Physical Cosmology

Problem set 6

Spring Semester 2025
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Issued: 4.4.2025
Due: 11.4.2025

Exercise 1 (3 points)

Consider a fluid made of N independent components labeled by the index $i = 1, .., N$. Each individual fluid has mass density ρ_i, pressure P_i, and velocity $\mathbf{v}_i$. The Euler and Poisson equations are given by:

$$\dot{\rho}_i + \nabla \cdot (\rho_i \mathbf{v}_i) = 0 \,, \tag{1}$$

$$\dot{\mathbf{v}}_i + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i = -\frac{1}{\rho_i}\nabla P_i - \nabla \Phi \,, \tag{2}$$

$$\Delta \Phi = 4\pi G \sum_{i=1}^{N} \rho_i \,, \tag{3}$$

where the spatial derivatives are taken with respect to the physical distance $\mathbf{r} := a(t)\mathbf{x}$. Introducing the perturbation variables δ_i, δP_i, $\mathbf{u}_i$ and $\delta\Phi$ as

$$\rho_i = \bar{\rho}_i + \delta\rho_i = \bar{\rho}_i \left(1 + \delta_i\right) \,, \qquad P_i = \bar{P}_i + \delta P_i \,, \qquad \mathbf{v}_i = H\mathbf{r} + \mathbf{u}_i \,, \qquad \Phi = \bar{\Phi} + \delta\Phi \,, \tag{4}$$

derive the Euler and Poisson equations at linear order in the perturbations:

$$\dot{\delta}_i + \frac{1}{a}\nabla \cdot \mathbf{u}_i = -\frac{1}{a}\nabla \cdot \left(\delta_i \mathbf{u}_i\right) \,, \tag{5}$$

$$\dot{\mathbf{u}}_i + H\mathbf{u}_i + \frac{1}{a}(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i = -\frac{1}{a\bar{\rho}_i}\frac{\nabla \delta P_i}{1 + \delta_i} - \frac{1}{a}\nabla \delta\Phi \,, \tag{6}$$

$$\frac{1}{a^2}\Delta \delta\Phi = 4\pi G \sum_{i=1}^{n} \bar{\rho}_i \delta_i \,, \tag{7}$$

where the spatial derivatives are now taken with respect to the comoving coordinates $\mathbf{x}$.

Hint: First, you need to find the relation between the operators $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{r}}$, and between the time derivatives $\frac{\partial}{\partial t}\big|_{\mathbf{r}}$ and $\frac{\partial}{\partial t}\big|_{\mathbf{x}}$. Furthermore, you need to use the background relations:

$$\dot{\bar{\rho}}_i + 3H\bar{\rho}_i = 0 \,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_{i=1}^{N} \bar{\rho}_i \,. \tag{8}$$

Exercise 2 (4 points)

We now consider equations (5)–(7) for a single pressure-less fluid, i.e., making the substitutions $\delta_i \equiv \delta$, $\bar{\rho}_i \equiv \bar{\rho}_m$, $\mathbf{u}_i \equiv \mathbf{u}$, $\delta P_i \equiv 0$. We then introduce the divergence and curl of the peculiar velocity $\mathbf{u}$:

$$\theta := -\frac{1}{a}\nabla \cdot \mathbf{u}\,, \qquad \boldsymbol{\omega} := \frac{1}{a}\nabla \times \mathbf{u}\,. \tag{9}$$

a) Use equation (6) to show that the linear-order solution $\boldsymbol{\omega}^{(1)}$ for the curl is decaying as the Universe expands.

b) Starting from the equations (5)–(7), show that the linear-order solutions $\delta^{(1)}(t,\mathbf{k})$ and $\theta^{(1)}(t,\mathbf{k})$ satisfy

$$\delta^{(1)}(t,\mathbf{k}) = D(t)\hat{\delta}(\mathbf{k})\,, \qquad \theta^{(1)}(t,\mathbf{k}) = HfD(t)\hat{\delta}(\mathbf{k})\,, \qquad f := \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\,, \tag{10}$$

where the *growth rate* D satisfies the differential equation

$$\ddot{D} + 2H\dot{D} - 4\pi G\bar{\rho}_m D = 0\,, \tag{11}$$

and is normalized to unity at some early epoch t_0, i.e., $D(t_0) = 1$ and $\delta(t_0,\mathbf{k}) =: \hat{\delta}(\mathbf{k})$.

Hint: Rewrite equation (5) and the divergence of (6) in terms of θ.

c) Show that in a matter-dominated universe, the density contrast grows as the scale factor, $D(t) = a(t)$.

Hint: Recall that in a matter-dominated universe $a(t) = (\frac{3}{2}H_0 t)^{2/3}$. You can neglect decaying solutions.

Exercise 3 (3 points)

Starting from the relations

$$\delta_s \approx \delta_g - \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}r}\,, \qquad \mathcal{V} = -f\frac{\partial}{\partial r}\Delta^{-1}\delta_m\,, \qquad \delta_g = b\delta_m\,, \tag{12}$$

prove that the following expression for the galaxy density fluctuations in redshift-space holds

$$\delta_s(\mathbf{s}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{s}} \left(b + f\mu_k^2\right)\delta_m(\mathbf{k})\,, \tag{13}$$

and hence, the galaxy power spectrum in redshift-space satisfies the *Kaiser formula*

$$P_s(k,\mu_k) = (b + f\mu_k^2)^2 P_m(k)\,, \tag{14}$$

where b is the galaxy bias and μ_k is the cosine angle between a given Fourier mode and the line-of-sight direction: $\mu_k := \hat{s}\cdot\hat{k}$.