Physical Cosmology

Exercise 0

Spring Semester 2025
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Issued: 21.02.2025
Due: -

This optional exercise sheet brings together a collection of reading assignments and exercises of General Relativity. It is intended for students who would like a review or an introduction to the topic.

Reading material

A suggested reading on the basics of General Relativity is Section 2.1 of Dodelson, S., *Modern Cosmology*, 2003, Academic Press.

Contraction of tensors

Let $T^{\mu\nu}{}_{\rho\lambda}$ be a rank (2,2) tensor. The contraction of the tensor is given by

$$S^{\mu}{}_{\lambda} = \sum_{\nu} T^{\mu\nu}{}_{\nu\lambda} = T^{\mu\nu}{}_{\nu\lambda}\,,$$

where we have used Einstein's summation convention in the last equality. Show that the contraction of a (2,2) tensor is again a tensor by showing that $S^{\mu}{}_{\nu}$ transforms like a tensor under coordinate transformations.

Metric on the n-sphere

1. A 2-dimensional sphere embedded in a 3-dimensional Euclidean space is described by the equation $x^2 + y^2 + z^2 = r^2$.

   Show that each point on the sphere can be expressed in terms of spherical coordinates (r, θ, ϕ) as:

   $$x = r \sin\theta \cos\phi$$
   $$y = r \sin\theta \sin\phi$$
   $$z = r \cos\theta\,.$$

   Derive the metric tensor g_{ij} describing the geometry on the surface of the 2-dimensional sphere in spherical coordinates.

2. Now imagine a 3-dimensional sphere of constant radius r in 4-dimensional Euclidean space, described by the equation $x^2 + y^2 + z^2 + w^2 = r^2$. In analogy with the previous question, show that each point on the surface of the 3-dimensional sphere can be expressed in terms of 4-dimensional spherical coordinates (r, χ, θ, ϕ) as

   $$x = r \sin\chi \sin\theta \cos\phi$$
   $$y = r \sin\chi \sin\theta \sin\phi$$
   $$z = r \sin\chi \cos\theta$$
   $$w = r \cos\chi\,.$$

Derive the metric tensor g_{ij} describing the geometry on the surface of the 3-dimensional sphere in spherical coordinates.

Geodesics on the 2-sphere

1. Compute the Christoffel symbols of the Lorentzian metric described by the line element

$$ds^2 = -dt^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \, .$$

   Notice that the spacial metric ($dt = 0$) describes a 2-dimensional sphere of fixed radius r.

2. Use the geodesic equation to find the equation of motion for a massive test particle moving on the sphere.

3. A great circle on the sphere can be described by the following equations

$$\theta(s) = s + a \qquad \phi(s) = b \, ,$$

   where s parametrises the path length along the curve and a and b are constants. Show that this family of great circles satisfy the geodesic equation derived in the previous question. Describe what happens for two particles traveling along geodesics when they start parallel to each other at the equator.

4. Find the Ricci tensor for this metric. Show that the contraction of the tensor leads to a constant Ricci scalar:

$$R \equiv g^{\mu\nu} R_{\mu\nu} = \frac{2}{r^2} \, .$$