Physical Cosmology

Problem set 10

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Exercise 1 [5 points]

Einstein rings form when the observer, lens and source are all exactly aligned – The photons emitted by the source are deflected by the lens (well-approximated by a point mass), forming a luminous ring around it. Starting from the deflection angle

$$\hat{\alpha} = \frac{4GM}{bc^2}\,, \tag{1}$$

show that the radius of this ring, called the Einstein radius, is (in radians) given by

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{\mathcal{D}_{ls}}{\mathcal{D}_l\mathcal{D}_s}}\,, \tag{2}$$

where M is the mass of the lens, $\mathcal{D}_l$ is the distance of the observer to the lens, $\mathcal{D}_s$ is the distance of the observer to the source, and $\mathcal{D}_{ls}$ is the distance between lens and source.

Hint: Note that equation (1) holds true only if the impact parameter is a lot larger than the Schwarzschild radius, $b \gg 2GM/c^2$, which implies that the deflection angle $\hat{\alpha}$ is small. Using this, derive the *lens equation*, $\mathcal{D}_s\hat{s} = \mathcal{D}_s\hat{n} - \mathcal{D}_{ls}\hat{\alpha}$. Then, set $\hat{s}$ to zero (which means that source, lens and observer are aligned).

Exercise 2 [5 points]

Consider the case in which light rays are deflected by density perturbations along the line of sight instead of a point mass (this is called "weak gravitational lensing"). The lens equation now reads

$$\hat{\mathbf{s}} = \hat{\mathbf{n}} - \hat{\nabla}\Phi\,, \tag{3}$$

where the *lensing potential* Φ is given by the integral

$$\Phi = \int_0^{\bar{r}_s} d\bar{r}\,\frac{g(\bar{r},\bar{r}_s)}{\bar{r}^2}2\Psi\,, \qquad g(\bar{r},\bar{r}_s) \equiv \bar{r}^2\left(\frac{\bar{r}_s-\bar{r}}{\bar{r}_s\bar{r}}\right)\,. \tag{4}$$

To quantify the weak gravitational lensing effects, we define the *distortion matrix*

$$\mathbb{D}_{ij} = \frac{\partial s_i}{\partial n_j} = \mathbb{I}_{ij} - \begin{pmatrix} \kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & \kappa-\gamma_1 \end{pmatrix}\,, \tag{5}$$

where κ is the *convergence* and $(\gamma_1,\,\gamma_2)$ are the shear components.

a) Show that the weak lensing observables are given by

$$\kappa = \int_0^{\bar{r}_s} \mathrm{d}\bar{r}\, \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2} \hat{\nabla}^2 \Psi \,, \tag{6}$$

$$\gamma_1 = \int_0^{\bar{r}_s} \mathrm{d}\bar{r}\, \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2} \left(\hat{\nabla}_1^2 - \hat{\nabla}_2^2 \right) \Psi \,, \tag{7}$$

$$\gamma_2 = 2 \int_0^{\bar{r}_s} \mathrm{d}\bar{r}\, \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2} \hat{\nabla}_1 \hat{\nabla}_2 \Psi \,. \tag{8}$$

b) By performing a 2D-Fourier transformation (assuming that the survey area is small), prove the relations

$$\kappa(\mathbf{l}) = -\frac{l^2}{2} \Phi(\mathbf{l}) \,, \qquad \gamma_1(\mathbf{l}) = \cos(2\phi_l) \kappa(\mathbf{l}) \,, \qquad \gamma_2(\mathbf{l}) = \sin(2\phi_l) \kappa(\mathbf{l}) \,, \tag{9}$$

where $\mathbf{l} = l\,(\cos\phi_l, \sin\phi_l)$.

c) The definitions of γ_1 and γ_2 depend on an arbitrary direction chosen on the 2D-sky. This results in the appearance of factors depending on the orientation of the **l**-mode in the equations for their power spectra. To construct lensing observables which do not depend on an arbitrary orientation, we define the E- and B-modes:

$$\begin{pmatrix} E(\mathbf{l}) \\ B(\mathbf{l}) \end{pmatrix} = R[-2\phi_l] \begin{pmatrix} \gamma_1(\mathbf{l}) \\ \gamma_2(\mathbf{l}) \end{pmatrix} \,, \qquad R[-2\varphi] = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \,. \tag{10}$$

Show that the E-modes are equal to the convergence, while the B-modes are vanishing,

$$E(\mathbf{l}) = \kappa(\mathbf{l}) \,, \qquad B(\mathbf{l}) = 0 \,, \qquad P_E(l) = P_\kappa(l) \,, \qquad P_B(l) = P_{EB}(l) = 0 \,. \tag{11}$$