Physical Cosmology

Spring Semester 2026
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Problem Set 8

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Exercise 1 (2 points)

Consider a stochastic field $X(\boldsymbol{r})$ living in N-dimensions (i.e. $\boldsymbol{r} = [r_1, \ldots, r_N]$); its power spectrum $P_X(k)$ is defined as:

$$\langle X^*(\boldsymbol{k}) X(\boldsymbol{k}') \rangle = (2\pi)^N \delta_D^{(N)}(\boldsymbol{k} - \boldsymbol{k}') P_X(k); \tag{1}$$

- Explain why a power spectrum $P_X(k) \propto 1/k^N$ is called *scale-invariant.*

  Hint: consider the two-point correlation function, namely the inverse Fourier transform of the power spectrum:

  $$\xi_X(\boldsymbol{r}) = \int \frac{d^N k}{(2\pi)^N} P_X(k) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}; \tag{2}$$

  and see what happens rescaling the distance, i.e. $\boldsymbol{r} \to \alpha \boldsymbol{r}$.

A power spectrum may seem a very abstract object; however, it is possible to build at least some intuition. Broadly speaking, $P_X(k)$ quantify the power associated with a wavenumber k, which is related to an inverse spatial scale as $\lambda \sim 2\pi/k$. Again, in N dimension, for a power law $P_X(k) \sim k^{-N+n}$, if $n > 0$ the spectrum is said to be *blue-tilted* (or simply blue), since there is more power at larger wavenumbers (i.e., smaller scales) than at smaller wavenumbers. On the opposite, if $n < 0$, the spectrum is staid to be *red.* If $n = 0$, as we already said, it is called *scale invariant.* A particular case is $n = N$, where it is called *white noise*, and it looks like the static noise of old cathode tube TVs.

- In Fig. 1 we show four gaussian fields in $N = 2$ dimensions generated using the following power spectra:

  $$\begin{aligned} &(a) : P_X(k) \sim \frac{1}{k^2}, \quad (b) : P_X(k) \sim \frac{1}{k^4}, \\ &(c) : P_X(k) \sim const, \quad (d) : P_X(k) \sim \exp\left(-\frac{1}{2\sigma^2}\log\left(\frac{k}{k_*}\right)^2\right), \quad \sigma = 0.01; \end{aligned} \tag{3}$$

  Can you tell which is which? Assign the correct power spectrum $(a - d)$ to each plot of the correspondent field $(1 - 4$, from top-left clock-wise), writing a brief sentence to justify your matches. For the log-normal peak, can you infer k_* just by eye? Hint: first try to infer λ_*.

- Compute the correlation function of the white noise. What is its physical meaning? Hint: to do this computation, remember how the Dirac delta is defined in terms of plane waves…

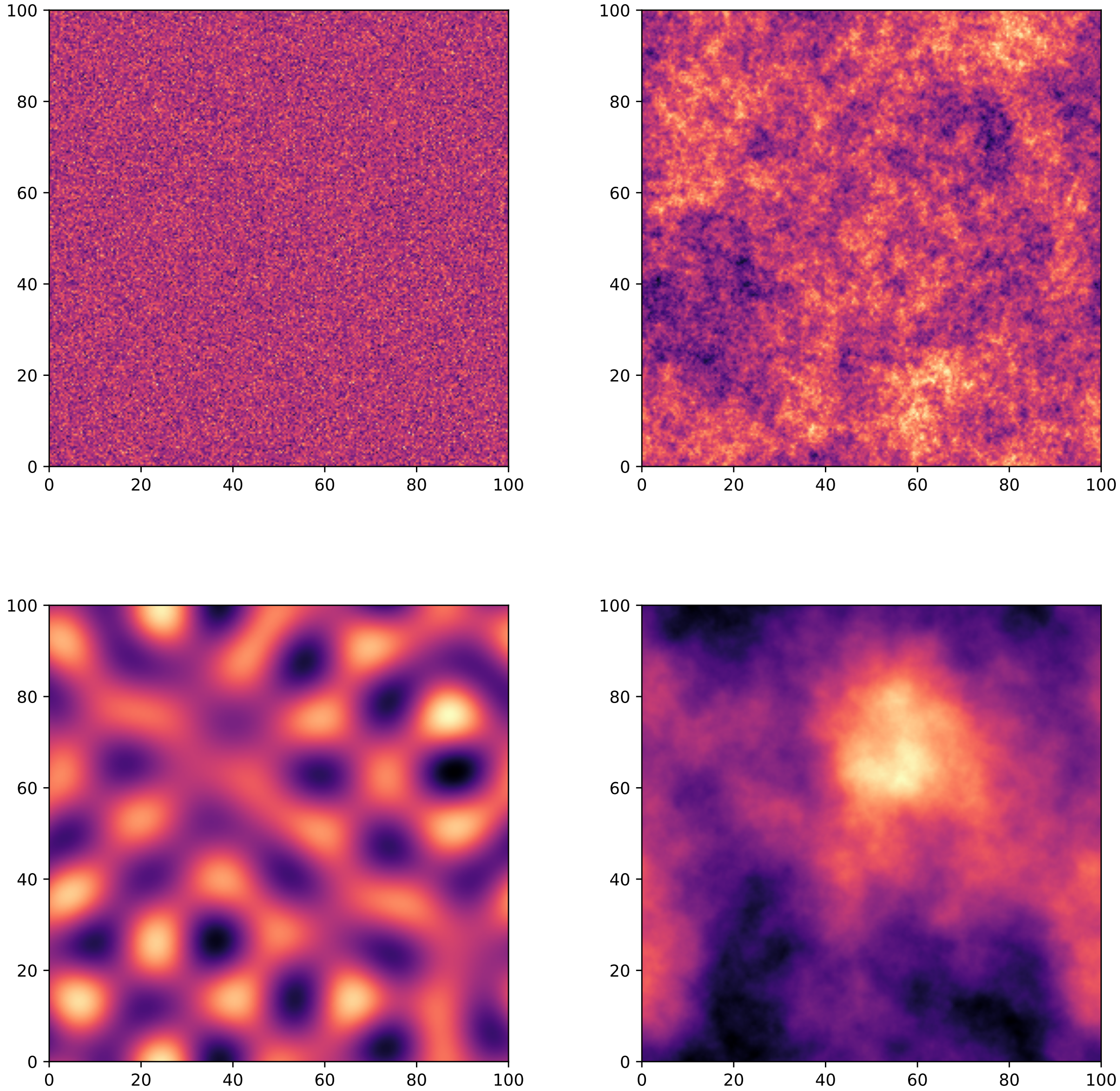

Figure 1: Two-dimensional gaussian fields generated using the power spectra of Eq. 3. Each field is numbered $1-4$, from top-left clock-wise

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Exercise 2 (3 points)

Starting from the relations

$$\delta_s \approx \delta_g - \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}r}\,, \qquad \mathcal{V} = -f\frac{\partial}{\partial r}\nabla^{-2}\delta_m\,, \qquad \delta_g = b\delta_m\,, \tag{4}$$

prove that the following expression for the galaxy density fluctuations in redshift-space holds

$$\delta_s(\mathbf{s}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{s}} \left(b + f\mu_k^2\right)\delta_m(\mathbf{k})\,, \tag{5}$$

and hence, the galaxy power spectrum in redshift-space satisfies the *Kaiser formula*

$$P_s(k,\mu_k) = (b + f\mu_k^2)^2 P_m(k)\,, \tag{6}$$

where b is the galaxy bias and μ_k is the cosine angle between a given Fourier mode and the line-of-sight direction: $\mu_k := \hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{k}}$.

Exercise 3 (5 points)

In this exercise, heavily inspired by Ex. 2 of Sec. 8 (Sec. 7 in the first edition) of Dodelson's book, we will obtain a numerical result for the growth of structures at the linear level, adopting some approximations. First of all, we neglect baryons-photon interactions[1]. We furthermore model both matter (baryons + dark matter) and radiation components (photons + neutrinos) as fluids, which amounts to track for each of them a density (δ for matter, Θ_0 for radiation) and a velocity potential (v for matter, Θ_1 for radiation). In the Newtonian gauge, the relevant system of equations describing the evolution of each Fourier mode k reads ($f' = \partial_\eta f$):

- $\Theta_0' + k\Theta_1 = -\Phi', \quad \Theta_1' - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi;$
- $\delta' + kv = -3\Phi', \quad v' + \mathcal{H}v = -k\Phi;$
- $k^2\Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2[\rho_m\delta + 4\rho_r\Theta_0].$

In this exercise, use the following parameters:

- $\Omega_r = \Omega_\gamma + \Omega_\nu = 9.22\times10^{-5};$
- $\Omega_m = \Omega_b + \Omega_{\mathrm{cdm}} = 0.314;$
- $h = 0.674.$

Considering of course a flat Universe, s.t. $\Omega_\Lambda = 1 - \Omega_r - \Omega_m$. Adopt adiabatic initial conditions for each Fourier mode:

$$\delta^{(i)} = \frac{3}{2}\Phi^{(i)}, \quad \Theta_0^{(i)} = \frac{1}{2}\Phi^{(i)}, \quad \Theta_1^{(i)} = -\frac{k}{6\mathcal{H}}\Phi^{(i)}, \quad v = -\frac{k}{2\mathcal{H}}\Phi^{(i)}$$

With $\mathcal{H}$ computed at the time when the initial conditions are set. Notice that for every k, one need to choose $\eta^{(i)}$ such that $k/\mathcal{H} \ll 1$ (i.e., when the mode is outside the Horizon). Since the equations are linear, one is free to rescale $\Phi^{(i)} = 1$. The true initial conditions can be re-introduced afterwards at the level of the power spectrum.

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[1] On paper, the equations would be just a little bit more complicated; their numerical solution becomes however particularly challenging, as the interaction term is very stiff.

- Numerically, using η as evolution variable is quite uncomfortable. Rewrite this system using as variable the e-folding time $x = \log a$.

- Solve numerically this simplified Einstein-Boltzmann system for various value of $k \in (10^{-4}, 1)$ Mpc^{-1}, setting the initial conditions at $a^{(i)} = 10^{-6}$, where all the modes of interests are outside the Horizon. Use a high order integrator, like Runge-Kutta 4. Hint: pay attention to Θ_0 and Θ_1 after matter-radiation equality. You could find numerical instabilities. It is not a real problem since at this point they don't influence the evolution of the potential anymore; therefore, for $a \gtrsim 10a_{\text{eq}}$ you can simply stop integrating them.

- Plot the temporal evolution of these quantities for some wavenumbers; try to see if you can reproduce Fig. 8.6 (Fig. 7.6 in the first edition) in Dodelson's book. On very small scales ($k \sim 1-10$ Mpc^{-1}), during radiation domination your solution should be identical to the following analytic expression:

$$\Phi(\eta, k) = \frac{3j_1(k\eta/\sqrt{3})}{k\eta/\sqrt{3}}\Phi(\eta^{(i)}, k); \tag{7}$$

  Compare it with the numerical result, recalling also that during RD $\eta = 1/\mathcal{H}$.

- Compute the matter power spectrum at $z = 0$; to do so, consider first that at late times the Poisson equation becomes:

$$k^2\Phi = 4\pi G a^2 \rho_m \delta; \tag{8}$$

  Use this equation to obtain a relation between $P_\Phi(z, k)$ and $P_\delta(z, k)$. The primordial power spectrum reads

$$P_\Phi^{(i)}(k) = \frac{4}{9} \times \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_{\text{pivot}}}\right)^{n_s - 1}, \tag{9}$$

  with $A_s = 2.1 \times 10^{-9}$, $n_s = 0.966$, $k_{\text{pivot}} = 0.05$ Mpc^{-1}. Compute and plot the power spectrum both for Φ and δ.

- Finally, you can compare your results to the solution of the full Einstein-Boltzmann system computed by the code CLASS (http://class-code.net/). Its Python wrapper can be easily installed in a Linux system by typing *pip install classy* in the terminal. You can plot the CLASS linear matter power spectrum with the following snippet:

```
import classy
from classy import Class
LambdaCDM = Class()
#setting the relevant cosmological parameters
LambdaCDM.set({'Omega_b':0.049,'Omega_cdm':0.265,'h':0.6732,
'ln10^{10}A_s':3.0448,'n_s':0.96605})
#telling class that I want the matter power spectrum
LambdaCDM.set({'output':'mPk','P_k_max_1/Mpc':10})
#run class
LambdaCDM.compute()
#wrapping out the result of the computation:
#this gives the matter power spectrum at a given z
P_m_class = np.vectorize(LambdaCDM.pk, excluded = 'z')
```

```
k = 10**np.arange(-4, 1, 0.01) #working in physical Mpc^{-1}
plt.loglog(k, P_m_class(k, 0))
plt.ylabel('Power Spectrum P(k) [(Mpc)$^3$]')
plt.xlabel('Wavenumber $k$ [Mpc$^{-1}$]')
plt.title('Matter Power Spectrum at $z = 0$')
plt.show()
```

Does the overall shape agree? What is the main difference with your result?