# Physical Cosmology

## Problem Set 1

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**Exercise 1** (5 points)

Consider the Friedmann equation

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{G}{a}\left(\frac{4\pi}{3}\rho a^3\right) = -\frac{K}{2}\,, \tag{1}$$

and treat it as an ordinary differential equation for the scale factor $a(t)$, with $t$ cosmic time. To obtain an exact solution we need to express the energy density $\rho$ as a function of $t$, or even better, as a function of $a(t)$. This can be achieved by integrating the continuity equation:

$$\frac{d\rho}{dt} + \frac{3}{a}\frac{da}{dt}(P+\rho) = 0\,, \tag{2}$$

and assuming an equation of state that relates the pressure $P$ with the energy density $\rho$.

Compute $a(t)$ in the simple scenario where the equation of state reads $P = w\rho$. In particular, for:

- $w = \frac{1}{3}$
- $w = 0$
- $w = -1$
- $w = 0$ and $\rho = 0$

Consider the Universe to be flat ($K = 0$) in the first three cases. Repeat the calculations for $a(\eta)$, where $\eta$ is the conformal time defined as $dt = a\,d\eta$. Discuss at which point of the Big Bang timeline we might regard the solutions above as useful.
*Hint: it is possible to find a simple way to obtain $a(\eta)$ from the functional form of $a(t)$, without having to repeat all the previous steps.*

**Exercise 2** (5 points)

A perfectly homogeneous and isotropic Universe is described by the FLRW metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\;\bar{g}_{\alpha\beta}\;dx^\alpha dx^\beta\;, \tag{3}$$

where $\bar{g}_{\alpha\beta}$ is a maximally symmetric three-metric, function of spatial coordinates only. In our notation we use Greek letters $\mu, \nu, ...$ for 4D spacetime components and $\alpha, \beta, ...$ for 3D spatial components.

Given the FLRW metric, compute the non-vanishing components of the following objects: (*you can assume* $\bar{g}_{\alpha\beta} = \delta_{\alpha\beta}$ if you are struggling)

- Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left( \frac{\partial}{\partial x^{\rho}} g_{\nu\sigma} + \frac{\partial}{\partial x^{\nu}} g_{\rho\sigma} - \frac{\partial}{\partial x^{\sigma}} g_{\nu\rho} \right) . \tag{4}$$

  Notice that the inverse metric tensor $g^{\mu\sigma}$ can be obtained via

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu} . \tag{5}$$

- Riemann tensor

$$R^{\mu}_{\nu\rho\sigma} = \frac{\partial}{\partial x^{\rho}} \Gamma^{\mu}_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}} \Gamma^{\mu}_{\nu\rho} + \Gamma^{\epsilon}_{\nu\sigma} \Gamma^{\mu}_{\rho\epsilon} - \Gamma^{\epsilon}_{\nu\rho} \Gamma^{\mu}_{\sigma\epsilon} . \tag{6}$$

- Ricci tensor

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} . \tag{7}$$

- Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} . \tag{8}$$

- Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} . \tag{9}$$

- Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} , \tag{10}$$

  where the energy-momentum tensor $T_{\mu\nu}$ is diagonal, with components:

$$T^{0}_{0} = -\rho , \qquad T^{\alpha}_{\beta} = P \, \delta^{\alpha}_{\beta} . \tag{11}$$

  You should recover the Friedmann equations.