Physical Cosmology

Problem Set 10

Spring Semester 2026
Prof. J. Yoo

Giovanni Piccoli

Issued: 26.5.2026
Due: 2.6.2026

Exercise 1 [5 points]

a) Starting from the energy-momentum tensor of the inflaton field,

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi - g_{\mu\nu}V(\phi)\,, \tag{1}$$

and considering scalar perturbations in the conformal Newtonian gauge,

$$\mathrm{d}s^2 = -a^2(1+2\alpha)\mathrm{d}\eta^2 + a^2\delta_{\alpha\beta}(1+2\varphi)\mathrm{d}x^\alpha\mathrm{d}x^\beta\,, \qquad u^\mu = \frac{1}{a}(1-\alpha, -U^{,\alpha})\,, \tag{2}$$

derive the linear-order fluid quantities:

$$\delta\rho = \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 + \partial_\phi V\delta\phi\,, \qquad \delta p = \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 - \partial_\phi V\delta\phi\,. \tag{3}$$

b) Derive the equation of motion for the linear perturbation in the inflaton field:

$$\delta\ddot\phi + 3H\delta\dot\phi + \left(\partial_\phi^2 V + \frac{k^2}{a^2}\right)\delta\phi = \dot\phi(\dot\alpha + \kappa) + (2\ddot\phi + 3H\dot\phi)\alpha\,. \tag{4}$$

Note that this equation is in Fourier space and k is the wavenumber. Also, recall that $\kappa = 3H\alpha - 3\dot\varphi$, and that $\alpha = -\varphi$ in a universe without anisotropic stress.

Exercise 2 [5 points]

Assume that the potential energy of the inflation field reads

$$V(\phi) = \lambda\,\phi^4\,, \tag{5}$$

and recall the slow roll condition

$$\epsilon(\phi) = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{1}{V}\frac{\partial V}{\partial\phi}\right)^2 \ll 1\,, \qquad M_{\mathrm{Pl}} = \frac{1}{\sqrt{8\pi G}}\,, \tag{6}$$

where the field is rolling towards $\phi = 0$ from the positive side.

a) At what value of ϕ is this condition broken? Assuming that inflation ends at $\epsilon = 1$, show that the number of e-foldings,

$$\ell(t_i) = \ln\left(\frac{a[t(\epsilon=1)]}{a(t_i)}\right)\,, \tag{7}$$

occuring during inflation is given by

$$\ell(t_i) = \pi G \left(\phi_i^2 - 8M_{\mathrm{Pl}}^2\right) . \tag{8}$$

for some initial value $\phi_i = \phi(t_i)$.

Hint: What is $\dot{\ell}$? Recall the evolution equation for the scalar field, and that we can neglect $\ddot{\phi}$ in the slow-roll approximation. Furthermore, recall that H is related to the energy density of the universe via the first Friedmann equation. What dominates the energy density during inflation?

b) Starting from equation (8) and the first Friedmann equation, show that

$$\phi = \phi_i \exp\left[-\sqrt{\frac{16\lambda M_{\mathrm{Pl}}^2}{3}}\,(t - t_i)\right] . \tag{9}$$

Then, again starting from the first Friedmann equation and using the solution above for ϕ, show that

$$\log\left(\frac{a(t)}{a_i}\right) = \frac{\phi_i^2}{8M_{\mathrm{Pl}}^2}\left[1 - \exp\left(-\sqrt{\frac{64\lambda M_{\mathrm{Pl}}^2}{3}}\,(t - t_i)\right)\right] . \tag{10}$$

c) Expand the solution above for small time differences $t - t_i$ and show that inflation is approximately exponential in the initial stage, i.e., $a \propto e^{Ht}$. Show that the constant ω is equal to the Hubble parameter during inflation.