3 Probes of Inhomogeneity

In cosmology, the initial condition is set in the early Universe with Gaussian random fluctuations in Fourier space, as the quantum fluctuations in vacuum are stretched beyond the horizon scales during the inflationary epoch. Since the Gaussian distribution is completely specified by the variance, the power spectrum contains all the information in the early Universe. However, the nonlinear growth in the late time complicates the interpretations. Here we focus on the linear theory and study various ways to measure the two-point statistics.

3.1 Basic Formalism

3.1.1 Two-Point Correlation Function and Power Spectrum

• ***3D information***.— Suppose that we use some cosmological probes such as galaxies and measure, say, the matter density fluctuation δ. Now imagine we have measurements of such probe over all positions $\mathbf{x}$. We can then measure the two-point correlation function $\xi(r)$ and its Fourier transform, the power spectrum $P(k)$:

$$\xi(\mathbf{r}) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}P(\mathbf{k}) , \qquad \big\langle\delta(\mathbf{k})\delta(\mathbf{k}')\big\rangle = (2\pi)^3\delta^{3D}(\mathbf{k}+\mathbf{k}')P(\mathbf{k}) , \tag{3.1}$$

and the variance is

$$\sigma^2 = \xi(0) = \int d\ln k\, \frac{k^3}{2\pi^2}P(k) , \qquad \Delta_k^2 := \frac{k^3}{2\pi^2}P(k) , \tag{3.2}$$

where Δ_k^2 is the dimensionless power spectrum and it is the contribution to the variance per each log k.

Note that different Fourier modes are not correlated in the initial condition and the power spectrum characterizes the Gaussian distribution at each Fourier mode.[1] Therefore, using cosmological probes, we need to measure the distribution map $\delta(\mathbf{x})$ and compute the two-point correlation function or the power spectrum.

• ***1D information***.— Spectroscopic measurements of distant quasars yield the density fluctuations of neutral hydrogens along the line-of-sight. In this case, we probe the density fluctuation, but only in terms of the line-of-sight separation, say, z-direction. Given the 1D map, we can measure the 1D correlation function, and it is related to the power spectrum as

$$\xi_{\rm 1D}(z) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{z})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{ik_z z}P_{\rm 3D}(\mathbf{k}) , \tag{3.3}$$

where the separation vector is $\mathbf{z} = z\hat{\mathbf{z}}$ along the line-of-sight direction. We can also define 1D power spectrum that is a Fourier counterpart of the 1D correlation function:

$$P_{\rm 1D}(k_z) \equiv \int dz\, e^{-ik_z z}\xi_{\rm 1D}(z) = \int \frac{d^3\mathbf{k}'}{(2\pi)^3}P(\mathbf{k}')\delta^{1D}(k'_z - k_z) = \int_0^\infty \frac{dk'_\parallel}{2\pi}\, k'_\parallel P(k'_\parallel, k_z) = \int_{k_z}^\infty \frac{dk}{2\pi}\, kP(k) , \tag{3.4}$$

where we again assumed that the 3D power spectrum is isotropic. The 1D power spectrum is the projection of the 3D power spectrum over 2D Fourier space. For sufficiently high k, it is largely one-to-one, though it has bias (called aliasing) on low k. This relation can be inverted as

$$P(k) = -\frac{2\pi}{k}\frac{d}{dk}P_{1D}(k) , \tag{3.5}$$

and the dimensionless power spectrum in 1D is

$$\sigma_{\rm 1D}^2 = \int d\ln k_z\, \frac{k_z}{\pi}P_{\rm 1D}(k_z) , \qquad \Delta_{k,\rm 1D}^2 := \frac{k_z}{\pi}\, P_{\rm 1D}(k_z) . \tag{3.6}$$

[1] However, as we studied in Section 2, the nonlinear evolution results in the mode coupling.

• ***2D information***.— Though the distance in cosmology is difficult to measure, it is easy to have 2D information on the sky. We define the 2D power spectrum in a similar way as the Fourier counterpart of the 2D correlation function:

$$P_{\rm 2D}(k_x,k_y)\equiv\int dx\int dy\, e^{-ik_xx}e^{-ik_yy}\xi_{\rm 2D}(x,y)=\int\frac{dk_z'}{2\pi}\,P(k_x,k_y,k_z')=\frac{1}{\pi}\int_{k_\perp}^\infty dk'\,\frac{k'P(k')}{\sqrt{k'^2-k_\perp^2}}\,, \tag{3.7}$$

where $k_\perp^2=k_x^2+k_y^2$. The 2D power spectrum is the projection over 1D Fourier space, and its similar relation to the 3D power spectrum exists. This relation can be again inverted by using the (non-trivial) Abell integral as

$$P(k)=-\frac{2}{k}\int_k^\infty dk_\perp\frac{P_{\rm 2D}(k_\perp)}{\sqrt{k_\perp^2-k^2}}\,, \tag{3.8}$$

and the dimensionless power spectrum in 2D is then

$$\sigma_{\rm 2D}^2=\int d\ln k_\perp\,\frac{k_\perp^2}{2\pi}\,P_{\rm 2D}(k_\perp)\,,\qquad\qquad \Delta_{k,{\rm 2D}}^2=\frac{k_\perp^2}{2\pi}P_{\rm 2D}(k_\perp)\,. \tag{3.9}$$

The projection-slice theorem says Fourier transformation of the projection is the slice of its Fourier transformation. It means exactly what we derived here. A similar relation holds in configuration space. The projected correlation function is related as

$$w_p(r_p):=\int dz\,\xi(r_p,z)=2\int_{r_p}^\infty dr\,\frac{\xi(r)}{\sqrt{r^2-r_p^2}}\,,\qquad\qquad \xi(r)=-\frac{1}{\pi}\int_r^\infty dr_p\,\frac{w_p(r_p)}{\sqrt{r_p^2-r^2}}\,. \tag{3.10}$$

3.1.2 Angular Correlation and Angular Power Spectrum

We briefly covered the statistics in a flat space. However, the sky is round, and we can only make observations by measuring the light signals. The cosmic microwave background anisotropies, for example, are measured only as a function of the angular position on the sky at the Earth. In cosmology, we often have angular information, but no distance measurements. Since this measurement $\delta(\hat\theta)$ is defined on a unit sphere, we can decompose it in terms of spherical harmonics as

$$\delta(\hat\theta):=\sum_{lm}a_{lm}Y_{lm}(\hat\theta)\,,\qquad\qquad a_{lm}\equiv\int d^2\hat\theta\,Y_{lm}^*(\hat\theta)\delta(\hat\theta)\,, \tag{3.11}$$

where we have discrete sum, instead of integral in Fourier space. The reality condition for δ imposes

$$a_{lm}^*=(-1)^m a_{l,-m}\,. \tag{3.12}$$

Similar to the case in 3D, we can define the angular correlation function and its Fourier counterpart:

$$w(\hat\gamma)=\left\langle\delta(\hat\theta)\delta(\hat\theta+\hat\gamma)\right\rangle=\sum_{lm}C_lY_{lm}(\hat\theta+\hat\gamma)Y_{lm}^*(\hat\theta)=\sum_l\frac{2l+1}{4\pi}C_lL_l(\cos\gamma)\,, \tag{3.13}$$

where we used the relation

$$\langle a_{lm}a_{l'm'}^*\rangle=\delta_{ll'}\delta_{mm'}C_l=\sum_m\frac{|a_{lm}|^2}{2l+1}\delta_{ll'}\delta_{mm'}\,, \tag{3.14}$$

and the Legendre polynomial is related to the spherical harmonics as

$$L_l(\mu)=\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,Y_{lm}(\hat\theta_1)Y_{lm}^*(\hat\theta_2)\,,\qquad\qquad \mu=\hat\theta_1\cdot\hat\theta_2\,. \tag{3.15}$$

The angular power spectrum can be obtained as

$$C_l=2\pi\int_{-1}^1 d\mu\,L_l(\mu)w(\theta)\,. \tag{3.16}$$

3.1.3 Flat-Sky Approximation

When the area of interest is relatively small in the sky, we can use the flat-sky approximation, and it often overlaps with the distant-observer approximation, in which the observer is so far away that the position angle is virtually constant, compared to their relative positions. In this case, the angular correlation and its power spectrum are closely related to those in flat space.

Now consider the 2D correlation function $\xi_{\rm 2D}$ and 2D power spectrum $P_{\rm 2D}(k)$:

$$\xi_{\rm 2D}(x,y)=\int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}P_{\rm 2D}(k_\perp)=\int\frac{d^2l}{(2\pi)^2}\,e^{i\mathbf{l}\cdot\boldsymbol{\theta}}P_l\,,\qquad P_l\equiv\frac{1}{r^2}P_{\rm 2D}\left(k_\perp=\frac{l}{r}\right)\,,\tag{3.17}$$

where we used $\mathbf{x}_\perp=r\boldsymbol{\theta}$ and defined the (flat-sky) angular power spectrum P_l. Note that the 2D power spectrum is dimensionful, but the angular power spectrum is dimensionless. Given the radial distance r, the 2D correlation function $\xi_{\rm 2D}$ can be considered as the angular correlation function, and assuming that the angular power spectrum is independent of its direction, we can further simply the relation:

$$w(\theta)=\xi_{\rm 2D}(r\boldsymbol{\theta})=\int\frac{dl}{2\pi}\,lP_l\,J_0(l\theta)\,,\tag{3.18}$$

where J_0 is the Bessel function. The (full-sky) angular power spectrum is then obtained as

$$C_l=2\pi\int d\mu\,L_l(\mu)w(\theta)\simeq\sum_{l'}l'P_{l'}\frac{2\delta_{ll'}}{2l+1}\simeq P_l\,,\tag{3.19}$$

where we manipulated the Bessel function for $l\gg1$ and $\theta\ll1$

$$J_0(l\theta)=\frac{1}{\pi}\int_0^\pi d\phi\,e^{il\theta\cos\phi}\simeq\frac{1}{\pi}\int_0^\pi d\phi\left(1+\frac{il\theta\cos\phi}{l}\right)^l\simeq\frac{1}{\pi}\int_0^\pi d\phi\,(\cos\theta+i\sin\theta\cos\phi)^l=L_l(\cos\theta)\,.\tag{3.20}$$

The angular quantities such as $w(\theta)$ and C_l are defined on a unit sphere, whereas the 2D quantities such as $\xi_{\rm 2D}$ and $P_{\rm 2D}$ are defined on a 2D flat space. Hence, the former is related to each other via spherical harmonics, and the latter via Fourier transformation. But they are defined in a way that the angular power spectrum C_l and its flat-sky counterpart P_l are equivalent in the limit of small sky.

3.1.4 Projection and Limber Approximation

We often measure some angular quantities in cosmology, but they are often the projection of the 3D quantities. For example, one can measure the angular map in a given galaxy survey, but the angular quantity $\delta_2(\theta)$ we measure indeed derives from the 3D quantity $\delta(\mathbf{x})$, but projected along the line-of-sight direction with some weighting $W(r)$:

$$\delta_2(\theta)=\int dr\,W(r)\delta(\mathbf{x})\,,\qquad\mathbf{x}=(r\boldsymbol{\theta},r)\,.\tag{3.21}$$

The weight function is normalized to unity and it is often parametrized in terms of redshift as

$$1=\int dr\,W_r(r)=\int dz\,W_z(z)\,,\tag{3.22}$$

where the weight function can be dimensionful, depending on its parametrization. The angular correlation is then

$$w(\theta)=\langle\delta_2(0)\delta_2(\theta)\rangle=\int dr_1\,W(r_1)\int dr_2\,W(r_2)\,\xi_{\rm 3D}(\mathbf{r})\,,\qquad\mathbf{r}\simeq(r_1\boldsymbol{\theta},r_2-r_1)\,,\tag{3.23}$$

where we assumed the flat-sky approximation. The angular power spectrum is

$$P_l=\int\frac{d^2\theta}{(2\pi)^2}e^{-i\mathbf{l}\cdot\boldsymbol{\theta}}w(\theta)=\int dr_1\,W(r_1)\int dr_2\,W(r_2)\int\frac{dk_z}{2\pi}\,e^{-ik_z(r_2-r_1)}\frac{1}{r_1^2}P\left[\left(k_\perp=\frac{l}{r_1},k_z\right)\right]\,.\tag{3.24}$$

Since we work in the flat-sky regime (or the distant observer), the radial distance is far larger than the transverse separation $r \gg r\boldsymbol{\theta}$. Hence, we have the separation of scale in Fourier space

$$k_\perp = \frac{l}{r} \simeq \frac{1}{r\theta} \gg \frac{1}{r} \simeq k_z \, , \tag{3.25}$$

and the power spectrum can be approximated as

$$P(k) \simeq P(k = k_\perp) + \frac{dP}{dk_z} k_z + \mathcal{O}(k_z)^2 \, , \qquad \frac{dP}{dk_z} k_z = \frac{dP}{dk} \frac{k_z^2}{k} \simeq \frac{P}{k^2 r^2} \simeq \frac{P}{l^2} \ll P \, . \tag{3.26}$$

Keeping the leading term in the power spectrum, we can integrate over k_z and approximate the angular power spectrum as

$$P_l \simeq \int dr \frac{W^2(r)}{r^2} \, P \left[\left(k_\perp = \frac{l}{r_1}, k_z \right) \right] \, . \tag{3.27}$$

This is sometimes called the Limber approximation. When the window function is sufficiently broad compared to the coherent length scale of the correlation, the Limber approximation is very accurate and useful. Its relation to the angular correlation is

$$w(\theta) = \int \frac{dl}{2\pi} \, l P_l \, J_0(l\theta) \equiv \int dk \, k P(k) F(k, \theta) \, , \tag{3.28}$$

where we defined the kernel

$$F(k, \theta) := \int \frac{dr}{2\pi} W^2(r) J_0(kr\theta) = \frac{1}{k} \int \frac{dl}{2\pi} W^2 \left(\frac{l}{k} \right) J_0(kr\theta) \, . \tag{3.29}$$

3.2 Matter Power Spectrum

The evolution equation (2.19) for the matter density growth yields simple solutions for the matter-dominated era (MDE) and the radiation-dominated era (RDE):

$$D_{\rm mde} \propto a \, , \qquad D_{\rm rde} \propto a^2 \, , \tag{3.30}$$

where the solution can be verified by direct substitutions. The growth in MDE is scale-independent, such that the perturbations on all scales grow equally in proportion to a. However, the growh in RDE is a bit different. In fact, the evolution equation is not valid in RDE, as we derived the equation by assuming the presureless medium, whereas the Universe in RDE is dominated by radiation (with large pressure). On small scales, the matter density cannot grow due to the radiation pressure, so no growth during the RDE, but on large scales (larger than the horizon scale in RDE) the evolution equation is valid, as the effect of pressure is negligible.[2]

Therefore, the matter density fluctuations on large scales can continuously growth throughout the periods of RDE and MDE, while those on small scales cannot grow, once they enter the horizon during RDE (remember that all modes were outside the horizon after inflation). So the scale of comparison is naturally the equality scale $k_{\rm EQ}$, where the epoch of equality is defined as $\bar\rho_m = \bar\rho_r$ at $t_{\rm EQ}$ (or $z_{\rm EQ} \simeq 3000$). The modes larger than the equality scale $k_A < k_{\rm EQ}$ stay outside the horizon during RDE, so that they continue to grow until today:

$$\delta(k_A; t_0) = \delta(k_A; t_i) \left(\frac{a_{\rm EQ}}{a_i} \right)^2 \left(\frac{a_0}{a_{\rm EQ}} \right) \, , \tag{3.31}$$

where t_i is the initial time after inflation and t_0 is the present time. Similarly for the mode $k_{\rm EQ}$, and hence the ratio of the power spectra at k_A and $k_{\rm EQ}$ is

$$\frac{P(k_A; t_0)}{P(k_{\rm EQ}; t_0)} = \left[\frac{P(k_A)}{P(k_{\rm EQ})} \right]_{t_i} = \left(\frac{k_A}{k_{\rm EQ}} \right)^{n_s} \, . \tag{3.32}$$

For a mode $k_A < k_{\rm EQ}$, the power spectrum is essentially primordial, up to the amplitude.

[2]For calculations outside the horizon, we need relativistic equations, so the validity of our Newtonian equation in this regime is a bit of coincident.

The modes smaller than the equality scale $k_B > k_{\rm EQ}$ start outside the horizon during RDE and grow for some time. However, after they enter the horizon during RDE, their growth freezes, until the Universe becomes MDE, so that their growth is

$$\delta(k_B;t_0)=\delta(k_B;t_i)\left(\frac{a_\star}{a_i}\right)^2\left(\frac{a_0}{a_{\rm EQ}}\right) , \tag{3.33}$$

where $a_\star$ is the time at which the modes k_B enter the horizon, i.e., $k_B=\mathcal{H}(t_\star)$. During RDE, the Hubble parameter H is proportional to a^{-2}, and the conformal Hubble parameter $\mathcal{H}:=aH$. Therefore, the scale factor at the horion crossing is $a_\star\propto 1/k_B$, and the ratio of the power spectra at k_B and $k_{\rm EQ}$ is then

$$\frac{P(k_B;t_0)}{P(k_{\rm EQ};t_0)}=\left[\frac{P(k_B)}{P(k_{\rm EQ})}\right]_{t_i}\left(\frac{a_\star}{a_{\rm EQ}}\right)^4=\left(\frac{k_B}{k_{\rm EQ}}\right)^{n_s-4} . \tag{3.34}$$

Compared to the initial condition, the growth in the matter power spectrum is suppressed on small scales due to the radiation pressure.

In fact, the dark matter density can still grows in RDE on small scales, as they do not feel the radiation pressure. However, the growth is indeed slowed due to the rapid Hubble expansion in RDE, so that the growth is only logarithmic, and the suppression on small scales is in fact $(\ln k/k^2)^2$, instead of $(1/k^2)^2$. This growth of dark matter density during RDE is important for structure formation today. CMB observations show that $\delta T/\bar T\sim\delta_b\sim 10^{-5}$ at $z=1100$. According to linear theory, this small matter density fluctuation can only grow by $D(z=1000)/D(z=0)\approx 1000$ to $\delta_b\sim 0.01$, which is not enough to form any nonlinear structure today. With dark matter already growing for a while, baryons can catch up quickly, once released from CMB.

3.3 Peculiar Velocity

3.3.1 Observations of Peculiar Velocities

The distant objects such as galaxies are receding from us due to the Hubble expansion, and this expansion (or the receding velocity v) is measured by the redshift z of the known line-emissions from the distant objects:

$$1+z=\frac{\lambda_{\rm obs}}{\lambda_{\rm rest}} . \tag{3.35}$$

If we interpret this measurement as the Doppler effect, we obtain the receding velocity

$$1+z\approx 1\pm\frac{v}{c} , \qquad\qquad v\equiv cz . \tag{3.36}$$

What happen to the objects at $z>1$? We can use the relativistic Doppler effect to obtain the receding velocity less than the speed of light, but this velocity is not really the physical velocity of the objects. The dominant contribution to the redshift is indeed the expansion of the Universe.

However, in addition to the Hubble expansion v_H, these objects are also moving, and this motion is referred to as the peculiar motion v_p. Due to the peculiar motion, the Doppler effect also contributes to the receding velocity, and the receding velocity can be written as

$$v=v_H+v_p , \qquad\qquad v_H=Hd=\mathcal{H}r , \tag{3.37}$$

where the object is assumed to be at the physical distance d (or comoving distance r). The redshift measurements (or the receding velocity) yield only the radial component of the receding velocity. The tangential peculiar motion can be measured. However, since this requires measurements of the angular motion of the distant objects over a long time, it is practically limited to the nearby objects such as stars in our own Galaxy. The measurements of the radial peculiar velocity also requires precise measurements of the distance d, which is very difficult in cosmology. For example, 10% error in the distance measurements at $d=50\ h^{-1}{\rm Mpc}$ yields the error of $500\ {\rm km\,s^{-1}}$ in the peculiar velocity measurement. Therefore, the peculiar velocity measurements are also limited to the low-redshift objects.

- receding velocity at $z>1$, gauge ambiguity, SN Ia or SZ measurements
- HW: derive Eq. (3.36) from Eq. (3.37)

3.3.2 Linear Theory

In Chapter 2, we learned that the velocity divergence is related to the density fluctuation:

$$\theta \equiv -\frac{1}{a}\nabla\cdot\boldsymbol{v} = Hf\delta\ . \tag{3.38}$$

Ignoring the vector perturbation, the velocity can be expressed in terms of the velocity potential U as

$$\boldsymbol{v} = -\nabla U\ , \qquad \theta = \frac{1}{a}\Delta U\ , \qquad U = \mathcal{H}f\Delta^{-1}\delta\ , \qquad \boldsymbol{v} = -\mathcal{H}f\nabla\Delta^{-1}\delta\ . \tag{3.39}$$

In Fourier space, the inverse Laplacian can be readily manipulated, and the velocity vector becomes

$$U(\mathbf{k}) = -\frac{\mathcal{H}f}{k^2}\delta(\mathbf{k})\ , \qquad \boldsymbol{v}(\mathbf{k}) = i\mathbf{k}\frac{\mathcal{H}f}{k^2}\delta(\mathbf{k})\ , \tag{3.40}$$

where we suppressed the time-dependence, for example,

$$\delta(\mathbf{k}) = D(t)\hat{\delta}(\mathbf{k},t_o)\ . \tag{3.41}$$

3.3.3 Two-Point Correlation of the Peculiar Velocities

Given the peculiar velocity (vector) field, we can compute the two-point correlation function of the peculiar velocities at two different points:

$$\Psi_{ij}(\mathbf{r}) = \langle v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})\rangle = \int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathcal{H}^2f^2P_m(k)\,\frac{k_ik_j}{k^4} \equiv \Psi_\perp(r)(\delta_{ij}-\hat{r}_i\hat{r}_j)+\hat{r}_i\hat{r}_j\Psi_\parallel(r)\ , \tag{3.42}$$

where $\hat{r}_i = \mathbf{r}_i/|\mathbf{r}|$, the matter density power spectrum is

$$\left\langle\delta(\mathbf{k})\delta(\mathbf{k}')\right\rangle = (2\pi)^3\delta^D(\mathbf{k}+\mathbf{k}')P_m(\mathbf{k})\ , \tag{3.43}$$

and we defined two velocity correlation functions, $\Psi_\parallel$ along the connecting direction and $\Psi_\perp$ perpendicular to it:

$$\Psi_\perp := \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\,\frac{j_1(kr)}{kr}\ , \qquad \Psi_\parallel := \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\left[j_0(kr)-\frac{2j_1(kr)}{kr}\right] = \frac{d}{dr}\left[r\Psi_\perp(r)\right]\ , \tag{3.44}$$

where we used

$$\int d\mu\,e^{\pm i\mu x} = 2j_0(x)\ , \qquad \int d\mu\,\mu^2\,e^{\pm i\mu x} = 2j_0(x)-\frac{4j_1(x)}{x}\ . \tag{3.45}$$

If we define the multipole correlation function of the matter as

$$\xi_l^n(x) := \int\frac{dk}{2\pi^2}\,k^nj_l(kx)P_m(k)\ , \tag{3.46}$$

we can show that the velocity correlation functions are

$$\Psi_\parallel \propto \frac{1}{3}\left(\xi_0^0-2\xi_2^0\right)\ , \qquad \Psi_\perp \propto \frac{1}{3}\left(\xi_0^0+\xi_2^0\right)\ . \tag{3.47}$$

The two-point correlation function of the velocity inner product is then

$$\langle\boldsymbol{v}(x)\cdot\boldsymbol{v}(x+r)\rangle = \Psi_\parallel(r)+2\Psi_\perp(r)\ , \tag{3.48}$$

and its variance is

$$\sigma_{3\mathrm{D}}^2 \equiv \langle\boldsymbol{v}(x)\cdot\boldsymbol{v}(x)\rangle = \int\frac{dk}{2\pi^2}\,\mathcal{H}^2f^2P(k)\ . \tag{3.49}$$

Since the peculiar velocity is often measured along the line-of-sight direction only, one-dimensional variance is often used in literature:

$$\sigma_{1\mathrm{D}}^2 = \frac{1}{3}\sigma_{3\mathrm{D}}^2\ . \tag{3.50}$$

For the same reason, the two-point correlation function of the line-of-sight velocities is often measured, and it is related to the velocity correlation Ψ_{ij} as

$$\langle V_1V_2\rangle = \hat{n}_{1i}\hat{n}_{2j}\Psi_{ij}\ , \qquad V_1 := \hat{n}_1^iv_i(x_1)\ , \qquad \hat{n}_1 = \mathbf{x}_1/|\mathbf{x}_1|\ , \tag{3.51}$$

where $\hat{n}_1$ is the line-of-sight direction for the position $\mathbf{x}_1$.

3.4 Redshift-Space Distortion

3.4.1 Redshift-Space Power Spectrum

In cosmology, we rarely know the physical distance to any of the cosmological objects, but we can measure their redshift z with relative ease. The redshift-space distance s is then assigned to the object as

$$s = \int_0^z \frac{dz'}{H} . \tag{3.52}$$

As we discussed in Section 3.3.1, the observed redshift is the sum of the Hubble expansion and the peculiar velocity. However, since it is measured in terms of wavelength, it is more convenient to express it as

$$1 + z \equiv (1 + \bar{z})(1 + \delta z) , \qquad z = \bar{z} + (1 + \bar{z})\delta z , \tag{3.53}$$

where the redshift $\bar{z}$ in the background would represent the comoving distance to the object in the background

$$r = \int_0^{\bar{z}} \frac{dz'}{H} , \qquad d = \frac{r}{1 + \bar{z}} , \tag{3.54}$$

and the peculiar velocity or any contributions to the observed redshift other than the Hubble expansion is described by the perturbation δz:

$$\delta z = v_p + \cdots . \tag{3.55}$$

To the linear order in perturbations, we can expand the redshift-space distance as

$$s \simeq r + \frac{1+z}{H}\delta z = r + \mathcal{V} , \qquad \mathcal{V} := \frac{v_p}{\mathcal{H}} = -f\frac{\partial}{\partial r}\Delta^{-1}\delta , \tag{3.56}$$

where we replaced $\bar{z}$ with z at the linear order. Despite the distortion in the radial distance, the number of galaxies we measure in a given area of the sky remains unaffected: $n_g(s)d^3s = n_g(r)d^3r$. Therefore, the observed galaxy fluctuation δ_s in redshift-space is related to the real-space fluctuation δ_g as

$$1 + \delta_s = \frac{n_g(r)}{n_g(s)} \left|\frac{d^3s}{d^3r}\right|^{-1} = \frac{r^2\bar{n}_g(r)}{s^2\bar{n}_g(s)} \left(1 + \frac{d\mathcal{V}}{dr}\right)^{-1} (1 + \delta_g) . \tag{3.57}$$

This relation is exact but assumes that the redshift-space distortion is purely radial, ignoring angular displacements.

One can make a progress by expanding equation (3.57) to the linear order in perturbations, and the redshift-space galaxy fluctuation is then

$$\delta_s = \delta_g - \left(\frac{d}{dr} + \frac{\alpha}{r}\right)\mathcal{V} , \tag{3.58}$$

where the selection function α is defined in terms of the (comoving) mean number density $\bar{n}_g$ of the galaxy sample as

$$\alpha := \frac{d\ln r^2\bar{n}_g}{d\ln r} = 2 + \frac{rH}{1+z}\,\frac{d\ln\bar{n}_g}{d\ln(1+z)} . \tag{3.59}$$

By adopting the distant-observer approximation ($r \to \infty$) and ignoring the velocity contributions, a further simplification can be made:

$$\delta_s \simeq \delta_g - \frac{d\mathcal{V}}{dr} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{s}}\,(b + f\mu_k^2)\,\delta_m(\mathbf{k}) , \tag{3.60}$$

where we used the linear bias approximation $\delta_g = b\,\delta_m$ and the cosine angle between the Fourier mode and the line-of-sight direction is $\mu_k = \hat{\mathbf{s}}\cdot\hat{\mathbf{k}}$. The galaxy power spectrum in redshift-space is then readily computed as

$$P_s(k,\mu_k) = (b + f\mu_k^2)^2 P_m(k) . \tag{3.61}$$

This redshift-space distortion effect was first derived by Nick Kaiser in 1987. Due to our redshift measurements as the radial distance, the Doppler effect affects our observation of the number density in redshift-space, such that the galaxy power spectrum becomes enhanced along the line-of-sight direction, representing the infall toward the overdense region.

• random motion on small scales, growth rate of structure

3.4.2 Multipole Expansion

The Kaiser formula for the redshift-space power spectrum indicates that the power spectrum is anisotropic, i.e., it depends not only a Fourier mode k, but also its direction. So, it is often convenient to expand $P_s(k,\mu_k)$ in terms of Legendre polynomials $L_l(x)$ as

$$P_s(k,\mu_k) = \sum_{l=0,2,4} L_l(\mu_k) P^s_l(k) \, , \tag{3.62}$$

and the corresponding multipole power spectra are

$$P^s_l(k) = \frac{2l+1}{2} \int_{-1}^{1} d\mu_k \; L_l(\mu_k) P_s(k,\mu_k) \, . \tag{3.63}$$

With its simple angular structure, the simple Kaiser formula in equation (3.61) is completely described by three multipole power spectra

$$P^s_0(k) = \left(b^2 + \frac{2fb}{3} + \frac{f^2}{5}\right) P_m(k) \, , \qquad P^s_2(k) = \left(\frac{4bf}{3} + \frac{4f^2}{7}\right) P_m(k) \, , \qquad P^s_4(k) = \frac{8}{35} f^2 P_m(k) \, , \tag{3.64}$$

while any deviation from the linearity or the distant-observer approximation can give rise to higher-order even multipoles ($l > 4$) and deviations of the lowest multipoles from the above equations.

The correlation function in redshift-space is the Fourier transform of the redshift-space power spectrum $P_s(k,\mu_k)$. With the distant-observer approximation the redshift-space correlation function can be computed and decomposed in terms of Legendre polynomials as

$$\xi_s(s,\mu) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{s}} \, P_s(k,\mu_k) = \sum_{l=0,2,4} L_l(\mu) \, \xi^s_l(s) \, , \tag{3.65}$$

and the multipole correlation functions are related to the multipole power spectra as

$$\xi^s_l(s) = i^l \int \frac{dk \; k^2}{2\pi^2} \, P^s_l(k) j_l(ks) \, , \tag{3.66}$$

$$P^s_l(k) = 4\pi(-i)^l \int dx \; x^2 \xi^s_l(x) j_l(kx) \, , \tag{3.67}$$

where $j_l(x)$ denotes the spherical Bessel functions and the cosine angle between the line-of-sight direction $\hat{\mathbf{n}}$ and the pair separation vector $\mathbf{s}$ is $\mu = \hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$. With the distant-observer approximation, there are no ambiguities associated with how to define the line-of-sight direction of the galaxy pair, as all angular directions are identical.

3.5 Galaxy Clusters

So far, we discussed the two-point statistics of some cosmological probes. One-point statistics such as the number density has also important cosmological information.

3.5.1 Spherical Collapse Model

A simple spherical collapse model was developed long time ago to serve as a toy model for dark matter halo formation. The idea is that a slightly overdense region in a flat universe evolves as if the region were a closed universe, such that it expands almost together with the background universe but eventually turns around and collapses. The overdense region described by the closed universe would collapse to a singularity, but in reality it virializes and stops contracting. By using the analytical solutions for the two universes, we can readily derive many useful relations about the evolution of such overdense regions.

Einstein-de Sitter Universe

A flat homogeneous universe dominated by pressureless matter is called the Einstein-de Sitter Universe:

$$H^2 = \frac{8\pi G}{3}\rho_m \,, \qquad \rho_m \propto \frac{1}{a^3} \,. \tag{3.68}$$

This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations are

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{\eta_0}\right)^2 \,, \qquad \frac{t}{t_0} = \left(\frac{\eta}{\eta_0}\right)^3 \,, \qquad \eta_0 = 3t_0 \,, \tag{3.69}$$

$$H = \frac{2}{3t} \,, \qquad \mathcal{H} = \frac{2}{\eta} \,, \qquad \rho_m = \frac{1}{6\pi G t^2} \,, \qquad r = \eta_0 - \eta = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \,, \tag{3.70}$$

where the reference point t_0 satisfies $a(t_0) = 1$, but it can be any time $t_0 \in (0, \infty)$. At a given epoch t_0, one can define a mass scale

$$M := \frac{4\pi}{3}\rho_0 = \frac{H_0^2}{2G} = \frac{2}{9Gt_0^2} \,, \qquad H_0 = \frac{2}{3t_0} \,, \tag{3.71}$$

Closed Homogeneous Universe

An analytic solution can be derived for a closed universe with again pressureless matter. The evolution equations for a closed universe are

$$\frac{\tilde{a}}{\tilde{a}_t} = \frac{1-\cos\theta}{2} \,, \qquad t = \frac{t_t}{\pi}(\theta - \sin\theta) = \frac{\tilde{a}_t^2(\theta - \sin\theta)}{2\sqrt{K}} \,, \qquad d\tilde{\eta} = \frac{\tilde{a}_t}{\sqrt{K}} d\theta \,, \tag{3.72}$$

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\rho}_m - \frac{K}{\tilde{a}^2} = \frac{K}{\tilde{a}^2}\left(\frac{\tilde{a}_t}{\tilde{a}} - 1\right) \,, \tag{3.73}$$

where we used tilde to distinguish quantities in the closed universe from the flat universe and the maximum expansion (or turn-around $\tilde{a}_t$) is reached at $\theta = \pi$ ($\tilde{H}_t = 0$). The density parameters are related to the curvature K of the universe as

$$\Omega_m - 1 = -\Omega_k = -\frac{K}{\tilde{a}^2\tilde{H}_0^2} \,, \qquad K = \frac{8\pi G}{3}\frac{\rho_0}{\tilde{a}_t} = \frac{H_0^2}{\tilde{a}_t} = \frac{2GM}{\tilde{a}_t} = \frac{\pi^2\tilde{a}_t^2}{4t_t^2} \,, \tag{3.74}$$

where H_0

Spherical Collapse Model

Matching the density equal at some early time, say t_0 (i.e., $\delta_0 = 0$), the time evolution of the overdense region can be derived in a non-perturbative way as

$$1 + \delta = \frac{\tilde{\rho}_m}{\rho_m} = \left(\frac{a}{\tilde{a}}\right)^3 = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} \,, \tag{3.75}$$

where we used

$$a^3 = \left(\frac{t}{t_0}\right)^2 = \left(\frac{t_t}{\pi t_0}\right)^2 (\theta - \sin\theta)^2 \,, \qquad \tilde{a}^3 = \left(\frac{\tilde{a}_t}{2}\right)^3 (1-\cos\theta)^3 = \frac{2}{9}\left(\frac{t_t}{\pi t_0}\right)^2 (1-\cos\theta)^3 \,. \tag{3.76}$$

Therefore, the density contrast δ_t at its maximum expansion

$$1 + \delta_t = \frac{9\pi^2}{16} \simeq 5.6 \,, \tag{3.77}$$

is about a few, while the density contrast δ_v at its virialization

$$1 + \delta_v = 18\pi^2 \simeq 177.7 \,, \tag{3.78}$$

is a few hundreds, under the assumption that the overdensity region virialized at the half of its maximum expansion. Note that the universe further expands and the background density is reduced by factor 4, until it collapses at $t_v = 2t_t$ (or $\theta = 2\pi$).

Finally, expanding the expressions to the linear order,

$$a = \frac{1}{36^{1/3}}\left(\frac{t_t}{\pi t_0}\right)^{2/3}\theta^2 + \cdots ~, \qquad \delta = \frac{3}{20}\theta^2 + \cdots ~, \tag{3.79}$$

the density contrast linearly extrapolated to today and its value at virialization are then derived as

$$\delta_L = \frac{D}{D_i}\delta_i = \frac{a}{a_i}\delta_i = \frac{3}{10}\left(\frac{9}{2}\right)^{1/3}(\theta - \sin\theta)^{2/3} ~, \qquad \delta_v \simeq 1.686 ~. \tag{3.80}$$

Assuming the non-perturbative expression is valid for $|\delta| \ll 1$, we have

$$\delta = \delta_L + \frac{17}{21}\delta_L^2 + \frac{341}{567}\delta_L^3 + \frac{55805}{130977}\delta_L^4 + \cdots ~, \qquad \delta_L = \delta - \frac{17}{21}\delta^2 + \frac{2815}{3969}\delta^3 - \frac{590725}{916839}\delta^4 + \cdots ~. \tag{3.81}$$

Biased Tracer

For any biased tracer δ_X, the Eulerian and the Lagrangian bias parameters can be written in a series

$$\delta_X = \sum_{n=1}^{\infty}\frac{b_n}{n!}\delta^n ~, \qquad \delta_X^L = \sum_{n=1}^{\infty}\frac{b_n^L}{n!}\delta_L^n ~, \tag{3.82}$$

where the superscript L represents quantities in the Lagrangian space. If the number density of the objects X is conserved

$$\rho\, d^3x = \bar\rho\, d^3q ~, \qquad \rho_X\, d^3x = \rho_X^L\, d^3q ~, \qquad \therefore~~ 1+\delta_X = (1+\delta)(1+\delta_X^L) ~, \tag{3.83}$$

the bias parameters are related as

$$b_1 = b_1^{\rm L} + 1 ~, \qquad b_2 = b_2^{\rm L} + \frac{8}{21}b_1^{\rm L} ~, \qquad b_3 = b_3^{\rm L} - \frac{13}{7}b_2^{\rm L} - \frac{796}{1323}b_1^{\rm L} ~, \qquad b_4 = b_4^{\rm L} - \frac{40}{7}b_3^{\rm L} + \frac{7220}{1323}b_2^{\rm L} + \frac{476320}{305613}b_1^{\rm L} ~. \tag{3.84}$$

This simple relation owes to the fact that the spherical collapse model is local in both Eulerian and Lagrangian spaces.

3.5.2 Dark Matter Halo Mass Function

Given the simple spherical collapse model, we would like to associate the collapsed region with some virialized objects like massive galaxy clusters or dark matter halos. Of our main interest is then the number density of such objects in a mass range $M \sim M + dM$, and this is called the mass function.

A simple model called, the excursion set approach, was developed: One starts with a smoothing scale R and its associated mass M. The density fluctuation δ_R after smoothing with R is very small ($\delta_R = 0$, if $R = \infty$), and this region has never reached the critical density threshold δ_c in its entire history. This implies that there is no virialized object associated with such mass. One then decreases the smoothing scale (or mass), and looks for the collapsed probability: Some overdense regions have at some point in the past reached the critical density, while some underdense regions have not. Therefore, the total fraction F_c of collapse can be obtained by using the survival probability P_s of a given scale, and it is related to the mass function as

$$F_c = 1 - \int_{-\infty}^{\delta_c} d\delta\, P_s = \int_M^\infty dM\,\frac{dn}{dM}\,\frac{M}{\bar\rho_m} ~, \qquad \therefore~~ \frac{dn}{dM} = \frac{\bar\rho_m}{M}\left(-\frac{\partial F_c}{\partial M}\right) \equiv \frac{\bar\rho_m}{M}\, f(\nu)\,\frac{d\ln\nu}{dM} ~, \tag{3.85}$$

where it is assumed that the mass function only depends on mass and we defined the multiplicity function f through the relation

$$\nu \equiv \frac{\delta_c(z)}{\sigma(M)} ~, \qquad \int_0^\infty \frac{d\nu}{\nu}\, f = 1 ~. \tag{3.86}$$

The task of obtaining the mass function boils down to computing the survival probability and expressing it in terms of the multiplicity function. The way to find the survival probability at a given mass scale M is to derive the evolution of the density fluctuation as we decrease the smoothing scale R. The reason is that the region may have already collapsed at a larger mass scale or smoothing scale, and this contribution should be removed in computing the survival probability at a lower mass scale. The survival probability at n-th step depends on the entire history of the trajectory (non-Markovian process) as

$$P_s(\delta_n,\sigma_n)d\delta_n = d\delta_n \int_{-\infty}^{\delta_c} d\delta_{n-1}\cdots\int_{-\infty}^{\delta_c} d\delta_1\; P_s(\delta_1,\cdots\delta_n,\sigma_1,\cdots,\sigma_n)\;, \tag{3.87}$$

it is notoriously difficult to solve, even numerically. However, once we assume that the fluctuations are independent at each smoothing and are Gaussian distributed (true only in Fourier space at linear order), the trajectory only depends on the previous step (Markovian process) and the survival probability becomes

$$P_s(\delta_n,\sigma_n) = \int_{-\infty}^{\delta_c} d\delta_{n-1} P_t(\delta_n,\sigma_n|\delta_{n-1},\sigma_{n-1})\; P_s(\delta_{n-1},\sigma_{n-1})\;, \tag{3.88}$$

where the transition probability P_t is nothing but a conditional probability. With the boundary condition $P_s = 0$ at $\delta = \delta_c$, the solution is (derived by Chandrasekhar for other purposes)

$$P_s = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\delta^2}{2\sigma^2}\right) - \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(2\delta_c-\delta)^2}{2\sigma^2}\right]\;. \tag{3.89}$$

The survival probability for its simplest case is described by a Gaussian distribution, but the second term reflects that there exist equally likely trajectories around the threshold that have reached the threshold in the past. The collapsed fraction is

$$F_c = 1 - \frac{1}{2}\mathrm{erf}\left(\frac{\nu_c}{\sqrt{2}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{\nu_c}{\sqrt{2}}\right) = \mathrm{erfc}\left(\frac{\nu_c}{\sqrt{2}}\right)\;, \tag{3.90}$$

and the multiplicity function is

$$f(\nu) = \sqrt{\frac{2}{\pi}}\;\nu\; e^{-\nu^2/2}\;. \tag{3.91}$$

Of course, this model relies on many approximations, and it is not accurate. However, it provides physical intuitions, connecting the complicated formation of galaxy clusters and the dynamical evolution of the matter density fluctuations. In general, numerical N-body simulations are run, and dark matter halos are identified by using some algorithm such as the friends-of-friends method or its variants to derive the mass function from the simulations.