Physical Cosmology

Problem Set 9

Spring Semester 2026
Prof. J. Yoo

Giovanni Piccoli

Issued: 12.5.2026
Due: 19.5.2026

Exercise 1 (4 points)

a) Derive the gauge transformations of the metric and matter scalar perturbations α, β, φ, γ, U and v

b) Check that the scalar variables φ_v, φ_χ, v_χ, φ_δ, δ_v and α_χ defined in the lecture notes are gauge-invariant.

Hint: You can solve a) by using that $\delta_\xi g_{\mu\nu}(x) = -\mathcal{L}_\xi g_{\mu\nu}$ and $\delta_\xi u^\mu = -\mathcal{L}_\xi u^\mu$ as described in the lecture notes. Having solved a), the fact that φ_v, φ_χ, v_χ and α_χ are gauge-invariant follows easily. To show that φ_δ and δ_v are gauge-invariant, you additionally need to consider the gauge-transformation of the stress-energy tensor, $\delta_\xi T_{\mu\nu}$, to determine how δP and $\delta\rho$ transform. Assume a single fluid with constant equation of state, so that $\bar{\rho} \propto a^{-3(1+w)}$.

Exercise 2 (4 points)

Consider a spatially flat inhomogeneous universe described by a metric fixed to the conformal Newtonian gauge:

$$\mathrm{d}s^2 = -a^2 e^{2\Psi}\mathrm{d}\eta^2 + a^2 e^{2\Phi}\delta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,. \tag{1}$$

Assuming the energy-momentum tensor of an imperfect fluid with equation of state w, derive the following relations, expanding Einstein's equations to linear order in perturbation theory:

$$\nabla^2\Phi = -4\pi G a^2 \bar{\rho}[\delta + 3\mathcal{H}v(1+w)]\,, \qquad \Phi + \Psi = -8\pi G\,\Pi\,. \tag{2}$$

Hint: : the Poisson-like equation can be obtained combining the $\eta\eta$ and the $\alpha\eta$ components of the Einstein's equation, while the constraint $\Phi + \Psi = -8\pi G\Pi$ (sometimes called gravitational slip equation) can be obtained from the $\alpha\beta$ components. This last computation can simplified notably if the terms proportional to $\delta_{\alpha\beta}$ are neglected. Why is this legit?

Exercise 3 (2 points)

Show that on scales $k \ll \mathcal{H}$ the comoving-gauge curvature perturbation $\varphi_v = \Phi - \mathcal{H}v$ is related to the Newtonian potential Φ as

$$\varphi_v = \frac{5+3w}{3(1+w)}\Phi\,. \tag{3}$$

Assume that there is no anisotropic stress, $\Pi = 0$, and use that $\dot{\Phi} = 0$ on super-Hubble scales.
Hint: Use the relation $12\pi G(\rho + p)av = 3H\Psi - 3\dot{\Phi}$ as given in the Lecture Notes.