tag{5.56}$$

There exists a very important conservation law on super horizon scales. First, we define a gauge-invariant variable Φ, which is essentially the comoving-gauge curvature:

$$\Phi := \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)}\varphi_\chi = \frac{H^2}{4\pi G(\rho+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{2H^2\Pi}{\rho+p} \,, \tag{5.57}$$

where the last equality can be readily verified by using Eq. (4.182). Taking the time derivative of the definition of Φ and using the Einstein equations (4.185) and (4.182) to remove $\dot\varphi_\chi$ and $\dot v_\chi$, we derive the governing equation for

$$\dot\Phi = \frac{H}{4\pi G(\rho+p)}\frac{c_s^2}{a^2}\Delta\varphi_\chi - \frac{H}{\rho+p}\left(e + \frac{2}{3a^2}\Delta\Pi\right) \,, \tag{5.58}$$

where the sound speed is $c_s^2 := \dot p/\dot\rho$ and the entropy perturbation is $\delta p =: c_s^2\delta\rho + e$. The derivation is fully based on the Einstein equation (no conservation equation at the perturbation level), so that the fluid components are for the total energy-momentum tensor. For the inflaton field with $\Pi = 0$ and $e = (1-c_s^2)\delta\rho_v$, we define the physical sound speed c_A for inflaton

$$\dot\Phi = \frac{H}{4\pi G(\rho+p)}\frac{c_A^2}{a^2}\Delta\varphi_\chi \,, \qquad c_A^2\Delta\varphi_\chi := c_s^2\Delta\varphi_\chi - 4\pi G a^2(1-c_s^2)\delta\rho_v = \Delta\left[1 + (1-c_s^2)\frac{3K}{\Delta^{-1}}\right]\varphi_\chi \,, \tag{5.59}$$

so that the physical sound speed for the inflaton is $c_A \equiv 1$ in a flat universe. In other words, when the scalar field is treated as a fluid, c_A appears in the fluid equations as the proper sound speed, instead of c_s^2, and it is a relativistic object. It is clear that the comoving-gauge curvature is conserved on super horizon scales.

5.2.6 Large-Scale Solution

The RHS of the equation (5.56) of motion can be re-arranged in a uniform-curvature gauge ($\varphi \equiv 0$) in a flat universe ($K = 0 = \Lambda$) with only scalar field but arbitrary potential, and the equation can be readily solved as Hwang (1993)

$$\delta\ddot\phi_\varphi + 3H\delta\dot\phi_\varphi + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot H}{H}\left(3H - \frac{\dot H}{H} + 2\frac{\ddot\phi}{\dot\phi}\right)\right]\delta\phi_\varphi = \frac{H}{a^3\dot\phi}\left[\frac{a^3\dot\phi^2}{H^2}\left(\frac{H}{\dot\phi}\,\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} + \frac{k^2}{a^2}\,\delta\phi_\varphi = 0 \,, \tag{5.60}$$

by using the following to replace $\dot\phi(\dot\alpha + \kappa) + (2\ddot\phi + 3H\dot\phi)\alpha$

$$\alpha_\varphi = 4\pi G\frac{\dot\phi}{H}\delta\phi_\varphi = -\frac{\dot H}{H}\frac{\delta\phi_\varphi}{\dot\phi} \,, \qquad \kappa_\varphi = -\frac{4\pi G}{H}\delta\rho_\phi \,, \qquad \kappa_\varphi + \dot\alpha_\varphi = \left(\frac{2\dot H^2}{H^2} - \frac{2\dot H}{H}\frac{\ddot\phi}{\dot\phi} - 3\dot H\right)\frac{\delta\phi_\varphi}{\dot\phi} \,, \tag{5.61}$$

where we used the background equations

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \,, \qquad \dot{H} = -4\pi G\,\dot{\phi}^2 \,. \tag{5.62}$$

On large scales, the solution is

$$\delta\phi_\varphi = -\frac{\dot{\phi}}{H}\left[C(\mathbf{k}) - D(\mathbf{k})\int_0^t dt'\,\frac{H^2}{a^3\dot{\phi}^2}\right] \,, \tag{5.63}$$

and

$$-\frac{2}{3}\alpha_\varphi = \frac{1}{3}\delta_\varphi = -\frac{2}{9}\frac{\kappa_\varphi}{H} = -\mathcal{H}(1+w)v_\varphi = -\frac{H^2}{\bar{\rho}}\frac{\dot{\phi}}{H}\delta\phi_\varphi = \frac{\dot{\phi}^2}{\bar{\rho}}\,C(\mathbf{k}) \,, \tag{5.64}$$

where C is the growing mode and D is the decaying mode.

5.3 Quantum Fluctuations in Quadratic Action

The background relation describes the inflationary expansion, and the equation of motion we derived describes the evolution of the perturbations at the linear order. Here we will derive their statistical properties. However, before we proceed, we need to better understand the structure of the theory. Even for the standard inflationary models of a single field, the theory is not a free-field, but an interacting field theory.

This can be illustrated as follows. To simplify the calculations, we choose the comoving gauge

$$0 = v_\phi = \frac{\delta\phi}{\phi'} \,, \qquad \phi(x) = \bar{\phi}(t) \,, \qquad \zeta := \varphi_v = \varphi_{\delta\phi} \,, \tag{5.65}$$

and it coincides with the uniform field gauge. Our main variable for scalar fluctuation is then the comoving gauge curvature ζ, as the scalar field is uniform. We can expand the action perturbatively to give

$$S = S_0[\bar{\phi}, \bar{g}_{ab}] + S_2[\zeta^2] + S_3[\zeta^3] + \cdots \,, \qquad H = H_0 + H_{\rm int} \,, \qquad H_{\rm int} = \sum_i F_i(\varepsilon, \eta, \cdots)\zeta^3(\tau) + \cdots \,, \tag{5.66}$$

where the background action S_0 defines the background evolution and its slow-roll parameters. Here we will study the quadratic action S_2 in great detail to derive the power spectrum of the scalar and tensor fluctuations, and the quadratic action is indeed a free-field action in the de-Sitter background (or with small deviations around it). However, remember that the full theory is interacting, and we cannot use the free-field theory to quantize the fluctuations, if we go beyond the quadratic action or compute the high-order correlation functions.

5.3.1 Quadratic Action for Scalars

To derive the linear-order equation of motion, we need to expand the action to the quadratic in perturbations. To simplify the calculations, we choose the comoving gauge. After some integrations by part of the quadratic action, the quadratic action for scalars in the comoving gauge becomes[2]

$$S_{(2)} = \frac{1}{2}\int dt\, d^3\mathbf{x}\; a^3\frac{\dot{\phi}^2}{H^2}\left[\dot{\zeta}^2 - \frac{1}{a^2}(\nabla\zeta)^2\right] = \frac{1}{2}\int d\eta\, d^3\mathbf{x}\;\left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right] \,, \tag{5.67}$$

where we integrate by part, $M_{\rm pl} = 1$, and we defined the canonically-normalized (Mukhanov-Sasaki) variable

$$v := z\zeta \,, \qquad \zeta := \varphi_v \,, \qquad z^2 := a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon \,. \tag{5.68}$$

At the quadratic action, scalar, vector, and tensor do not mix, while they do mix in general quadratic terms. In the action, their indices need to be contracted, e.g., $\phi_{i|j}h^{ij}$, and an integration by parts yields vanishing contribution due to the divergence free condition for vector and tensor contributions. The Lagrangian now takes the form of the simple harmonic oscillator, but with time-dependent mass term

$$m^2(\eta) := -\frac{z''}{z} \quad \underset{\rm dS}{\longrightarrow} \quad -\frac{a''}{a} = -\frac{2}{\eta^2} \,. \tag{5.69}$$

[2]Here, "scalars" are used to refer to the scalar fluctuations, not to be confused with the scalar field.

where we took the de-Sitter limit ($\varepsilon = z = 0$). The canonical momentum and the Hamiltonian are then

$$\pi = \frac{\delta\mathcal{L}}{\delta v'} = v' , \qquad \mathcal{H} = \pi v' - \mathcal{L} = \frac{1}{2}\left[(v')^2 + (\nabla v)^2 + m^2 v^2\right] . \tag{5.70}$$

The equation of motion for the Mukhanov-Sasaki variable is the Klein-Gordon equation:

$$\left(\Box - m^2\right) v = 0 , \qquad v''_{\mathbf{k}} + \omega_k^2 v_{\mathbf{k}} = 0 , \qquad \omega_k^2 := k^2 + m^2 , \qquad v_{\mathbf{k}} = v^*_{-\mathbf{k}} . \tag{5.71}$$

The mode functions take the simple solution for the time-dependence under the assumption that $\omega_k \simeq k$ is time-independent in the limit $\eta \to -\infty$:

$$v_{\mathbf{k}}(\eta) \equiv v^+_{\mathbf{k}} e^{i\omega_k\eta} + v^-_{\mathbf{k}} e^{-i\omega_k\eta} := v^+_{\mathbf{k}}(\eta) + v^-_{\mathbf{k}}(\eta) , \qquad v^+_{\mathbf{k}} = (v^-_{-\mathbf{k}})^\dagger , \tag{5.72}$$

where the amplitude of the mode functions are undetermined. Therefore, the general solution can be written as

$$v(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_{\mathbf{k}} e^{i\omega_k\eta} + v^-_{\mathbf{k}} e^{-i\omega_k\eta}\right) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(v^+_{-\mathbf{k}}\, e^{-ikx} + v^-_{\mathbf{k}} e^{ikx}\right) , \qquad k := (\omega_k, \mathbf{k}) . \tag{5.73}$$

5.3.2 Canonical Quantization

So far, we have derived a classical solution of the quadratic action for scalars. By promoting the Mukhanov-Sasaki field v and its canonical momentum field π to quantum fields, we need to impose the canonical quantization relation ($\hbar = 1$)

$$[\hat{v}(\eta,\mathbf{x}), \hat{\pi}(\eta,\mathbf{y})] = i\delta^{3\mathrm{D}}(\mathbf{x}-\mathbf{y}) , \qquad [\hat{v}(\eta,\mathbf{x}), \hat{v}(\eta,\mathbf{y})] = [\hat{\pi}(\eta,\mathbf{x}), \hat{\pi}(\eta,\mathbf{y})] = 0 , \tag{5.74}$$

where we work in the Heisenberg picture for the time-dependent operators. Apparent from the notation, we want to define the creation and annihilation operators as

$$v^-_{\mathbf{k}} := \hat{a}_{\mathbf{k}} v^-_k , \qquad \left(v^-_{\mathbf{k}}\right)^\dagger = v^+_{-\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} v^+_k , \qquad (v^-_k)^* = v^+_k , \tag{5.75}$$

such that we derive

$$\hat{v}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\hat{a}^\dagger_{\mathbf{k}} v^+_k\, e^{-ikx} + \hat{a}_{\mathbf{k}} v^-_k\, e^{ikx}\right) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\hat{a}^\dagger_{\mathbf{k}} v^+_k(\eta)\, e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}} v^-_k(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}}\right] , \tag{5.76}$$

where we defined

$$v^\pm_k(\eta) := v^\pm_k e^{\pm i\omega_k\eta} . \tag{5.77}$$

By substituting into the canonical quantization relation, we can derive that the ladder operators indeed satisfy the standard quantization relation at the equal time

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{3\mathrm{D}}(\mathbf{k}-\mathbf{k}') , \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0 , \tag{5.78}$$

if the normalization for the mode functions is properly chosen

$$W[v^-_k, v^+_k] := v^-_k v^{+\prime}_k - v^{-\prime}_k v^+_k := i . \tag{5.79}$$

With the properly normalized operators, we obtain the usual relations

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 , \qquad \langle 0|0\rangle = 1 , \qquad |n_{\mathbf{k}}\rangle = \sqrt{\frac{2E_k}{n!}}\,[(\hat{a}^\dagger_{\mathbf{k}})^n]|0\rangle , \tag{5.80}$$

where $\sqrt{2E}$ is put to make it Lorentz invariant. One can quantize the field, starting with the time-independent Harmonic oscillators, then applying the Heisenberg picture with the free-field Hamiltonian, as in Peskin & Schröder.

5.3.3 Vacuum Expectation Value

While we imposed the normalization condition for the mode functions $v_k^\pm(\eta)$ in terms of their Wronskian, the physical vacuum is yet to be fully determined, due to the arbitrariness in the mode functions. Note that we can change $v_k^\pm$ and $\hat{a}_k$ together, while $\hat{v}(x)$ remains unchanged. Consider a different set of mode functions $u_k^\pm$ that are related to the original mode functions as

$$u_k^-(\eta) = \alpha_k v_k^-(\eta) + \beta_k v_k^+(\eta) , \tag{5.81}$$

and construct the creation and annihilation operators $\hat{b}_{\mathbf{k}}^\pm$ with $u_k^\pm$

$$u_{\mathbf{k}}^- := \hat{b}_{\mathbf{k}} u_k^- . \tag{5.82}$$

Using this relation, we can write the operator $\hat{v}$ and its canonical momentum $\hat{\pi}$ in terms of $\hat{b}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}^\dagger$. These two sets of quantum operators are then related as by, so called, the Bogolyubov transformation:

$$\hat{a}_{\mathbf{k}} = \alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}_{-\mathbf{k}}^\dagger , \qquad \hat{a}_{\mathbf{k}}^\dagger = \alpha_k \hat{b}_{\mathbf{k}}^\dagger + \beta_k^* \hat{b}_{-\mathbf{k}} , \qquad |\alpha_k|^2 - |\beta_k|^2 = 1 , \tag{5.83}$$

where the normalization for the transformation coefficients is due to the Wronskian normalization. Note that the vacuum defined by one set of operators $\hat{a}_{\mathbf{k}}$ is not the vacuum with respect to the other set of operators $\hat{b}_{\mathbf{k}}$. To properly determine the physical vacuum, we need to fix the mode function completely.

In terms of the mode functions, the Hamiltonian in Minkowski spacetime is

$$\hat{H} = \int d^3\mathbf{x}\, \hat{\mathcal{H}} , \qquad \hat{\mathcal{H}} = \frac{1}{2}\left[\hat{\pi}^2 + (\nabla\hat{v})^2\right] , \qquad m \to 0 . \tag{5.84}$$

Using the expression for the mode function in Eq. (5.76), we derive the Hamiltonian

$$\hat{H} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\left((v_k^{+\prime})^2 + k^2(v_k^+)^2\right)\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger + \left((v_k^{-\prime})^2 + k^2(v_k^-)^2\right)\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \left(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\right)\left(|v_k^{-\prime}|^2 + k^2|v_k^-|^2\right)\right] , \tag{5.85}$$

acting on the vacuum $|0\rangle$ as

$$\hat{H}|0\rangle = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\left((v_k^{+\prime})^2 + k^2(v_k^+)^2\right)\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ + \left(|v_k^{-\prime}|^2 + k^2|v_k^-|^2\right)(2\pi)^3\delta^{3\mathrm{D}}(0)\right]|0\rangle . \tag{5.86}$$

The vacuum $|0\rangle$ is an eigenstate of the Hamiltonian, and indeed the first round bracket vanishes. The remaining term in the Hamiltonian should be minimized by a proper choice of the mode function. Given the normalization of the Wronskian and the time dependence of the mode function, the physical mode function is then found to be[3]

$$W[v_k^-, v_k^+] = 2ik|v_k^-|^2 = i , \qquad v_k^-(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} . \tag{5.87}$$

Therefore, we derive the vacuum expectation values

$$\left\langle 0|\hat{v}_{\mathbf{k}}^\dagger \hat{v}_{\mathbf{k}'}|0\right\rangle = (2\pi)^3\delta^{3D}(\mathbf{k}-\mathbf{k}')P_v(k) , \qquad P_v(k) = |v_k^-|^2 = \frac{1}{2k} . \tag{5.88}$$

5.3.4 Scalar Fluctuations

Now we consider the time-dependent mass term in the equation of motion, and following the same procedure we pick the vacuum that corresponds to the solution in the Minkowski spacetime as the modes were deep inside the horizon in the far past

$$v_k(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}}\, e^{-i\omega_k(\eta)\eta} , \qquad \lim_{\eta\to-\infty} v_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} , \tag{5.89}$$

and this choice of the mode function is called the Bunch-Davis vacuum. Note that with time-dependent mass term (or spacetime) the vacuum defined as the minimum of the Hamiltonian is also evolving in time, i.e., the vacuum state a moment ago is not a vacuum, but a state of particles.

[3] In fact, given the Bogolyubov transformation, one cannot use the time-dependence $\pm i\omega_k\eta$ to find the physical mode function, rather one has to find a general solution with $v_k^- = A_k e^{iB_k\eta}$.

To the zero-th order in the slow-roll approximation ($\varepsilon = 0$), the inflationary period is the de-Sitter spacetime, in which

$$m^2(\eta) = -\frac{a''}{a} = -\frac{2}{\eta^2} \,, \qquad \omega_k^2 = k^2 - \frac{2}{\eta^2} \,, \tag{5.90}$$

and we can derive the exact solution for the mode functions:

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) \,. \tag{5.91}$$

When the k-mode is stretched beyond the horizon, the amplitude of the mode function is

$$\lim_{k\eta\to 0} v_k(\eta) = \frac{1}{i\sqrt{2}} \frac{1}{k^{3/2}\eta} \,, \qquad \lim_{k\eta\to 0} k^3|v_k|^2 = \frac{1}{2\eta^2} = \frac{a^2H^2}{2} \,, \tag{5.92}$$

and the power spectra of the mode function and the comoving-gauge curvature are

$$P_v \equiv |v_k|^2 = \frac{a^2H^2}{2k^3} \,, \qquad \Delta_\zeta^2 := \frac{k^3}{2\pi^2} P_\zeta = \frac{1}{2a^2\varepsilon} \Delta_v^2 = \frac{H^2}{8\pi^2\varepsilon} \,. \tag{5.93}$$

5.3.5 Tensor Fluctuations: Gravity Waves

We can repeat the exercise for the scalar fluctuations to derive the tensor fluctuations. The tensor perturbations are decomposed in terms of two helicity eigenstates as

$$h_{ij} := 2C_{ij}^{(t)} = 2h^{(\pm 2)} Q_{ij}^{(\pm 2)} \,, \tag{5.94}$$

and the quadratic action for tensor is

$$S_{(2)} = \frac{M_{\rm pl}^2}{8} \int d\eta \, d^3\mathbf{x} \, a^2 \left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] = \sum_{s=\pm 2} \int d\eta \, d^3\mathbf{k} \, \frac{a^2}{4} M_{\rm pl}^2 \left[(h_{\mathbf{k}}^{s\prime})^2 - k^2 (h_{\mathbf{k}}^s)^2\right] \,. \tag{5.95}$$

Using the Mukhanov-Sasaki variable for tensor fluctuations, the quadratic action is

$$v_{\mathbf{k}}^s := \frac{a}{2} M_{\rm pl} h_{\mathbf{k}}^s \,, \qquad S_{(2)} = \sum_{s=\pm 2} \frac{1}{2} \int d\eta \, d^3\mathbf{k} \left[(v_{\mathbf{k}}^{s\prime})^2 - k^2 (v_{\mathbf{k}}^s)^2 + \frac{a''}{a} (v_{\mathbf{k}}^s)^2\right] \,, \qquad m^2 = -\frac{a''}{a} \,, \tag{5.96}$$

where we integrate by part and used $\mathcal{H}^2 + \mathcal{H}' = a''/a$. The quadratic action for tensor is identical, and we can readily derive the tensor power spectrum

$$P_v = \frac{(aH)^2}{2k^3} \,, \qquad P_T := 2P_{h_k^s} = 2\left(\frac{2}{aM_{\rm pl}}\right)^2 P_v = \frac{4}{k^3} \frac{H^2}{M_{\rm pl}^2} \,. \tag{5.97}$$

The amplitude of the tensor power spectrum is the energy scale of the inflation in the early Universe, and its ratio to the scalar power spectrum is

$$r := \frac{\Delta_t^2}{\Delta_s^2} = \frac{8}{M_{\rm pl}^2} \frac{\dot{\phi}^2}{H^2} = 16\varepsilon \,, \tag{5.98}$$

slow-roll suppressed. Note that GR is a proper low-energy EFT of quantum gravity and hence there are no issues in quantizing gravity in the standard QFT as above. The problems arise only when the energy scale approaches the Planck scale.

5.4 Predictions of the Standard Inflationary Models

5.4.1 Consistency Relations

For the standard single field inflationary models with the slow-roll approximation, we summarize the predictions for scalar fluctuations

$$A_s := \frac{H^2}{8\pi^2\varepsilon M_{\rm pl}^2} = \frac{1}{24\pi^2\varepsilon}\frac{V}{M_{\rm pl}^4} , \qquad \Delta_\zeta^2 = \frac{k^3 P_\zeta}{2\pi^2} = A_s\left(\frac{k}{k_*}\right)^{n_s-1} . \tag{5.99}$$

Given the relation $k = aH$ for the fluctuations:

$$d\ln k = d\ln a + d\ln H = Hdt + \frac{\dot H}{H}dt = Hdt(1-\varepsilon) , \tag{5.100}$$

we derive the spectral index as

$$n_s - 1 := \frac{d\ln A_s}{d\ln k} = \frac{d\ln H^2}{d\ln k} - \frac{d\ln\varepsilon}{d\ln k} = -\frac{2\varepsilon_1}{1-\varepsilon_1} - \frac{\varepsilon_2}{1-\varepsilon_1} \approx -2\varepsilon_1 - \varepsilon_2 \approx 2\eta_V - 6\varepsilon_V , \tag{5.101}$$

where we used $\varepsilon_1 \approx \varepsilon_V$ and $\varepsilon_2 = 2(\varepsilon_1 - \eta) \approx 4\varepsilon_V - 2\eta_V$. The predictions for tensor fluctuations are

$$P_T = \frac{4}{k^3}\frac{H_*^2}{M_{\rm pl}^2} = \left(\frac{2\pi^2}{k^3}\right)A_T , \qquad A_T := \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2} = \frac{2V}{3\pi^2 M_{\rm pl}^4} , \qquad n_t := \frac{d\ln k^3 P_T}{d\ln k} \simeq -2\varepsilon , \tag{5.102}$$

and the consistency relations

$$r := \frac{A_T}{A_s} = \frac{8\dot\phi_*^2}{H_*^2} = 16\varepsilon = -8n_t . \tag{5.103}$$

By measuring the power spectrum amplitude and its slope for both scalar and tensor fluctuations, we can ensure that the fluctuations are indeed generated by a single field inflaton or rule out the standard inflationary models. There exist other predictions in the standard inflationary models (and of course, for the beyond the standard models) that can be used to test models, such as the primordial non-Gaussianity and so on.

5.4.2 Lyth Bound

Given the definition of the e-folds, we can further manipulate it by using the inflaton as a time clock:

$$N(\phi_k) = \int_{\phi_k}^{\phi_{\rm end}} d\phi\,\frac{H}{\dot\phi} = \int_{\phi_k}^{\phi_{\rm end}}\frac{d\phi}{M_{\rm pl}\sqrt{2\varepsilon}} , \qquad r = 16\varepsilon = \frac{8}{M_{\rm pl}^2}\left(\frac{d\phi}{dN}\right)^2 , \tag{5.104}$$

and this relation further implies that the excursion of the inflaton field is related to the tensor-to-scalar ratio as

$$\frac{\Delta\phi_k}{M_{\rm pl}} \simeq \int_{N_{\rm end}}^{N_{\rm cmb}} dN\,\sqrt{\frac{r}{8}} , \tag{5.105}$$

where $\varepsilon(\phi_{\rm end}) \equiv 1$. To solve the horizon problem, the mode k should have expanded at least 40−60 in e-folds. So, this consistency relation (Lyth, 1997) implies that an inflationary field variation of the order of the Planck mass is needed to produce $r > 0.01$. From the theoretical point of view, this sets the upper bound on the amplitude of gravitational waves. Indeed, the standard inflationary model predictions are very small.

Note that the uncertainty in e-folds N is due to our ignorance in the reheating era: After the inflationary period ends, the inflaton field decays into other particles and reheats the Universe. This period is expected to be described by a matter-dominated era, as the inflaton oscillates around the minimum of the potential, effectively acting as a matter. However, we know very little about this period.

The current observational constraint is

$$A_s \simeq 2.2\times10^{-9} , \qquad n_s \simeq 0.96 , \qquad \varepsilon \simeq 0.01 . \tag{5.106}$$

indicating the energy scale of the inflation is

$$A_T = \frac{2V}{3\pi^2 M_{\rm pl}^4} = 16\varepsilon A_s , \qquad H^2 = \frac{V}{3M_{\rm pl}^2} = \varepsilon\left(2\times10^{14}\ {\rm GeV}\right)^2 . \tag{5.107}$$

5.4.3 A Worked Example

Here we consider a very simple inflationary model with a power-law potential:

$$V = \frac{1}{2} m^{4-\alpha} \phi^\alpha \,, \tag{5.108}$$

where the mass m and the slope α are the free parameters of the model. It chaotically starts everywhere at any time in field configurations, and its predictions are then

$$\varepsilon_V = \frac{\alpha^2}{2}\left(\frac{M_{\rm pl}}{\phi}\right)^2 \,, \qquad \eta_V = \alpha(\alpha-1)\left(\frac{M_{\rm pl}}{\phi}\right)^2 \,, \tag{5.109}$$

$$N \simeq \int \frac{d\phi}{M_{\rm pl}^2}\frac{V}{V'} = \frac{\phi^2 - \phi_{\rm end}^2}{2M_{\rm pl}^2 \alpha} \,, \qquad r \simeq 16\varepsilon_V \,, \qquad n_s - 1 \simeq 2\eta_V - 6\varepsilon_V \,. \tag{5.110}$$

Approximating $\phi_{\rm end} \simeq 0$, we further derive

$$N \simeq \frac{1}{2\alpha}\left(\frac{\phi}{M_{\rm pl}}\right)^2 \,, \qquad \varepsilon_V \simeq \frac{\alpha}{4N} \,, \qquad \eta_V = \frac{\alpha-1}{2N} \,, \qquad 1-n_s \simeq \frac{\alpha+2}{2N} \,, \qquad r \simeq \tag{5.111}$$

5.5 Adiabatic Modes and Isocurvature Modes

• ***Adiabatic modes***.— Assuming a flat Universe, we can arrange Eq. (5.58) to show

$$\dot\varphi_v = \Xi - \frac{H}{\rho+p}\frac{k^2}{a^2}\left(\frac{c_s^2}{4\pi G}\,\varphi_\chi - \frac{2}{3}\Pi\right) \,, \qquad \Xi := \frac{\dot{\bar\rho}\,\delta p - \dot{\bar p}\,\delta\rho}{3(\bar\rho+\bar p)^2} \equiv -\frac{He}{\rho+p} \,, \tag{5.112}$$

where the entropy perturbation is gauge invariant at the linear order

$$e := \delta p - c_s^2 \delta\rho \,. \tag{5.113}$$

If the pressure of a fluid is just a function of the density, it satisfies the adiabatic condition ($\Xi \equiv e \equiv 0$):

$$p = p(\rho) = p(\bar\rho) + \left.\frac{dp}{d\rho}\right|_0 \delta\rho + \cdots = \bar p + c_s^2\delta\rho + \cdots \,, \qquad c_s^2 := \frac{\dot{\bar p}}{\dot{\bar\rho}} \,, \qquad e := 0 \,. \tag{5.114}$$

Therefore, in the limit $k \to 0$, if $\Xi = 0$ vanishes, the comoving-gauge curvature perturbation is conserved, regardless of contents in the Universe.

$$\lim_{k\to 0} \varphi_v = \text{constant in time} \quad \text{if} \;\; \Xi = 0 \,. \tag{5.115}$$

Indeed, the adiabatic condition $\Xi = 0$ is satisfied for the matter-dominated era, the radiation-dominated era, and for the single field inflation,[4] which are essentially cases with a single fluid.

For multi-fluid cases, the adiabatic condition can be imposed for individual components, *fluctuating at the same rate at a given point*:

$$\frac{\delta\rho_i}{\dot{\bar\rho}_i} = \frac{\delta\rho_{\rm tot}}{\dot{\bar\rho}_{\rm tot}} = \frac{\delta p_i}{\dot{\bar p}_i} = \frac{\delta p_{\rm tot}}{\dot{\bar p}_{\rm tot}} =: -\varphi_v\,\mathcal{I} \,, \qquad \frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \quad \text{for } \forall\, a, b \,, \tag{5.116}$$

and in the limit $k \to 0$ we can indeed derive the adiabatic condition

$$\mathcal{I} := \frac{1}{a}\int_{t_i}^t dt\; a(t) \,, \qquad v_\chi \equiv -\frac{1}{a}\mathcal{I}\,\varphi_v \,. \tag{5.117}$$

This is a non-trivial condition, as opposed to the single-fluid case. For example, consider radiation and matter components:

$$\dot{\bar\rho}_\gamma = -4H\bar\rho_\gamma \,, \qquad \dot{\bar\rho}_m = -3H\bar\rho_m \,, \qquad \Xi = \frac{H\bar\rho_m\bar\rho_\gamma}{(3\bar\rho_m + 4\bar\rho_\gamma)^2}(4\delta_m - 3\delta_\gamma) \,, \tag{5.118}$$

[4] It vanishes only in the limit $k = 0$ for single field models.

such that φ_v is conserved, only when the adiabatic condition

$$\delta_m = \frac{\delta_\gamma}{1+1/3} \tag{5.119}$$

is satisfied in the limit $k \to 0$. Even for single-field inflationary scenarios, there should have existed many other matter fields, and some energy transfer to these fields are inevitable. However, these non-adiabatic perturbations decay fast as the inflation proceeds, and they become exponentially suppressed when these matter fields dominate the energy budget during the reheating era. For adiabatic case, the curvature fluctuations for each fluid are identical:

$$\varphi_{\delta_a} := \varphi + \frac{\delta_a}{3(1+w_a)} = \varphi_{\delta_b} , \quad \text{for} \, \forall \, a, \ b . \tag{5.120}$$

• ***Isocurvature/entropy mode***.— Isocurvature perturbations represent non-adiabatic fluctuations that arise from a decay of a single source (or the inflaton). With the specific definition of $\mathcal{S}_{XY}$ below, one can set up the initial conditions for N-component fluids in terms of one adiabatic fluctuation and $N-1$ isocurvature fluctuations (no isocurvature fluctuations for a single component). Note, however, that this independent setup is valid only at the initial time. The evolution of isocurvature perturbations depends not only on inflationary dynamics, but also on post-inflationary evolution. For example, if all particles thermalize after inflation, all isocurvature perturbations become adiabatic perturbations eventually. The details are in CPT.pdf.

The isocurvature perturbations and the entropy perturbations are interchangeably used, because they do represent the perturbations between species and it does conserve the curvature. In practice, the entropy perturbations are parametrized by two free parameters at some pivot scale k_0 (0.002/Mpc in WMAP), i.e., ratio α of the isocurvature to the adiabatic perturbations in their amplitudes and their correlation β

$$\frac{P_\mathcal{S}}{P_\zeta} := \frac{\alpha}{1-\alpha} , \qquad \beta := \frac{P_{\mathcal{S}\zeta}}{\sqrt{P_\mathcal{S} P_\zeta}} , \tag{5.121}$$

where the relative entropy perturbation (or specific entropy) is defined as

$$\mathcal{S}_{XY} \equiv \delta\left(\frac{n_X}{n_Y}\right) \Big/ \left(\frac{n_X}{n_Y}\right) = \frac{\delta n_X}{n_X} - \frac{\delta n_Y}{n_Y} = \frac{\delta_X}{1+w_X} - \frac{\delta_Y}{1+w_Y} . \tag{5.122}$$

By defining the gauge-invariant curvature perturbation in the uniform-density gauge

$$\varphi_\delta = \varphi - H\frac{\delta\rho}{\dot\rho} = \varphi + \frac{\delta}{3(1+w)} , \tag{5.123}$$

we can readily show that the entropy perturbation is gauge invariant and conserved on large scales in the absence of mutual interactions.

$$\mathcal{S}_{XY} = 3\left(\varphi_\delta^X - \varphi_\delta^Y\right) . \tag{5.124}$$

In literature, it is often the case that the reference species is set for photons, and three extra components are considered such as baryon, cdm, neutrino (sometimes neutrino velocity) for isocurvature perturbations. In the most general case, we need to consider $\mathcal{S}_{X\gamma}$ with $X = b, \ \mathrm{cdm}, \ \nu$ in addition to P_ζ, such that the initial power spectra are characterized by 4-4 matrix of auto and cross power spectra, each of which is described by the initial amplitude and the slope. A pure isocurvature model is ruled out, because the Sachs-Wolfe plateau is six times larger than in the adiabatic case and the contributions on small scales are further suppressed $\mathcal{T}_l \propto (k/k_\mathrm{eq})^{-2}$ in the isocurvature case.