6 Weak Gravitational Lensing

6.1 Gravitational Lensing by a Point Mass

In classical mechanics, the gravitational interaction due to a point mass M provides a perturbation along the transverse direction to a test particle moving with the relative speed $v_{\rm rel}$:

$$\Delta v_\perp = \frac{2GM}{b\, v_{\rm rel}} \,, \tag{6.1}$$

where G is the Newton's constant and b is the transverse separation (or the impact parameter). The prediction for the light deflection angle $\hat\alpha$ in Einstein's general relativity is well-known to follow the same result in classical mechanics, but with additional factor two:

$$\hat\alpha = \frac{4GM}{b\, c^2} = 8.155\times 10^{-3}\ \text{arcsec}\left(\frac{M}{M_\odot}\right)\left(\frac{b}{\rm AU}\right)^{-1} \,. \tag{6.2}$$

Given the deflection angle $\hat\alpha$, we can readily write down *the lens equation* in terms of the angular diameter distances

$$\mathcal{D}_s\hat s = \mathcal{D}_s\hat n - \mathcal{D}_{ls}\hat\alpha \,, \qquad \hat s = \hat n - \theta_E^2/\hat n \,, \tag{6.3}$$

where the Einstein radius is

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{\mathcal{D}_{ls}}{\mathcal{D}_l\mathcal{D}_s}} = 2.853\times 10^{-3}\ \text{arcsec}\left(\frac{M}{M_\odot}\right)\left(\frac{\mathcal{D}_l}{\rm kpc}\right)^{-1/2}\left(1-\frac{\mathcal{D}_l}{\mathcal{D}_s}\right)^{1/2} \,. \tag{6.4}$$

For a point mass lens and a point source, two lensed image positions are readily obtained as

$$\hat n_1 = \frac{1}{2}\left(\hat s + \sqrt{\hat s^2 + 4\theta_E^2}\right) \,, \qquad \hat n_2 = \frac{1}{2}\left(\hat s - \sqrt{\hat s^2 + 4\theta_E^2}\right) < 0 \,, \qquad \hat n_1 + \hat n_2 = \hat s \,, \tag{6.5}$$

and when the source and the lens are aligned, the lensed images form a ring with radius θ_E.

• microlensing, probe of MACHOs or exoplanets

6.2 Standard Weak Lensing Formalism

6.2.1 Lens Equation and Distortion Matrix

This light deflection due to a point mass can be generalized to derive the standard weak lensing formalism by considering the gravitational potential fluctuation $\psi = -GM/r$ of the general matter distribution ρ (but still a single lens plane), instead of a point mass (ψ indeed corresponds to the metric fluctuation α_χ). The lensing potential Φ is the line-of-sight integration of the metric fluctuation,

$$\Phi := \frac{1}{c^2}\frac{\mathcal{D}_{ls}}{\mathcal{D}_l\mathcal{D}_s}\int dz\ 2\psi \,, \tag{6.6}$$

and using the Poisson equation, we can relate the lensing potential with the surface density Σ as

$$\nabla^2\psi = 4\pi G\bar\rho a^2\delta \,, \qquad \hat\nabla^2\Phi = 2\frac{\Sigma}{\Sigma_c} \,, \qquad \Phi(\hat n) = \int d^2\hat n'\,\frac{\Sigma}{\pi\Sigma_c}\ln|\hat n - \hat n'| \,, \tag{6.7}$$

where we ignored the boundary term when the Poisson equation is integrated and the critical surface density is defined as

$$\Sigma_c^{-1} := \frac{4\pi G}{c^2}\frac{\mathcal{D}_{ls}\mathcal{D}_l}{\mathcal{D}_s} \,, \qquad \Sigma_c = 1.663\times 10^6\ hM_\odot\ {\rm pc}^{-2}\left(\frac{\mathcal{D}_s}{\mathcal{D}_{ls}}\right)\left(\frac{\mathcal{D}_l}{h^{-1}{\rm Mpc}}\right)^{-1} \,. \tag{6.8}$$

Though the lensing potential is formally divergent for a point mass, its angular derivative is well defined:

$$\hat\alpha = \left(\frac{\mathcal{D}_l\mathcal{D}_s}{\mathcal{D}_{ls}}\right)\nabla_\perp\Phi = \frac{\mathcal{D}_s}{\mathcal{D}_{ls}}\hat\nabla\Phi \,, \tag{6.9}$$

such that the lens equation becomes

$$\hat{\boldsymbol{s}} = \hat{\boldsymbol{n}} - \hat{\nabla}\Phi \,, \tag{6.10}$$

where $\hat{\nabla}$ is the angular gradient. When the lensing material is distributed over the redshift, the lensing potential is then obtained by integrating the potential fluctuation over the line-of-sight distance as

$$\Phi = \int_0^{\bar{r}_s} d\bar{r} \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) 2\psi \equiv \int_0^{\bar{r}_s} d\bar{r}\, \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2}\, 2\psi \,, \tag{6.11}$$

where we switched to a comoving angular diameter distance $\bar{r}$ and we defined the weight function g for later convenience

$$g := \bar{r}^2 \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) . \tag{6.12}$$

When the background source galaxies are also spread over some redshift with the distribution $n_g(r_s)$, the lensing potential can be readily generalized by replacing the weight function with

$$g := \bar{r}^2 \int_{\bar{r}}^{\infty} d\bar{r}_s \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}}\right) n_g(\bar{r}_s) \,, \qquad \Phi = \int_0^{\infty} d\bar{r}\, \frac{g(\bar{r})}{\bar{r}^2}\, 2\psi \,, \qquad 1 = \int_0^{\infty} d\bar{r}_s\, n_g(\bar{r}_s) \,, \tag{6.13}$$

where the upper limit for the integration is indeed $\bar{r}(z = \infty)$ and the source distribution is normalized.[1]

Using the lens equation, the distortion matrix $\mathbb{D}$ (or sometimes called the amplification matrix) is defined as

$$\mathbb{D}_{ij} \equiv \frac{\partial s_i}{\partial n_j} = \mathbb{I}_{ij} - \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} , \qquad \Phi_{ij} := \hat{\nabla}_j \hat{\nabla}_i \Phi \,, \tag{6.14}$$

where $\mathbb{I}$ is the two-dimensional identity matrix and we defined a short hand notation for the angular derivatives of the lensing potential. The distortion matrix is conventionally decomposed into the trace, the traceless symmetric and the anti-symmetric matrices:

$$\mathbb{D} := \mathbb{I} - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} , \qquad \det \mathbb{D} = (1-\kappa)^2 - \gamma^2 + \omega^2 \,, \tag{6.15}$$

where the trace is the gravitational lensing convergence κ and the symmetric traceless part is the lensing shear $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$:

$$\kappa \equiv 1 - \frac{1}{2}\mathrm{Tr}\,\mathbb{D} = \frac{1}{2}\left(\Phi_{11} + \Phi_{22}\right) \,, \qquad \omega \equiv \frac{\mathbb{D}_{21} - \mathbb{D}_{12}}{2} = 0 \,, \tag{6.16}$$

$$\gamma_1 \equiv \frac{\mathbb{D}_{22} - \mathbb{D}_{11}}{2} = \frac{1}{2}\left(\Phi_{11} - \Phi_{22}\right) \,, \qquad \gamma_2 \equiv -\frac{\mathbb{D}_{12} + \mathbb{D}_{21}}{2} = \Phi_{12} = \Phi_{21} \,. \tag{6.17}$$

Since the distortion matrix in Eq. (6.14) is symmetric, the rotation ω vanishes in the standard formalism at all orders.

The standard lensing formalism is based on the lens equation and the lensing potential in Eq. (6.10). However, the source angular position $\hat{\boldsymbol{s}} := (\theta + \delta\theta, \phi + \delta\phi)$ is gauge-dependent, and the lensing potential that is responsible for the angular distortion $(\delta\theta, \delta\phi)$ is also gauge-dependent. Indeed, we already know that 2ψ in Eq. (6.10) should be $(\alpha_\chi - \varphi_\chi)$ to match the leading terms for $\delta\theta$ and the Poisson equation in Eq. (6.21) is indeed an Einstein equation with ψ there replaced by $-\varphi_\chi$.[2] Furthermore, there exist no contributions from the vector and the tensor perturbations in the standard lensing formalism. Finally, while the derivations in this subsection assume no linearity, all formulas of the standard lensing formalism turn out to be valid only at the linear order in perturbations.

6.2.2 Convergence and Shear

While the distortion matrix is defined in terms of angles, it is often assumed in literature that the line-of-sight direction is along z-axis ($\hat{\boldsymbol{n}} /\!/ \hat{\boldsymbol{z}}$, i.e., $\theta = 0$), and two angles are aligned with x-y plane. In such a setting, consider two small angular vectors at the source position subtended respectively by $d\theta$ and $d\phi$ at the observer position

$$\Delta s_i^{d\theta} = \mathbb{D}_{i1} d\theta \,, \qquad \Delta s_i^{d\phi} = \mathbb{D}_{i2} d\phi \,. \tag{6.18}$$

[1] Sometimes it is normalized when integrated over redshift.

[2] Additional condition of a vanishing anisotropic pressure is needed to guarantee $\alpha_\chi = -\varphi_\chi$ and hence the consistency in the lensing equation.

The solid angle at the source subtended by these two angular vectors is then related to the solid angle at the observer as

$$d\Omega_s = \left|\Delta\boldsymbol{s}^{d\theta} \times \Delta\boldsymbol{s}^{d\phi}\right| = \det\mathbb{D}\; d\theta d\phi = \det\mathbb{D}\; d\Omega_o\;, \tag{6.19}$$

and hence the gravitational lensing magnification μ is then

$$\mu^{-1} \equiv \frac{d\Omega_s}{d\Omega_o} = \det\mathbb{D}\;. \tag{6.20}$$

For this reason, the distortion matrix is often called the inverse magnification matrix. Using the Poisson equation in cosmology,

$$\nabla^2\psi = 4\pi G\bar\rho a^2\delta_m = \frac{3H_o^2}{2}\Omega_m\frac{\delta_m}{a}\;, \tag{6.21}$$

the gravitational lensing convergence can be computed in terms of the matter density fluctuation δ_m in the comoving gauge as

$$\kappa = \int_0^{\bar r_s} d\bar r\; g(\bar r,\bar r_s)\nabla_\perp^2\psi = \frac{3H_o^2}{2}\Omega_m\int_0^{\bar r_s} d\bar r\;\frac{g(\bar r,\bar r_s)}{a}\;\delta_m \tag{6.22}$$

where we used the Laplacian in cylindrical coordinate

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial\bar r^2}\;, \qquad \nabla_\perp^2 = \frac{1}{\bar r^2}\hat\nabla^2\;, \tag{6.23}$$

and ignored the boundary terms.

In general, we can compute the individual components of the distortion matrix

$$\gamma_1 = \int_0^{\bar r_s} d\bar r\; g(\bar r,\bar r_s)\left(\nabla_1^2 - \nabla_2^2\right)\psi\;, \qquad \gamma_2 = 2\int_0^{\bar r_s} d\bar r\; g(\bar r,\bar r_s)\nabla_1\nabla_2\psi\;, \tag{6.24}$$

by using

$$\frac{1}{2}\Phi_{ij} \equiv \frac{1}{2}\hat\nabla_i\hat\nabla_j\Phi = \int_0^{\bar r_s} d\bar r\; g(\bar r,\bar r_s)\nabla_i\nabla_j\psi\;, \tag{6.25}$$

where the indices i, j represent the perpendicular components.

6.2.3 Angular Power Spectrum and Angular Correlation

Assuming that the survey area is small, we will utilize the angular Fourier transformation in Eq. (3.17) by again computing

$$\Phi(\mathbf{l}) = \int d^2\theta\; e^{-i\mathbf{l}\cdot\boldsymbol\theta}\int_0^{\bar r_s} d\bar r\;\frac{g(\bar r,\bar r_s)}{\bar r^2}\;2\psi = \int_0^{\bar r_s} d\bar r\;\frac{g(\bar r,\bar r_s)}{\bar r^4}\int\frac{dk_\parallel}{2\pi}\;2\psi\left(k_\perp = \frac{l}{\bar r}\right)e^{ik_\parallel\bar r}\;, \tag{6.26}$$

such that the lensing observables are

$$\kappa(\mathbf{l}) = -\frac{l^2}{2}\Phi(l)\;, \qquad \gamma_1(\mathbf{l}) = -\frac{l_1^2 - l_2^2}{2}\Phi(l) = \cos 2\phi_l\;\kappa(\mathbf{l})\;, \qquad \gamma_2(\mathbf{l}) = -l_1l_2\Phi(l) = \sin 2\phi_l\;\kappa(\mathbf{l})\;, \tag{6.27}$$

where we used $\mathbf{l} = (l_1, l_2) = l(\cos\phi_l, \sin\phi_l)$. Therefore, the angular power spectra can be readily derived as

$$P_\kappa(l) = \frac{l^4}{4}P_\Phi(l)\;, \qquad P_{\gamma_1} = \cos^2 2\phi_l\;P_\kappa(l)\;, \qquad P_{\gamma_2} = \sin^2 2\phi_l\;P_\kappa(l)\;, \tag{6.28}$$

where the angular power spectrum of the lensing potential is obtained by using the Limber approximation as

$$P_\Phi(l) = \int_0^{\bar r_s} d\bar r\;\frac{g^2(\bar r,\bar r_s)}{\bar r^6}\;4P_\psi\left(k_\perp = \frac{l}{\bar r}\right)\;. \tag{6.29}$$

From the relation of the lensing observables, we find it useful to construct E and B-modes as

$$E(\mathbf{l}) := \cos 2\phi_l\;\gamma_1(\mathbf{l}) + \sin 2\phi_l\;\gamma_2(\mathbf{l})\;, \qquad B(\mathbf{l}) := -\sin 2\phi_l\;\gamma_1(\mathbf{l}) + \cos 2\phi_l\;\gamma_2(\mathbf{l})\;. \tag{6.30}$$

We can readily derive

$$E(\mathbf{l}) = \kappa(\mathbf{l}) \,, \qquad B(\mathbf{l}) = 0 \,, \qquad P_E(l) = P_\kappa(l) \,, \qquad P_B(l) = P_{EB}(l) = 0 \,, \tag{6.31}$$

in the absence of any systematics and/or physics other than the gravitational lensing, such that it provides a consistency check of the measurements, where the convergence power spectrum is again related to the matter power spectrum as

$$P_\kappa(l) = \left(\frac{3H_0^2}{2}\Omega_m\right)^2 \int_0^{\bar r_s} d\bar r \, \frac{g^2(\bar r, \bar r_s)}{\bar r^2 a^2} \, P_m\left(k_\perp = \frac{l}{\bar r}\right) \,. \tag{6.32}$$

Now we compute the angular correlation function by Fourier transforming the angular power spectrum. Out of two shear components, we construct three angular correlation functions as

$$\begin{aligned} w_{ij}(\theta) &:= \langle\gamma_i(0)\gamma_j(\theta)\rangle = \int \frac{d^2l}{(2\pi)^2}\, e^{i\mathbf{l}\cdot\boldsymbol{\theta}} \begin{pmatrix} \cos^2 2\phi_l & \cos 2\phi_l \sin 2\phi_l \\ \cos 2\phi_l \sin 2\phi_l & \sin^2 2\phi_l \end{pmatrix} P_\kappa(l) & (6.33) \\ &= \frac{1}{2}\int_0^\infty \frac{dl\, l}{2\pi}\, P_\kappa(l) \begin{pmatrix} J_0(l\theta) + J_4(l\theta) & 0 \\ 0 & J_0(l\theta) - J_4(l\theta) \end{pmatrix} \,, & (6.34) \end{aligned}$$

where J_n is the Bessel function and used its integral representation

$$J_0(x) = \int \frac{d\phi}{2\pi}\, e^{ix\cos\phi} \,, \qquad J_4(x) = \int \frac{d\phi}{2\pi}\, e^{ix\cos\phi} \cos 4\phi \,. \tag{6.35}$$

6.2.4 Worked Examples

For the simplest case, where the lens and the source are at two definite redshift slices, the lensing observables can be written in a polar coordinate as

$$\begin{aligned} 2\kappa &= \Phi_{rr} + \frac{\Phi_r}{r} + \frac{\Phi_{\theta\theta}}{r^2} = 2\frac{\Sigma}{\Sigma_c} \,, & (6.36) \\ 2\gamma_1 &= \cos 2\theta \Phi_{rr} - \frac{2\sin 2\theta}{r}\Phi_{r\theta} - \frac{\cos 2\theta}{r}\Phi_r - \frac{\cos 2\theta}{r^2}\Phi_{\theta\theta} + \frac{2\sin 2\theta}{r^2}\Phi_\theta \,, & (6.37) \\ 2\gamma_2 &= \sin 2\theta \Phi_{rr} + \frac{2\cos 2\theta}{r}\Phi_{r\theta} - \frac{\sin 2\theta}{r}\Phi_r - \frac{\sin 2\theta}{r^2}\Phi_{\theta\theta} - \frac{2\cos 2\theta}{r^2}\Phi_\theta \,, & (6.38) \\ \gamma^2 &:= \gamma_1^2 + \gamma_2^2 = \frac{1}{4}\left(\Phi_{rr} - \frac{\Phi_r}{r} - \frac{\Phi_{\theta\theta}}{r^2}\right)^2 + \left(\frac{\Phi_{r\theta}}{r} - \frac{\Phi_\theta}{r^2}\right)^2 \,. & (6.39) \end{aligned}$$

For an axisymmetric lens, the lensing observables are further simplified, and the convergence and shear are

$$2\kappa = \hat\nabla^2\Phi = \Phi_{rr} + \frac{\Phi_r}{r} = 2\,\frac{\Sigma}{\Sigma_c} \,, \qquad \gamma = \frac{1}{2}\left(\frac{\Phi_r}{r} - \Phi_{rr}\right) = \frac{\Phi_r}{r} - \frac{\Sigma}{\Sigma_c} = \frac{\bar\Sigma(<r) - \Sigma}{\Sigma_c} \,, \tag{6.40}$$

where $\bar\Sigma(<r)$ is the average surface density enclosed in radius r and $\bar\Sigma(<r) = \Phi_r/r$ from the first relation. The magnification is determined by the surface density of the lensing material, and the gravitational shear is set by the excess surface density of the enclosed mass $\Delta\Sigma := \bar\Sigma(<r) - \Sigma(r)$.

For a point mass, the convergence and the shear are

$$\begin{aligned} \psi &= -\frac{GM}{r} \,, \qquad \nabla^2\psi = 4\pi GM\delta^D(x) \,, \qquad \kappa = \frac{\Sigma}{\Sigma_c} = \frac{M\delta^D(R)}{\Sigma_c} \,, & (6.41) \\ \gamma &= \frac{\bar\Sigma(<R) - \Sigma}{\Sigma_c} = \frac{\bar\Sigma}{\Sigma_c} = \frac{\theta_E^2}{\theta^2} \,, \qquad \bar\Sigma := \frac{M}{\pi R^2} \,. & (6.42) \end{aligned}$$

The lensing magnification is then

$$\mu^{-1} = \det\, \mathbb{D} = 1 - \Phi_{rr} - \frac{\Phi_r}{r} + \frac{\Phi_{rr}\Phi_r}{r} = \frac{s}{r}\frac{\partial s}{\partial r} \,, \tag{6.43}$$

where we used the lens equation

$$s = r - \Phi_r \,, \qquad \partial_r s = 1 - \Phi_{rr} \,. \tag{6.44}$$

For a point mass, there exist two lensed images. When two images are not spatially resolved, the magnification of the lensed images is the sum of two, and we derive the master equation for microlening

$$\mu = \left(\frac{s}{r}\frac{\partial s}{\partial r}\right)_1^{-1} + \left(\frac{s}{r}\frac{\partial s}{\partial r}\right)_2^{-1} = \frac{u^2+2}{u\sqrt{u^2+4}} \,, \qquad u := \frac{s}{\theta_E} \,. \tag{6.45}$$

6.2.5 Galaxy-Galaxy Lensing

Galaxy-galaxy lensing is used to refer to the two-point correlation of the galaxies at one point and the lensing signal measured by background galaxies at the other point. In short, it measures the galaxy-matter cross-correlation. Compared to the cosmic shear measurements, the advantage here is that we have well-defined lenses (lens galaxies) in the foreground, such that the shear measurements in galaxy-galaxy lensing are less susceptible to other systematics.

Assuming spherical symmetry, we can readily derive the lensing convergence and the shear as

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c} , \qquad \gamma(\theta) = \bar\kappa(\theta) - \kappa(\theta) , \tag{6.46}$$

where the "comoving" critical surface density is[3]

$$\Sigma_c(z_1, z_2) = \frac{c^2}{4\pi G}\frac{r_s}{r_l r_{ls}}\frac{1}{1+z_l} = 1.663\times 10^{18}\ hM_\odot\text{Mpc}^{-2}\ \frac{r_s}{r_l}\left(\frac{r_{ls}}{h^{-1}\text{Mpc}}\right)^{-1}\frac{1}{1+z_l} . \tag{6.47}$$

Since we are measuring the excess matter around galaxies, the lensing observables are related to the projected galaxy-matter correlation function:

$$w(R) = \int_{-\infty}^{\infty} dz\ \xi_{\rm gm}\left(r = \sqrt{R^2+z^2}\right) = \int_0^\infty \frac{dk_\perp k_\perp}{2\pi}\ P_{\rm gm}(k_\perp)\ J_0(k_\perp R) , \tag{6.48}$$

where the integration along the line-of-sight is performed. However, note that this is valid only on small angle, as the observed angular separation θ, not the physical separation R is kept fixed. Under the small-angle approximation, the convergence at a given separation can be derived as

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c} = \int dz\ \frac{\bar\rho_m}{\Sigma_c}\ (1+\xi_{gm}) = \frac{\bar\rho_m}{\Sigma_c}\ w(R) , \tag{6.49}$$

and the remaining lensing observables are

$$\bar\kappa(\theta) = \frac{2\pi}{\pi\theta^2}\int_0^\theta d\theta\ \theta\ \kappa(\theta) = \frac{3H_0^2\Omega_m}{2}\frac{(\bar r_s - \bar r_l)\bar r_l}{a_l \bar r_s}\int \frac{dk_\perp\ k_\perp}{2\pi}\ P_{gm}(k_\perp;\bar r_l)\frac{2J_1(k_\perp \bar r\theta)}{k_\perp \bar r\theta} , \tag{6.50}$$

$$\gamma_T(\theta) = \bar\kappa(\theta) - \kappa(\theta) = \frac{3H_0^2\Omega_m}{2}\frac{(\bar r_s - \bar r_l)\bar r_l}{a_l \bar r_s}\int \frac{dk_\perp\ k_\perp}{2\pi}\ P_{gm}(k_\perp;\bar r_l)\ J_2(k_\perp \bar r\theta) , \tag{6.51}$$

$$\Delta\Sigma(R) = \Sigma_c\gamma_T = \bar\rho_m\int_{-\infty}^{\infty} dz\left[\frac{2}{R^2}\int_0^R dR'\ R'\ \xi_{\rm gm}(R',z) - \xi_{\rm gm}(R,z)\right] . \tag{6.52}$$

6.3 Weak Lensing Observables

6.3.1 Ellipticity of Galaxies

The ellipticity ϵ of galaxies is measured in terms of its semi-major axis a and the semi-minor axis b or in terms of the axis ratio q as

$$\epsilon := \frac{a^2-b^2}{a^2+b^2} \equiv \frac{1-q^2}{1+q^2} \equiv \frac{\delta - \frac{1}{2}\delta^2}{1-\delta+\frac{1}{2}\delta^2} \simeq \delta , \qquad q := \frac{b}{a} := 1-\delta . \tag{6.53}$$

In an idealized case of round galaxies, the ellipticity ϵ, the axis ratio q, and the distortion δ are a measure of gravitational lensing effects of intervening matter, and they are equivalent in the weak lensing regime. In observations, the center of the galaxy and its ellipticity moment are measured by using some weight function $W[I_\nu(\hat n)]$ of the observed intensity as

$$\hat n_o := \frac{\int d^2\hat n\ \hat n\ W[\hat n]}{\int d^2\hat n\ W[\hat n]} , \qquad \mathcal{M}_{ij} := \frac{\int d^2\hat n\ (\hat n - \hat n_o)_i(\hat n - \hat n_o)_j\ W[\hat n]}{\int d^2\hat n\ W[\hat n]} , \tag{6.54}$$

[3] Multiply by $(1+z_l)^2$ for physical critical surface density. Note that sometimes people use the physical angular diameter distances, while using comoving coordinates for other quantities, in which $(1+z_l)^2$ appears in the equation, instead of $(1+z_l)$.

where the simplest weight function is just the observed intensity $W = I[\hat{n}]$. Given the ellipticity moment, we can define the ellipticity vector and the position angle as

$$\boldsymbol{\epsilon} := \left(\frac{\mathcal{M}_{xx} - \mathcal{M}_{yy}}{\mathcal{M}_{xx} + \mathcal{M}_{yy}} \,,\ \frac{2\mathcal{M}_{xy}}{\mathcal{M}_{xx} + \mathcal{M}_{yy}}\right) := (\epsilon_+, \epsilon_\times) = \epsilon(\cos 2\Theta,\ \sin 2\Theta) \,, \qquad \tan 2\Theta \equiv \frac{2\mathcal{M}_{xy}}{\mathcal{M}_{xx} - \mathcal{M}_{yy}} \,. \tag{6.55}$$

Note that the ellipticity vector is headless, such that it is identical under 180 degree rotation, or spin 2. Since only the ellipticity vector matters, the ellipticity moments $\mathcal{M}$ are often defined without the denominator.

6.3.2 Lensing Polarization

The ellipticity moments of the source galaxies would be what we measure in the absence of gravitational lensing. However, the gravitational lensing changes the observed ellipticity moments. Now, for simplicity, we will ignore rotation ($\omega = 0$) and express the distortion matrix in our coordinate:

$$\mathbb{D} = \mathbb{I} - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \,, \tag{6.56}$$

and the magnification matrix is then the inverse of the distortion matrix:

$$\mathbf{M}_{ij} := \mathbb{D}^{-1}_{ij} = \frac{1}{|\mathbb{D}|}\begin{pmatrix} 1-\kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & 1-\kappa-\gamma_1 \end{pmatrix} \,, \qquad \mu := |\mathbf{M}| = |\mathbb{D}|^{-1} = \frac{1}{(1-\kappa)^2 - \gamma^2} \,. \tag{6.57}$$

Further, assuming the surface brightness conservation due to gravitational lensing (i.e., no frequency change), the observed ellipticity moments are related to those in the source rest-frame as

$$\mathcal{M}^I_{ij} := \int d^2\hat{n}\ \hat{n}_i\hat{n}_j W[\hat{n}] = \int |\mathbf{M}| d^2\hat{s}\ \ \mathbf{M}_{ik}\hat{s}_k\ \mathbf{M}_{jl}\hat{s}_l\ W[\hat{s}] \simeq \mu\, \mathbf{M}_{ik}\mathbf{M}_{jl}\mathcal{M}^s_{kl} \,, \tag{6.58}$$

where we assumed $\hat{n}_o = 0$ and the source size is small that the magnification matrix is constant over the area. Using the definition of the source ellipticity moments

$$\mathcal{M}^s_{11} = \frac{1+\epsilon^s_+}{2}\mathcal{M} \,, \qquad \mathcal{M}^s_{22} = \frac{1-\epsilon^s_+}{2}\mathcal{M} \,, \qquad \mathcal{M}^s_{12} = \frac{\epsilon^s_\times}{2}\mathcal{M} \,, \qquad \mathcal{M} := \mathcal{M}^s_{11} + \mathcal{M}^s_{22} \,, \tag{6.59}$$

the observed ellipticity can be derived in terms of the magnification matrix as

$$\epsilon^I_+ = \frac{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times\mathbf{M}_{12}(\mathbf{M}_{11} - \mathbf{M}_{22}) - 2\epsilon^s_+\mathbf{M}^2_{12} - (1-\epsilon^s_+)\mathbf{M}^2_{22}}{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times\mathbf{M}_{12}(\mathbf{M}_{11} + \mathbf{M}_{22}) + 2\epsilon^s_+\mathbf{M}^2_{12} + (1-\epsilon^s_+)\mathbf{M}^2_{22}} \,, \tag{6.60}$$

$$\epsilon^I_\times = \frac{2\mathbf{M}_{12}\left[\epsilon^s_\times\mathbf{M}_{12} + (1-\epsilon^s_+)\mathbf{M}_{22}\right] + 2\mathbf{M}_{11}\left[(1+\epsilon^s_+)\mathbf{M}_{12} + \epsilon^s_\times\mathbf{M}_{22}\right]}{(1+\epsilon^s_+)\mathbf{M}^2_{11} + 2\epsilon^s_\times\mathbf{M}_{12}(\mathbf{M}_{11} + \mathbf{M}_{22}) + 2\mathbf{M}^2_{12} + (1-\epsilon^s_+)\mathbf{M}^2_{22}} \,, \tag{6.61}$$

where the relation is exact. In terms of the lensing convergence and shear,[4] we derive

$$\epsilon^I_+ = \frac{\epsilon^s_+\left[(1-\kappa)^2 + \gamma_1^2 - \gamma_2^2\right] + 2\epsilon^s_\times\gamma_1\gamma_2 + 2\gamma_1(1-\kappa)}{2\epsilon^s_+\gamma_1(1-\kappa) + 2\epsilon^s_\times\gamma_2(1-\kappa) + (1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \,, \tag{6.62}$$

$$\epsilon^I_\times = \frac{\epsilon^s_\times\left[(1-\kappa)^2 - \gamma_1^2 + \gamma_2^2\right] + 2\epsilon^s_+\gamma_1\gamma_2 + 2\gamma_2(1-\kappa)}{2\epsilon^s_+\gamma_1(1-\kappa) + 2\epsilon^s_\times\gamma_2(1-\kappa) + (1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \,. \tag{6.63}$$

For circular sources, where $\mathcal{M}^s_{11} = \mathcal{M}^s_{22} \neq 0$, and $\mathcal{M}_{12} = 0$ (or $\epsilon^s_+ = \epsilon^s_\times = 0$), the observed ellipticity becomes

$$\epsilon^I_+ = \frac{2\gamma_1(1-\kappa)}{(1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_1 \,, \qquad \epsilon^I_\times = \frac{2\gamma_2(1-\kappa)}{(1-\kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_2 \,, \tag{6.64}$$

where we expanded to the linear order in the last step. Assuming there is no ellipticity correlation of the source galaxies

$$\langle\epsilon_+\rangle = \langle\epsilon_\times\rangle = \langle\epsilon_+\epsilon_\times\rangle = \frac{1}{2}\left\langle\epsilon^2\right\rangle\langle\sin 4\phi\rangle = 0 \,, \tag{6.65}$$

$$\left\langle\epsilon_+^2\right\rangle = \left\langle\epsilon_\times^2\right\rangle = \left\langle\epsilon^2\right\rangle\left\langle\cos^2 2\phi\right\rangle = \frac{1}{2}\left\langle\epsilon^2\right\rangle \,, \tag{6.66}$$

[4]With rotation, the magnification matrix is not symmetric, $\mathbf{M}_{12} \neq \mathbf{M}_{21}$.

the observed ellipticity correlation becomes

$$\xi_{\epsilon_+^I}(\theta) = \left\langle \epsilon_+^I(0)\epsilon_+^I(\theta) \right\rangle = \xi_\delta(\theta)\left(1 - \sigma_\epsilon^2 + \frac{1}{4}\sigma_\epsilon^4\right) = \xi_{\epsilon_\times^I} \ , \tag{6.67}$$

to be compared to the typical ellipticity

$$\sigma_\epsilon = \langle \epsilon^s \rangle^{1/2} \simeq 0.3 \ . \tag{6.68}$$

• modified gravity, no galaxy bias