Physical Cosmology

Problem set 3

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Exercise 1 (3 points)

In the early Universe, neutrinos and anti-neutrinos are kept in thermal equilibrium with the hot plasma by interacting with electrons, neutrons and protons via the weak interaction. Moreover, the creation and annihilation of neutrinos and anti-neutrinos is balanced in thermal equilibrium:

$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \,. \tag{1}$$

As the Universe cools down, the weak interaction is no longer efficient and neutrinos decouple from the thermal bath.

Using the cross section for the weak interaction:

$$\sigma_{\text{weak}} = 5 \cdot 10^{-43} \left(\frac{k_B T}{1\ \text{MeV}} \right)^2 \text{cm}^2 \,, \tag{2}$$

and assuming massless neutrinos, compute the following quantities:

- Temperature, time and redshift corresponding to neutrino decoupling
- Temperature of the neutrino background during the matter-dominated-era at $z = 1000$

Exercise 2 (3 points)

Assume that the Universe had no baryon asymmetry and that the radiation number density at $z = 0$ was $n_{\gamma,0} = 400\,\text{cm}^{-3}$. Further assume that the expectation value of the product of interaction cross section and particle velocity for annihilation reactions is given by $\langle \sigma v \rangle \approx \frac{\hbar^2}{m_\pi^2 c}$, where m_π denotes the mass of the pion.

- What is the temperature of the thermal bath when baryons and anti-baryons stop to annihilate significantly?
- Compute the ratio of baryonic and anti-baryonic number density to photon number density. What would be the relic number density of baryons and anti-baryons at the present time? An order-of-magnitude calculation is sufficient
- According to the ΛCDM model, the Universe undergoes different evolutionary phases: first, radiation dominates the dynamics, then non-relativistic matter, and finally dark energy as the cosmological constant. How would the succession of phases change and how would their periods be modified if the Universe had no asymmetry? Explain qualitatively

Exercise 3 (4 points)

Start from the phase-space distribution function (in natural units)

$$f(p,t) = \frac{g}{(2\pi)^3} \frac{1}{\exp\left[(E-\mu)/T\right] \pm 1} \ , \qquad \begin{cases} + : \text{fermions} \\ - : \text{bosons} \end{cases} \tag{3}$$

to derive the number density $n(t)$, the energy density $\rho(t)$ and the pressure $P(t)$

$$\begin{aligned} n(t) &= \int d^3\mathbf{p}\ f(p,t) = \frac{g}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2-m^2}E\ dE}{\exp\left[(E-\mu)/T\right] \pm 1} \ , \\ \rho(t) &= \int d^3\mathbf{p}\ E\ f(p,t) = \frac{g}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2-m^2}E^2\ dE}{\exp\left[(E-\mu)/T\right] \pm 1} \ , \\ P(t) &= \int d^3\mathbf{p}\, \frac{p^2}{3E} f(p,t) = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2-m^2)^{3/2} dE}{\exp\left[(E-\mu)/T\right] \pm 1} \ . \end{aligned} \tag{4}$$

In the non-relativistic limit, derive

$$f(p,t) = \frac{g}{(2\pi)^3} \exp\left(-\frac{m-\mu}{T}\right) \exp\left(-\frac{p^2}{2mT}\right) \ , \tag{5}$$

and

$$n(t) = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-\frac{m-\mu}{T}\right] \ , \qquad \rho(t) = mn(t) \ , \qquad P(t) = n(t)T \ , \tag{6}$$

and in the relativistic limit, derive

$$n(t) = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & : \text{bosons} \\ \frac{3}{4} & : \text{fermions} \end{cases} \ , \qquad \rho(t) = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & : \text{bosons} \\ \frac{7}{8} & : \text{fermions} \end{cases} \ , \qquad P(t) = \frac{1}{3}\rho(t) \ . \tag{7}$$