



Physical Cosmology

Spring Semester 2026

Prof. J. Yoo

Problem set 7

Giovanni Piccoli

Issued: 21.4.2026

Due: 28.4.2026

Exercise 1 (5 points)

Consider a fluid made of N independent components labeled by the index $i = 1, \dots, N$. Each individual fluid has mass density ρ_i , pressure P_i , and velocity \mathbf{v}_i . The Euler and Poisson equations are given by:

$$\dot{\rho}_i + \nabla \cdot (\rho_i \mathbf{v}_i) = 0, \quad (1)$$

$$\dot{\mathbf{v}}_i + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i = -\frac{1}{\rho_i} \nabla P_i - \nabla \Phi, \quad (2)$$

$$\Delta \Phi = 4\pi G \sum_{i=1}^N \rho_i, \quad (3)$$

where the spatial derivatives are taken with respect to the physical distance $\mathbf{r} := a(t)\mathbf{x}$. Introducing the perturbation variables δ_i , δP_i , \mathbf{u}_i and $\delta\Phi$ as

$$\rho_i = \bar{\rho}_i + \delta\rho_i = \bar{\rho}_i (1 + \delta_i), \quad P_i = \bar{P}_i + \delta P_i, \quad \mathbf{v}_i = H\mathbf{r} + \mathbf{u}_i, \quad \Phi = \bar{\Phi} + \delta\Phi, \quad (4)$$

derive the Euler and Poisson equations at linear order in the perturbations:

$$\dot{\delta}_i + \frac{1}{a} \nabla \cdot \mathbf{u}_i = -\frac{1}{a} \nabla \cdot (\delta_i \mathbf{u}_i), \quad (5)$$

$$\dot{\mathbf{u}}_i + H\mathbf{u}_i + \frac{1}{a} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i = -\frac{1}{a\bar{\rho}_i} \frac{\nabla \delta P_i}{1 + \delta_i} - \frac{1}{a} \nabla \delta\Phi, \quad (6)$$

$$\frac{1}{a^2} \Delta \delta\Phi = 4\pi G \sum_{i=1}^n \bar{\rho}_i \delta_i, \quad (7)$$

where the spatial derivatives are now taken with respect to the comoving coordinates \mathbf{x} .

Hint: First, you need to find the relation between the operators $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{r}}$, and between the time derivatives $\left. \frac{\partial}{\partial t} \right|_{\mathbf{r}}$ and $\left. \frac{\partial}{\partial t} \right|_{\mathbf{x}}$. Furthermore, you need to use the background relations:

$$\dot{\bar{\rho}}_i + 3H\bar{\rho}_i = 0, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_{i=1}^N \bar{\rho}_i. \quad (8)$$

Exercise 2 (5 points)

We now consider equations (5)–(7) for a single pressure-less fluid, i.e., making the substitutions $\delta_i \equiv \delta$, $\bar{\rho}_i \equiv \bar{\rho}_m$, $\mathbf{u}_i \equiv \mathbf{u}$, $\delta P_i \equiv 0$. We then introduce the divergence and curl of the peculiar velocity \mathbf{u} :

$$\theta := -\frac{1}{a}\nabla \cdot \mathbf{u}, \quad \boldsymbol{\omega} := \frac{1}{a}\nabla \times \mathbf{u}. \quad (9)$$

- a) Use equation (6) to show that the linear-order solution $\boldsymbol{\omega}^{(1)}$ for the curl is decaying as the Universe expands.
- b) Starting from the equations (5)–(7), show that the linear-order solutions $\delta^{(1)}(t, \mathbf{k})$ and $\theta^{(1)}(t, \mathbf{k})$ satisfy

$$\delta^{(1)}(t, \mathbf{k}) = D(t)\hat{\delta}(\mathbf{k}), \quad \theta^{(1)}(t, \mathbf{k}) = HfD(t)\hat{\delta}(\mathbf{k}), \quad f := \frac{d \ln D}{d \ln a}, \quad (10)$$

where the *growth rate* D satisfies the differential equation

$$\ddot{D} + 2H\dot{D} - 4\pi G\bar{\rho}_m D = 0, \quad (11)$$

and is normalized to unity at some early epoch t_0 , i.e., $D(t_0) = 1$ and $\delta(t_0, \mathbf{k}) =: \hat{\delta}(\mathbf{k})$.

Hint: Rewrite equation (5) and the divergence of (6) in terms of θ .

- c) Show that in a matter-dominated universe, the density contrast grows as the scale factor, $D(t) = a(t)$.

Hint: Recall that in a matter-dominated universe $a(t) = (\frac{3}{2}H_0t)^{2/3}$. You can neglect decaying solutions.