Weak gravitational lensing

strong lensing, micro lensing, weak lensing

* strong : large mass, multiple images, Einstein ring, giant arc, ...
  cosmological distance. static over many years, time delay
  mainly about the system: dark matter, & distance, Hubble Para.

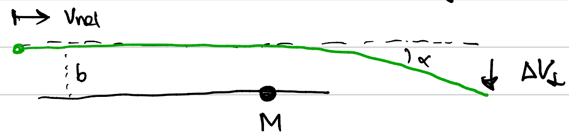
* micro : in our galaxy. lens by BH or stars, multiple images at < micro arcsec
  not spatially resolved. but brightness changes in real time
  1 in million stars, monitor several billion stars every day
  → planets produce small perturbations (different para space from radial velo. transit)
  BHs in our galaxy.

* weak : cosmological area → cosmological parameter. Ω_m, Ω_Λ, $P(k)$, ...
  but signals are $\sim 10^{-5}$, small change in ellipticity of galaxies

very simple basic description of gravitational lensing

impulse approximation
or gravitational kick

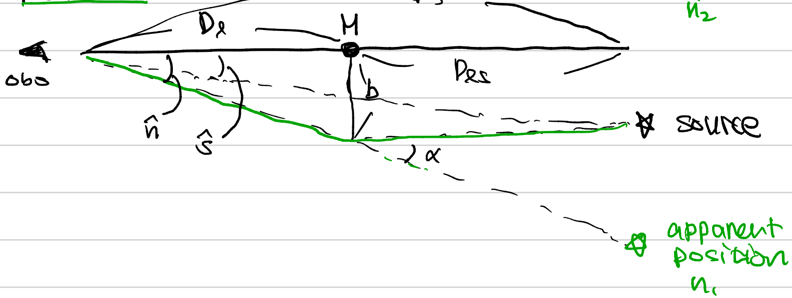
$$\Delta V_\perp = \frac{2GM}{b \cdot v_{rel}}$$



light $m=0$ & $v=c$ $\Rightarrow$ $\hat\alpha = \frac{2GM}{b c^2} \to \frac{4GM}{b \cdot c^2}$ with relativistic correction factor 2

$\hat\alpha = \frac{4GM}{bc^2} = 2.8 \times 10^{-3}$ arcsec $M_\odot \cdot b_{AU}^{-1}$. Eddington exp. 1.75 arcsec during total eclipse of Sun compared to positions at night

Lens EQ



$D_s \cdot \hat s = D_s \cdot \hat n - D_{ls} \cdot \hat\alpha$ $\therefore$ $\hat s = \hat n - \Delta \hat n$

deflection angle

$$\Delta \hat n = \frac{D_{ls}}{D_s}\hat\alpha = \frac{4GM}{c^2}\frac{D_{ls}}{D_s \cdot D_l}\frac{1}{\hat n}$$

Einstein radius $\theta_E^2 = \frac{4GM}{c^2}\frac{D_{ls}}{D_l \cdot D_s}$

always two solutions

$\hat n_1 = \frac{1}{2}\left(\hat s + \sqrt{s^2 + 4\theta_E^2}\right)$, $\hat n_2 = \frac{1}{2}\left(\hat s - \sqrt{s^2 + 4\theta_E^2}\right) < 0$

$\hat n_1 + \hat n_2 = \hat s$

Generalization of Lens EQ

$\hat{s} = \hat{n} - \hat{\Delta n}$, $\Delta \hat{n} = \frac{4GM}{c^2} \frac{D_{\ell s}}{D_\ell D_s} \hat{n}$: point mass $\rightarrow$ $\hat{s} = \hat{n} - \hat{\nabla}\Phi$

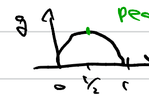
better to use grav. potential

* mass distribution at D_ℓ (fixed) : $\Phi = \frac{1}{c^2} \frac{D_{\ell s}}{D_\ell D_s} \int_{-R}^{R} dz \cdot 2\psi$ $\psi = -\frac{GM}{r}$ point mass
* mass distribution over ranges : $\Phi = \frac{1}{c^2} \int_0^{r_s} dr_\ell \left(\frac{r_{\ell s}}{r_\ell r_s} \right) 2\psi \equiv \frac{1}{c^2} \int_0^{\infty} dr_\ell \frac{g(r_\ell, r_\ell)}{r_\ell^2} 2\psi$
* source distribution over ranges :

$g(r_\ell, r_s) = r_\ell^2 \int_{r_\ell}^{\infty} dr_s \left(\frac{r_{\ell s}}{r_\ell r_s} \right) n_g(r_s)$ $\int_0^\infty dr_s\, n_g(r_s) = 1$ $[g] = L$

include r_ℓ^2 volume weight on 2ψ

for a fixed src $g = \frac{r_\ell r_{\ell s}}{r_s}$



Distortion matrix

$\hat{s} = \hat{n} - \hat{\nabla}\Phi$: geodesic eq

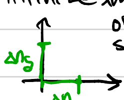
$d\hat{s} = d\hat{n} - d(\hat{\nabla}\Phi) = \mathbb{D}_{ij} dn_j$: geodesic deviation eq

$$\mathbb{D}_{ij} = \frac{ds_i}{dn_j} = \mathbb{I}_{ij} - \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} = I - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$ $\Phi_{ij} \equiv \hat{\nabla}_i \hat{\nabla}_j \Phi$

$\kappa \equiv 1 - \mathrm{Tr}\,\mathbb{D} = \frac{1}{2}(\Phi_{11} + \Phi_{22})$, $\gamma_1 \equiv \frac{1}{2}(\mathbb{D}_{22} - \mathbb{D}_{11}) = \frac{1}{2}(\Phi_{11} - \Phi_{22})$ $\det \mathbb{D} = (1-\kappa)^2 - \gamma^2 + \omega^2$

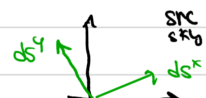
$\gamma_2 \equiv -\frac{\mathbb{D}_{12} + \mathbb{D}_{21}}{2} = \Phi_{12} = \Phi_{21}$, $\omega = \frac{1}{2}(\mathbb{D}_{21} - \mathbb{D}_{12}) = 0$ $\simeq 1 - 2\kappa$

infinitesimal length $dn \rightarrow ds$



$\Delta n_x = \Delta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\rightarrow$ $ds_i^x = \Delta \begin{pmatrix} \mathbb{D}_{11} \\ \mathbb{D}_{21} \end{pmatrix}$

$\Delta n_y = \Delta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $ds_i^y = \Delta \begin{pmatrix} \mathbb{D}_{12} \\ \mathbb{D}_{22} \end{pmatrix}$



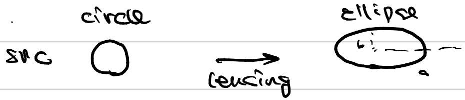
$dA_{src} = ds^x \times ds^y = \Delta^2 \cdot \det \mathbb{D}$

magnification $\mu = \frac{dA_{obs}}{dA_{src}} = \det^{-1}\mathbb{D} \simeq 1 + 2\kappa$

ω : rotation . γ_1, γ_2 : shear . similar to GW polarization

e.g. only $\gamma_1 \neq 0$. $ds^x = \Delta \begin{pmatrix} 1-\gamma_1 \\ 0 \end{pmatrix}$ $ds^y = \Delta \begin{pmatrix} 0 \\ 1+\gamma_1 \end{pmatrix}$ squeeze in n_x, n_y expand

Weak Lensing Observables



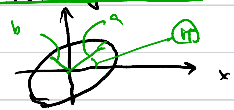
ellipticity $\varepsilon = \frac{a^2-b^2}{a^2+b^2} = 0 - 1$

measure of gravitational lensing

magnification matrix $M_{ij} \equiv (\mathbb{D}^{-1})_{ij} = \frac{1}{|\mathbb{D}|}\begin{pmatrix} 1-\kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & 1-\kappa-\gamma_1 \end{pmatrix}$ $\quad \omega = 0 \quad . \quad \det \mathbb{D} = (1-\kappa)^2 - \gamma^2 + \omega^2$

$d\hat{n}_i = M_{ij}\, d\hat{s}_j$ mapping from src to obs

Ellipticity vector



$\vec{\varepsilon} = (\varepsilon_+ , \varepsilon_\times) = \varepsilon(\cos 2\Theta , \sin 2\Theta) = \left(\frac{M_{xx}-M_{yy}}{M_{xx}+M_{yy}} , \frac{2M_{xy}}{M_{xx}+M_{yy}}\right)$

invariant under rotation π (spin 2)

ellipticity moment $M_{ij}^{obs} = \int d^2\hat{n}\; n_i n_j\, W(\hat{n})$ galaxies are fuzzy

$\simeq \int d^2\hat{s} \cdot \mu \cdot (M_{ik} s_k) \cdot (M_{je} s_e)\, W(\hat{s}) \simeq \mu M_{ik} M_{je} M_{ke}^{src}$

$\therefore\; \varepsilon_+^{obs} \doteq \varepsilon_+^{int} + 2\gamma_1 \;,\; \varepsilon_\times^{obs} \doteq \varepsilon_\times^{int} + 2\gamma_2$ $\quad \gamma \sim 10^{-5} \;,\; \varepsilon^{int} \sim 0.3$

1pt : $\langle \varepsilon_+^{obs} \rangle = \langle \varepsilon_+^{int} \rangle + 0 = \langle \varepsilon \rangle \langle \cos 2\Theta \rangle = 0 = \langle \varepsilon_\times^{obs} \rangle$ $\quad \langle \varepsilon_+^{obs} \varepsilon_\times^{obs} \rangle = 0$

$\langle (\varepsilon_+^{obs})^2 \rangle = \langle (\varepsilon_+^{int})^2 \rangle + 2\langle \gamma_1 \rangle \langle \varepsilon_+^{int} \rangle + 0 = \langle \varepsilon_{int}^2 \rangle \langle \cos^2 2\Theta \rangle + 0 = \frac{1}{2}\langle \varepsilon_{int}^2 \rangle \sim \frac{1}{2}\, 0.3^2$

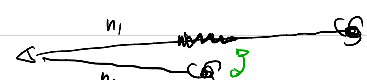
2pt : $\langle \varepsilon_+^{obs}(n_1)\, \varepsilon_+^{obs}(n_2) \rangle = \underbrace{\langle \varepsilon_+^{int}(n_1)\, \varepsilon_+^{int}(n_2) \rangle}_{\to 0} + \underbrace{4\langle \gamma_1(n_1)\, \gamma_2(n_2) \rangle}_{= \text{power spectrum}} + \underbrace{2\langle \gamma_1(n_1)\, \varepsilon_+^{int}(n_2) \rangle}_{\to 0} + \underbrace{(\leftrightarrow)}_{\to 0}$

$\gamma_1(\hat{n}) = \frac{1}{2}(\Phi_{11} - \Phi_{22}) = \frac{1}{2}(\hat{\nabla}_1\hat{\nabla}_1 - \hat{\nabla}_2\hat{\nabla}_2) \int_0^\infty dr_e \frac{g(r_e r_s)}{r_e^2}\, 2\psi$

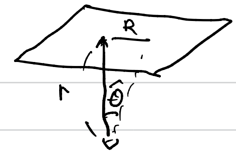
$\gamma_2(\hat{n}) = \Phi_{12} = \Phi_{21} = \hat{\nabla}_1\hat{\nabla}_2 \int_0^\infty dr_e \frac{g(r_e r_s)}{r_e^2}\, 2\psi$

no galaxy bias

systematic errors : intrinsic alignment



Angular power spectrum



* full sky vs flat sky

* 2D FT $\xi_{2D}(x,y) = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i k_\perp \cdot R} P_{2D}(k_\perp)$ $\vec{k}_\perp \cdot \vec{R} = \vec{\ell} \cdot \vec{\theta}$ $\vec{R} = r \cdot \vec{\theta}$, $\vec{\ell} = r \cdot \vec{k}_\perp$ dimensionless

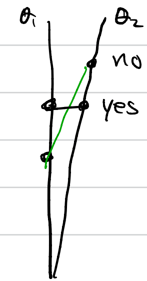
angular corr ↑ $= \int \frac{d^2\ell}{(2\pi)^2} e^{i \ell \cdot \theta} P_\ell$ $\therefore P_\ell = \frac{1}{r^2} P_{2D}(k_\perp = \ell / r)$ dimensionless

→ C_ℓ in full sky

* Limber approximation

distance $r \gg R = r\theta$ → $k_\parallel \sim \frac{1}{r} \ll k_\perp \sim \frac{1}{R}$

$\therefore P(k) \simeq P(k_\perp) + \frac{dP}{dk_\parallel} \cdot k_\parallel + \cdots \simeq \underline{P(k_\perp)}$ $\because \frac{dP}{dk_\parallel} k_\parallel \sim \frac{dP}{dk} \frac{k_\parallel^2}{k} \sim \frac{P}{k^2 r^2} \sim \frac{P}{\ell^2} \ll P$



$\Phi(\hat\theta) = \int^s dr \frac{g}{r^2} 2\Psi(r \cdot \hat\theta)$

$\Phi(\ell) = \int d^2\theta\, e^{-i\ell \cdot \theta} \int^s dr \frac{g}{r^2} 2\Psi$

$= \int dr \frac{g}{r^4} \int \frac{dk_\parallel}{2\pi} e^{+ik_\parallel r} 2\Psi(k_\perp = \frac{\ell}{r}, k_\parallel)$

$r_2 - r_1 \sim r$ → no contribution

$r_2 - r_1 \sim R$ → yes

$\langle \Phi(\ell_1) \Phi^*(\ell_2) \rangle = (2\pi)^2 \delta^D(\ell_1 - \ell_2) P_\ell$

$= \int dr_1 \int dr_2 \frac{g_1}{r_1^4} \frac{g_2}{r_2^4} \times \underbrace{\int \frac{dk_1^\parallel}{2\pi} \int \frac{dk_2^\parallel}{2\pi} e^{ik_1^\parallel r_1 - ik_2^\parallel r_2}}_{A} \times 4 P_\Psi(k_\perp, k_\parallel) (2\pi)^3 \underbrace{\delta^D(k_1^\parallel - k_2^\parallel)}_{A} \delta^D(k_1^\perp - k_2^\perp)$

$\delta^D(k_1^\perp - k_2^\perp) \rightarrow r_1^2 \delta^D(\ell_1 - \frac{r_1}{r_2} \ell_2)$

Ⓐ $\Rightarrow \int \frac{dk_\parallel}{2\pi} e^{ik_\parallel(r_2 - r_1)}$

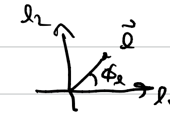
Ⓒ integrate over $k_\parallel \rightarrow \delta^D(r_2 - r_1)$

Ⓑ then Limber approx.

$P_\Psi(k_\perp, k_\parallel) \simeq P_\Psi(k_\perp)$

$\therefore P_\ell = \int dr \frac{g^2}{r^6} 4 P_\Psi(k_\perp = \frac{\ell}{r})$

E-B decomposition



$$\kappa = \frac{1}{2}\nabla^2 \Phi \qquad\qquad \kappa_\ell = -\frac{1}{2}\ell^2 \Phi_\ell$$

$$\gamma_1 = \frac{1}{2}(\nabla_1^2 - \nabla_2^2)\Phi \quad \rightarrow \quad \gamma_{1\ell} = -\frac{1}{2}(\ell_1^2 - \ell_2^2)\Phi_\ell = \cos 2\phi_\ell \cdot \kappa_\ell$$

$$\gamma_2 = \nabla_1 \nabla_2 \Phi \qquad\qquad \gamma_{2\ell} = -\ell_1 \ell_2 \Phi_\ell = \sin 2\phi_\ell \cdot \kappa_\ell$$

$$E_\ell \equiv \cos 2\phi_\ell \cdot \gamma_1(\ell) + \sin 2\phi_\ell \cdot \gamma_2(\ell) = \kappa_\ell$$

$$B_\ell = -\sin 2\phi_\ell \cdot \gamma_1(\ell) + \cos 2\phi_\ell \cdot \gamma_2(\ell) = 0$$

$$\therefore \quad P_\kappa = P_E = \frac{1}{4}\ell^4 P_\Phi \quad , \quad P_B = 0 = P_{EB}$$

$$\nabla^2 \Psi = 4\pi G \bar{\rho} a^2 \delta = \frac{3}{2} H_0^2 \Omega_m \frac{\delta}{a} \quad . \quad P_\Psi = \left(\frac{3H_0^2}{2}\Omega_m\right)^2 \frac{1}{a^2 k^4} P_m$$

$$= \left(\frac{3H_0^2}{2}\Omega_m\right)^2 \int dr \frac{g^2}{r^2 a^2} \cdot P_m\left(k_\perp = \frac{\ell}{r}\right)$$