Carroll 9712019 p. 217 —

$G_{\mu\nu}$. $g_{\mu\nu}$: dynamic variable vs $\eta_{\mu\nu}$ static
spacetime ⇔ matter

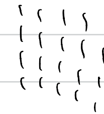
LHS of Einstein EQ.

Robertson-Walker metric

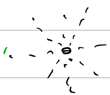
1920~30

* Homogeneity & isotropy : mathematical definition is there, but intuitively
  cosmological principle
  for 3-space, not for 4-spacetime.

homo. not iso



iso. not homo.



* time evolution is not symmetric. no off-diagonal

$$\Rightarrow ds^2 = -dt^2 + a^2(t)\,\bar{g}_{ij}\,dx^i dx^j \qquad i,j = 1,2,3$$

↳ 3 metric w/o a

a : scale factor x^i : comoving coord

largest possible # of   6 : for $n=3$. 3 rot. 3 trans.

* maximally symmetric space $R_{ijkl} = K(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})$, $n(n+1)/2$ Killing vectors

* Robertson-Walker metric $ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$

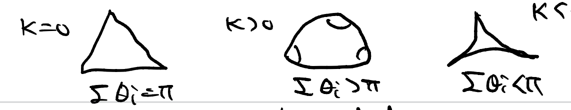
$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$
K : constant $\sim L^{-2}$

→ solution of Einstein Eq for expanding universe

homo. iso expansion
$a(t)$ affect ds^2 : real

$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ $g^{RW}_{\mu\nu}$

$g^{\mu\nu}$: inverse. $g^\mu{}_\nu = \delta^\mu{}_\nu$



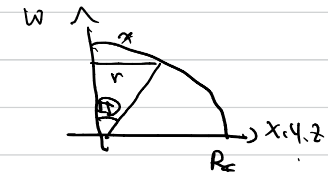
bounded space

* How to find $\tilde{g}_{ij}$? $\quad ds_3^2 = \tilde{g}_{ij}\, dx^i dx^j = A(r)dr^2 + r^2 d\Omega^2$

$\tilde{g}_{ij}$ cannot depend on direction. depend only on r

intuitively. Euclidean. sphere or hyperbola

* consider 3-sphere in 4D w/ radius R (fixed)



$0 < \frac{1}{K} = R_K^2 = x^2 + y^2 + z^2 + \omega^2,$ $\quad x = R_K \sin\Theta \cdot \sin\theta \cdot \cos\phi \ \cdots$

$r^2 \equiv x^2 + y^2 + z^2 = (R_K \sin\Theta)^2 \le R_K^2$

4th coord

$\omega = R_K \cos\Theta$

$$\therefore \tilde{g}_{ij}\, dx^i dx^j = \underbrace{dx^2 + dy^2 + dz^2}_{= dr^2 + r^2 d\Omega^2} + d\omega^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{R^2 - r^2}$$

$\chi_K \equiv R_K \Theta$ are length

$$= \frac{dr^2}{1 - r^2/R^2} + r^2 d\Omega^2$$

$$= \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 = d\chi_K^2 + r^2 d\Omega^2$$

valid for K<0

$$d\chi_K \equiv \frac{dr}{\sqrt{1 - Kr^2}}$$

summary

$K=0$ flat. $\quad ds^2 = -dt^2 + a^2(dr^2 + r^2 d\Omega^2)$

$K = \pm 1$. open. closed $\quad ds^2 = -dt^2 + a^2(d\chi_K^2 + r^2 d\Omega^2)$

$$\begin{cases} r = \chi & \text{Euclidean 3-space} \\ \underline{r = \frac{1}{\sqrt{K}} \sin\sqrt{K}\chi_K} & \text{3-sphere in 4D} \\ \underline{r = \frac{1}{\sqrt{-K}} \sinh\sqrt{-K}\chi_K} & \text{3-hyperbole in 4D} \end{cases}$$

* open hyperbolic univ ($K<0$)

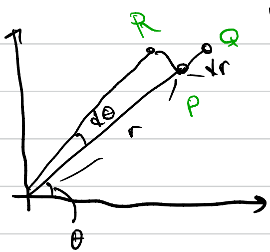
$0 > \frac{1}{K} \approx -R_K^2 = x^2 + y^2 + z^2 - \omega^2$

$r = R_K \sinh \Theta$. $x_K = R_K \Theta$. $\omega = R_K \cosh \Theta$

* $K \to \frac{K}{|K| L^2}$. $r \to \sqrt{|K|} r L$. $a \to a / \sqrt{|K|} L$ metric is invariant $\therefore K = 0, \pm 1$

for a unit length scale

but $K = \frac{1}{R_K^2} = \# \; a^{-2}$



$r = (x, y, z)$

radial length : $a\,dr$

transverse length : $a r d\Omega$

$P = (t, x, y, z) = (t, r, \theta, \phi)$

$Q = (t, r + dr, \theta, \phi)$

$R = (t, r, \theta + d\theta, \phi)$