# Physical Cosmology

## Problem set 6

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Prof. J. Yoo

Giovanni Piccoli

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**Exercise 1** (5 points)

An inflationary phase can be described taking the dominant contribution to the energy-momentum tensor to be given by a homogeneous scalar field, whose amplitude slowly rolls down some potential $V(\phi)$. While observations are not able to pinpoint its precise form, remarkably lot of potentials (such as $V \propto \phi^n$) has already been ruled out. The matter Lagrangian of a scalar field minimally coupled to gravity is given as usual as:

$$S_\phi = \int d^4x \sqrt{-g} \Big[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \Big]; \tag{1}$$

(a) obtain the energy momentum tensor $T^{(\phi)}_{\mu\nu}$ from this action for a generic $g_{\mu\nu}$.

(b) specialize now to a homogeneous Universe, assuming $\phi = \phi(t)$ only ($t$ being cosmic time) and assuming a FRWL metric. From the components of the energy-momentum tensor read off the energy density and the pressure associated to the field. Under which condition does the *slow-roll* regime hold, namely $p \approx -\rho$?

(c) show that $\phi$ satisfies the following equations of motion, namely the Klein-Gordon equations in an expanding Universe:

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{2}$$

denoting with $\dot{f} = \partial_t f$, $g' = \partial_\phi g$. The Hubble factor is given as usual by the Friedmann equation:

$$H^2 = \frac{\rho}{3M_{\rm pl}^2}, \tag{3}$$

where $M_{\rm pl}^2 = 1/8\pi G$.

(d) consider the slow-roll condition you obtained previously, and differentiate it with respect of time; show that it lead to the following expression for the field velocity:

$$\dot{\phi} \approx -\frac{V'}{3H}, \tag{4}$$

which means that the slow-roll dynamics is confined on a one-dimensional trajectory in the space $(\phi, \dot{\phi})$.

**Exercise 2** (5 points)

Cosmic time is not a convenient evolution variable, given that during inflation the background expands exponentially: it is much better to consider the e-folding time $N = \log(a/a_{\rm ref})$, in terms of an unspecified reference scale factor $a_{\rm ref}$.

(a) From Eq. 4, obtain an integral expression for the number of e-folds that it takes for the field to go from $\phi_i$ to $\phi_f$. *Hint*: change variable, recalling that $\dot{N} = H$. Finally, use the slow-roll approximation to express $H$.

(b) Inflation is characterized as a phase of approximate exponential expansion. To quantify how much it deviates from the ideal case, it is useful to introduce the first slow-roll parameter (notice that it is an adimensional number):

$$\varepsilon := \frac{d}{dt} H^{-1}; \tag{5}$$

defining the momentum $\pi = d\phi/dN$, show the following *exact* relation:

$$\varepsilon = \frac{\pi^2}{2M_{\rm pl}^2}. \tag{6}$$

(c) Finally, show that the Klein-Gordon equation can be written as follows (a form particularly useful for numerical studies):

$$\frac{d\phi}{dN} = \pi, \quad \frac{d\pi}{dN} = -(3-\varepsilon)\pi - \frac{V'}{H^2}, \tag{7}$$

and that the Friedman equation becomes:

$$H^2 = \frac{V(\phi)}{M_{\rm pl}^2(3-\varepsilon)}. \tag{8}$$