# Physical Cosmology

## Problem set 3

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**Exercise 1** (3 points)

In the early Universe, free protons and free neutrons were in thermal equilibrium with the hot plasma. Equilibrium was maintained by the reaction with electronic neutrinos and antineutrinos:

$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \,. \tag{1}$$

Using the cross section for the weak interaction given by

$$\sigma_{\text{weak}} = 5 \cdot 10^{-43} \left( \frac{k_B T}{1 \text{ MeV}} \right)^2 \text{cm}^2 \,, \tag{2}$$

and assuming massless neutrinos, compute the following quantities:

- temperature, time and redshift at which neutrinos decouple from the thermal bath
- temperature of the neutrino background during the matter-dominated epoch at $z = 1000$

**Exercise 2** (3 points)

Assume that our Universe has no baryon–antibaryon asymmetry and that the radiation number density at $z = 0$ is $n = 400\,\text{cm}^{-3}$. Further assume that the expectation value of the product of interaction cross section and particle velocity $\sigma v$ for annihilation reactions is given by $\langle \sigma v \rangle \approx \frac{1}{m_\pi^2}$, where $m_\pi$ denotes the mass of the pion and we are working in units in which $c = \hbar = 1$.

- What is the temperature of the photon field when the baryons and antibaryons stop to annihilate significantly?
- What would be the relic number density of baryons and antibaryons at the present time? Calculate the corresponding ratio of baryonic and antibaryonic number density to radiation number density. An order-of-magnitude calculation is sufficient.
- The universe undergoes different evolutionary phases: At first, radiation dominates its dynamics. Soon after, then matter, and in the later epoche the dark energy component grows in significance. How would the succession of phases change and how would their periods be modified, if the universe had no asymmetry? Explain qualitatively.

**Exercise 3** (4 points)

Use the phase-space distribution function

$$f(p,t) = \frac{g}{(2\pi)^3} \frac{1}{\exp\left[(E-\mu)/T\right] \pm 1} \,, \qquad \begin{cases} + : \text{Fermion} \\ - : \text{Boson} \end{cases} \tag{3}$$

to derive the number density $n(t)$, the energy density $\rho(t)$ and the pressure $P(t)$

$$
\begin{aligned}
n(t) &= \int d^3\mathbf{p}\ f(p,t) = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{E^2-m^2}E\ dE}{\exp\left[(E-\mu)/T\right]\pm 1}\ , \\
\rho(t) &= \int d^3\mathbf{p}\ E\ f(p,t) = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{E^2-m^2}E^2\ dE}{\exp\left[(E-\mu)/T\right]\pm 1}\ , \\
P(t) &= \int d^3\mathbf{p}\frac{1}{3}\frac{p^2}{E} f(p,t) = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}dE}{\exp\left[(E-\mu)/T\right]\pm 1}\ .
\end{aligned}
\tag{4}
$$

In the non-relativistic limit, derive

$$
f(p,t) = \frac{g}{(2\pi)^3}\exp\left(-\frac{m-\mu}{T}\right)\exp\left(-\frac{p^2}{2mT}\right)\ , \tag{5}
$$

and

$$
n(t) = g\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left[-\frac{m-\mu}{kT}\right]\ , \qquad \rho(t) = mn(t)\ , \qquad P(t) = n(t)kT\ , \tag{6}
$$

and in the relativistic limit, derive

$$
n(t) = \begin{cases} \frac{g}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3 \\ \frac{3g}{4\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3 \end{cases}, \qquad \rho(t) = \begin{cases} \frac{g\pi^2}{30}kT\left(\frac{kT}{\hbar c}\right)^3 & : \text{Boson} \\ \frac{7}{8}\frac{g\pi^2}{30}kT\left(\frac{kT}{\hbar c}\right)^3 & : \text{Fermion} \end{cases}, \qquad P(t) = \frac{1}{3}\rho(t)\ . \tag{7}
$$

See Sec. 1.2.2 in the lecture notes for more details.