Physical Cosmology

Problem Set 7

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Exercise 1 (4 points)

a) Derive the gauge transformations of the metric and matter scalar perturbations α, β, φ, γ, U and v

b) Check that the scalar variables φ_v, φ_χ, v_χ, φ_δ, δ_v and α_χ defined in the lecture notes are gauge-invariant.

Hint: You can solve a) by using that $\delta_\xi g_{\mu\nu}(x) = -\mathcal{L}_\xi g_{\mu\nu}$ and $\delta_\xi u^\mu = -\mathcal{L}_\xi u^\mu$ as described in the lecture notes. Having solved a), the fact that φ_v, φ_χ, v_χ and α_χ are gauge-invariant follows easily. To show that φ_δ and δ_v are gauge-invariant, you additionally need to consider the gauge-transformation of the stress-energy tensor, $\delta_\xi T_{\mu\nu}$, to determine how δP and $\delta\rho$ transform. Also, recall that $\bar{\rho} \propto a^{3(1+w)}$ and $\bar{P} + \bar{\rho} = \bar{\rho}(1+w)$.

Exercise 2 (4 points)

Consider a spatially flat inhomogeneous universe described by a metric fixed to the conformal Newtonian gauge:

$$\mathrm{d}s^2 = -a^2(1+2\psi)\mathrm{d}\eta^2 + a^2(1+2\phi)\delta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \tag{1}$$

evolving according to the Einstein equations. Derive the following relations

$$\Delta\phi = -4\pi G a^2 \bar{\rho}[\delta + 3\mathcal{H}v(1+w)] \,, \qquad \phi + \psi = -8\pi G\,\Pi \,. \tag{2}$$

Exercise 3 (2 points)

Show that on scales $k \ll \mathcal{H}$ the comoving-gauge curvature perturbation $\varphi_v = \phi - \mathcal{H}v$ is related to the Newtonian potential ϕ as

$$\varphi_v = \frac{5+3w}{3(1+w)}\phi \,. \tag{3}$$

Assume that $\Pi = 0$ and use that $\dot{\phi} = 0$ on super-Hubble scales.