Standard Inflationary Model

* single scalar field. canonical kinetic term. slow-roll potential & GR

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi - V(\phi) \right] \qquad \varepsilon \equiv \frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) < 0$$

(decreasing comoving horizon $\frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) < 0$)

Scalar field action

EoM : $\Box \phi - V_{,\phi} = 0$. $\quad T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\rho}\phi^{,\rho} - V g_{\mu\nu} = \rho u_\mu u_\nu + p H_{\mu\nu} + \pi_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu$

$\Rightarrow$ BG. perturbations . $\quad \pi_{\mu\nu} = q_\mu = 0$. $\quad v_\phi = \delta\phi / \phi'$ $\quad \rightarrow v = 0 = \delta\phi \quad$ comoving = unitary

$$e = \delta p - c_s^2 \delta\rho = (1 - c_s^2)\delta\rho_v$$

$$\begin{cases} \bar\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi) , \quad \bar p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi) . \quad \text{others} = 0 . \\ \ddot\phi + 3H\dot\phi + V_{,\phi} = 0 \end{cases}$$

rolls over the potential + Hubble friction

$$H^2 = \frac{8\pi G}{3}\rho_\phi \ , \ \dot H = -4\pi G \dot\phi^2 \ , \ H^2 + \dot H = \frac{\ddot a}{a} = \frac{8\pi G}{3}(V - \dot\phi^2)$$

Slow-roll approximation

$\dot\phi^2 \ll V \quad \rightarrow \quad \bar\rho_\phi = -\bar p_\phi \quad \& \quad V(\phi) = \text{constant} \quad \rightarrow \quad w = -1$. exponential expansion

slow-roll parameter $\varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot H}{H^2} = \frac{1}{2}\frac{\dot\phi^2}{H^2 M_p^2}$. $\varepsilon = 0 \Leftrightarrow \dot\phi = 0$ slow-roll

condition for inflation : decreasing comoving horizon $\quad 0 > \frac{d}{dt}\left(\frac{1}{\mathcal{H}}\right) = -\frac{1}{a}\left(1 + \frac{\dot H}{H^2}\right) = -\frac{1-\varepsilon}{a}$

and slow-roll $\eta \equiv -\frac{\ddot\phi}{H\dot\phi}$ $\qquad$ ok if $|\varepsilon| < 0$

other slow-roll parameters $\quad \varepsilon_V \equiv \frac{M_p^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 , \quad \eta_V = M_p^2 \left(\frac{V_{,\phi\phi}}{V}\right)$

$$\ddot\phi \simeq 0 \quad V_{,\phi} \sim 3H\dot\phi \quad \Rightarrow \quad \varepsilon_V \simeq \varepsilon$$

Conservation on super horizon scales

$\Phi = \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)} \varphi_x \to \varphi_v$ for $K=0$

$\dot{\Phi} = \cdots = -\frac{H}{4\pi G(\rho+p)} \frac{k^2}{a^2} \varphi_x \to 0$ for $k \to 0$ $\quad c_A = 1$ super-horizon conservation for φ_v

de Sitter spacetime

perfect symmetry : expanding forever . $H^2 = \frac{1}{3}\Lambda$ constant

$H = \frac{d}{dt}\ln a$, $a d\eta = dt$, $a = e^{Ht} = -\frac{1}{H\eta}$, $a = (0, \infty)$, $t = (-\infty, \infty)$, $\eta = (-\infty, 0)$

$\varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = 0$. non-vanishing ε ~~characterize deviation from de Sitter~~ inflation must end

past inside horizon future super horizon

* $H \sim \sqrt{\varepsilon}\, 10^{14}\,\text{GeV}$ $(\simeq 10^{-5} M_{Pl})$, $E_{CMB} \sim 0.3\,\text{meV} \to \frac{2}{3} \cdot 10^{27}$ at end of inf.

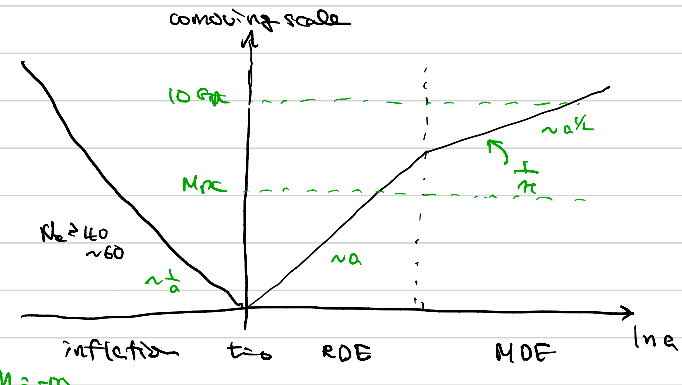
$10\,\text{Gpc} \to 100\,\text{cm}$ at end of inf. , then inf. expansion

$N \geq 40 \sim 60 \to 10^{17} \sim 10^{26}$ $\left(\lg x = \frac{N}{\ln 10}\right)$ < proton ~ fm ≃ $10^{-15}\,m$

$\frac{1}{H} \sim \frac{10^5}{\sqrt{\varepsilon}} t_{Pl} \sim 10^{-37}\,\text{sec}$. $N_e = \int dt\, H \to \Delta t \sim N_e/H \sim 10^{-36}\,\text{sec}$

* quantum fluctuations in vacuum : stretched and frozen at $k \sim \mathcal{H}$ with same amplitude H

lose quantum nature . scalar field & gravity



* prediction : exponential expansion. $N \gtrsim 40 \sim 60$

generation of cosmological fluctuations $\sim 10^{-5}$, scale-invariant ($n_s \simeq 1$), Gaussian

$$\zeta = \varphi_v \qquad \Delta_\zeta^2 = \frac{k^3 P_\zeta}{2\pi^2} = A_s \left(\frac{k}{k_0}\right)^{n_s-1}, \qquad A_s = \frac{H^2}{8\pi^2 \epsilon M_{pl}^2} \sim 10^{-9}, \qquad n_s - 1 = \frac{d}{d\ln k}\Delta_\zeta^2 \simeq -2\epsilon, \qquad \epsilon \sim 0.01$$

$$\Delta_T^2 = \frac{k^3 P_T}{2\pi^2} = A_T \left(\frac{k}{k_0}\right)^{n_T}, \qquad A_T = \frac{2H^2}{\pi^2 M_{pl}^2}, \qquad n_T \simeq -2\epsilon, \qquad r \equiv A_T/A_s = 16\epsilon \sim 0.1$$

consistency

Quantum fluctuations

no dynamics in ϕ

comoving gauge $v_\phi = 0 \to \delta\phi = 0$ uniform field gauge, unitary gauge $\phi(x,t) = \bar{\phi}(t) + 0$

scalar dynamics $\to \zeta = \varphi_v$ quadratic cubic + tensor (gravity waves)

$$S[\phi, g_{\mu\nu}] = \bar{S}[\bar{\phi}, \bar{g}_{\mu\nu}]^{BG} + 0 + S_2[\bar{\phi}, \bar{g}_{\mu\nu}]\zeta^2 + S_3[\bar{\phi}, \bar{g}_{\mu\nu}]\zeta^3 + \cdots$$

Hamiltonian $H = H_0 + H_{int}$, H_0 : free-field theory, H_{int} : interaction

kinetic + mass

standard single-field inflation : interacting theory, fluctuations, not Gaussian

small interaction but in fact very Gaussian

$\frac{1}{8\pi G} = M_{pl} = 1$

Quadratic action in comoving gauge

$\sim \epsilon$ suppressed

$$S_2 = \frac{1}{2}\int dt\, d^3x \cdot a^3 \frac{\dot\phi^2}{H^2}\left[\dot\zeta^2 - \frac{1}{a^2}(\nabla\zeta)^2\right]$$

$$= \frac{1}{2}\int d\eta\, d^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z} v^2\right]$$

cosmology $\uparrow$

$v = z \cdot \zeta$ Mukhanov-Sasaki $z^2 = a^2 \frac{\dot\phi^2}{H^2} = 2a^2\epsilon$

free scalar field action $m^2 \equiv -\frac{z''}{z} \xrightarrow{dS} -\frac{a''}{a} = -\frac{2}{\eta^2}$

$\eta \to -\infty$. Minkowski. massless $m \to 0$

canonical form of SHO

$\pi = \frac{\delta \mathcal{L}}{\delta v'} = v'$ canonical momentum, $\mathcal{H}_v = \frac{1}{2}\left[\pi^2 + (\nabla v)^2\right] + \frac{1}{2}m^2 v^2$. simple-harmonic oscillator

time-dependent mass

EoM : $(\Box - m^2)\, v(x,\eta) = 0 \xrightarrow{FT} v_k''(\eta) + \omega_k^2 v_k(\eta) = 0$, $\omega_k^2 \equiv k^2 + m^2(\eta)$

time-dependent

two solutions $v_k^\pm(\eta)$

$\to$ creation & annihilation operators

Quantization (simple overview)

find solution for mode fct v, π $\to$ promote them to quantum operators

impose canonical quantization relation $[\hat v, \hat\pi] = i\delta(x-y)$

some ambiguity in mode fct

compute Hamiltonian H, find the vacuum that minimize H $\leftarrow$

$\uparrow$ fixed by

vacuum $\hat a_k |0\rangle = 0$ for $\forall \vec k$ non-local. but deep inside Minkowski $\to$ ordinary QFT Bunch-Davies vacuum

However, the mode fct & vacuum : time-dependent in time-dependent spacetime

vacuum at $t-\epsilon$ is not vacuum at t. $\to$ creation of particles

simply quantum fluctuations in vacuum get stretched to super horizon scales

Power spectrum

analytical solution for mode fcts in de-Sitter

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \xrightarrow{\eta\to-\infty} \frac{1}{\sqrt{2k}} e^{-ik\eta}$$ free field in Minkowski

$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k}} e^{ikx} v_k$ QFT

as $a \to \infty$ expands, given mode k goes outside horizon & $\varphi_v = \zeta$ freezes

we want to compute the $\sigma_\zeta^2(k)$ or $P_\zeta(k)$ at this time (no more evolution)

at horizon crossing $k = \mathcal{H}$ or $k\eta \to 0$ $\quad \left(a = -\frac{1}{H\eta}, \ \mathcal{H} = -\frac{1}{\eta} \right)$

$v_k \to \frac{-i}{\sqrt{2}\, k^{3/2}\eta}$, $\quad \langle 0 | \hat{v}_k^+ \hat{v}_{k'} | 0 \rangle = (2\pi)^3 \delta^D(k-k') P_v(k) \Rightarrow P_v(k) = |v_k|^2 = \frac{1}{2k^3\eta^2} = \frac{a^2H^2}{2k^3}$

VEV at crossing

$$\Delta_\zeta^2 = \frac{1}{z^2}\Delta_v^2 = \frac{k^3}{2\pi^2}\left(\frac{a^2H^2}{2k^3}\right)\frac{1}{2a^2\varepsilon} = \frac{H^2}{8\pi^2\varepsilon}$$ scale-invariant

Quantization (details)

mode fct $\to \frac{1}{\sqrt{2k}} e^{-ik\eta}$ in Minkowski $\quad \frac{d^3k}{2E}$: Lorentz invariant volume

$v(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x} [v_k^+(\eta) + v_k^-(\eta)]$, same for $\pi(x)$, then $\hat{v}, \hat{\pi}$, impose canonical quantization

$[\hat{v}(x), \hat{\pi}(y)] = i\delta^D(x-y)$, then ladder operators a_k, a_k^+, $[a_k^+, a_{k'}] = (2\pi)^3 \delta^D(k-k')$

$[\hat{v}, \hat{v}] = [\hat{\pi}, \hat{\pi}] = 0 \qquad v_k^+ = \hat{a}_k^+ v_k^+(\eta) \qquad$ if $W[v^-, v^+] = v_k^- v_k^{+\prime} - v_k^{-\prime} v_k^+ = i$

vacuum $\hat{a}_k |0\rangle = 0$ for $\forall \vec{k}$ $\quad$ non-local, but deep inside Minkowski

Spectral index

at horizon crossing $k \simeq aH$. $d\ln k = d\ln a + \cancel{d\ln H}^{\text{small}} = H dt$

$$n_s - 1 = \frac{d}{d\ln k} \ln \Delta_s^2 = \frac{d\ln H^2}{d\ln k} - \cancel{\frac{d\ln \epsilon}{d\ln t}}^{\text{small}} \doteq 2\frac{d\ln H}{d\ln t} = 2\frac{\dot{H}}{H^2} = -2\epsilon$$

Tensor fluctuations : gravity waves

$$S_2 = \frac{1}{8}\int d\eta \cdot d^3x \cdot a^2\left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] \simeq \sum_{s=\pm 2} \frac{1}{2}\int d\eta\, d^3x \left[(v^{s\prime})^2 - (\nabla v^s)^2 + \frac{a''}{a} v^2\right]$$

$$h_{ij} \equiv 2 C_{ij}^{(T)} = 2 h^{(\pm 2)} Q_{ij}^{(\pm 2)} \quad , \quad v_k^s \equiv \frac{1}{2} a h_k^s$$

$$\Rightarrow P_v = \frac{a^2 H^2}{2k^3} \quad , \quad \Delta_T^2 = \frac{k^3}{2\pi^2}\left(2 P_{h_k^s}\right) = \frac{k^3}{2\pi^2} \cdot 2 \cdot \frac{4}{a^2} \cdot \frac{a^2 H^2}{2k^3} = \frac{2H^2}{\pi^2} \quad , \quad r = \frac{\Delta_T^2}{\Delta_s^2} = 16\epsilon$$

low energy QFT of gravity : no issues

Inflaton decay & Reheating

$\epsilon \to 1$. inflaton decay into radiation, dm, baryons, ... & universe heat up, begins RDE, BBN

highly non-linear, model-dependent (lattice QCD), no good understanding

BUT super-horizon modes ζ : frozen, ind. of events inside horizon

particle horizon size at $E \sim 10^{14}$ GeV ?

Issues of standard inflation model

* no inflaton found, instability due to quantum correction $\epsilon \to \infty$)
* trans-Planckian issue

$$N_e = \int dt \cdot H = \int d\phi \cdot \frac{H}{\dot\phi} \sim \int \frac{d\phi}{\sqrt{2\epsilon}} \to \frac{\Delta\phi}{M_{pl}} \sim \sqrt{\epsilon}\, N \sim 4-6 > 1$$

Beyond the standard

* non-canonical kinetic term
* multi-field models (isocurvature)
* modified gravity theories

$\rightarrow$ deviations in higher-order statistics

bispectrum, non-Gaussianity