Physical Cosmology

Problem Set 8

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Exercise 1: Linear evolution of the inflaton fluid quantities (3 points)

a) Starting from the energy-momentum tensor of the inflaton field,

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi - g_{\mu\nu}V(\phi)\,, \tag{1}$$

and considering scalar perturbations in the conformal Newtonian gauge,

$$\mathrm{d}s^2 = -a^2(1+2\alpha)\mathrm{d}\eta^2 + a^2\delta_{\alpha\beta}(1+2\varphi)\mathrm{d}x^\alpha\mathrm{d}x^\beta\,, \qquad u^\mu = \frac{1}{a}(1-\alpha, -U^{,\alpha})\,, \tag{2}$$

derive the linear-order fluid quantities:

$$\delta\rho = \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 + \partial_\phi V\delta\phi\,, \qquad \delta p = \dot\phi\,\delta\dot\phi - \alpha\,\dot\phi^2 - \partial_\phi V\delta\phi\,. \tag{3}$$

b) Derive the equation of motion for the linear perturbation in the inflaton field:

$$\delta\ddot\phi + 3H\delta\dot\phi + \left(\partial^2_\phi V + \frac{k^2}{a^2}\right)\delta\phi = \dot\phi(\dot\alpha+\kappa) + (2\ddot\phi + 3H\dot\phi)\alpha\,. \tag{4}$$

Note that this equation is in Fourier space and k is the wavenumber. Also, recall that $\kappa = 3H\alpha - 3\dot\varphi$, and that $\alpha = -\varphi$ in a universe without anisotropic stress.

Exercise 2: Linear evolution of the curvature perturbation (4 points)

We define the gauge-invariant Mukhanov variable Φ as

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\rho+p)}\varphi_\chi\,, \tag{5}$$

where K is the spatial curvature and the gauge-invariant variables φ_v and φ_χ are defined in the lecture notes. Using the Einstein equations, derive the governing equations for the Mukhanov variable

$$\begin{aligned}\Phi &= \frac{H^2}{4\pi G(\rho+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{2H^2\,\Pi}{\rho+p}\,,\\ \dot\Phi &= -\frac{H}{4\pi G(\rho+p)}\frac{k^2c_s^2}{a^2}\varphi_\chi - \frac{H}{\rho+p}\left(e - \frac{2k^2}{3a^2}\Pi\right),\end{aligned} \tag{6}$$

where $e = \delta p - c_s^2 \delta\rho$, $c_s^2 = \dot{p}/\dot{\rho}$ is the speed of sound, and Π is the scalar part of the anisotropic stress tensor.

Furthermore, assuming a flat universe with $K = 0$, show that the second equation can be rearranged as

$$\dot{\Phi} = \Xi - \frac{H}{\rho + p}\frac{k^2}{a^2}\left(\frac{c_s^2}{4\pi G}\varphi_\chi - \frac{2}{3}\Pi\right), \qquad \Xi \equiv \frac{\dot{\bar{\rho}}\,\delta p - \dot{\bar{p}}\,\delta\rho}{3(\bar{\rho} + \bar{p})^2}\,. \tag{7}$$

This means that the comoving gauge curvature perturbation is conserved in the super horizon limit ($k \to 0$) if $\Xi = 0$, i.e., if the initial conditions are adiabatic.

Exercise 3: Inflation parameters for a given model (3 points)

Derive the equations in Section ***5.3.3 A Worked Example*** of the lecture notes for $\alpha = 2$.

Bonus Exercise [1]: Numerical solution of a simplified Einstein-Boltzmann system (5 points)

Consider a universe where all the matter behaves as CDM, and radiation (photons + neutrinos) is described by a perfect fluid. The Einstein-Boltsmann system in the Newtonian gauge reads

$$\begin{cases} \Theta_0' + k\Theta_1 = -\Phi' \\ \Theta_1' - \dfrac{k}{3}\Theta_0 = -\dfrac{k}{3}\Phi \\ \delta' + kv = -3\Phi' \\ v' + \mathcal{H}v = -k\Phi \\ k^2\Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2\left(\rho_m\delta + 4\rho_r\Theta_0\right)\,, \end{cases}$$

where the monopole Θ_0 and the dipole Θ_1 are the first two moments of the radiation distribution function.

Assume adiabatic initial conditions for each Fourier mode:

$$\delta^{(i)} = \frac{3}{2}\Phi^{(i)}, \quad \Theta_0^{(i)} = \frac{1}{2}\Phi^{(i)}, \quad \Theta_1^{(i)} = -\frac{k}{6\mathcal{H}}\Phi^{(i)}, \quad v = -\frac{k}{2\mathcal{H}}\Phi^{(i)}\,,$$

with $\mathcal{H}$ computed at the time when the initial conditions are set. Notice that for every k, one needs to choose $\eta^{(i)}$ such that $k/\mathcal{H} \ll 1$ (i.e., when the mode is outside the horizon). Since the equations are linear, we have the freedom to re-scale $\Phi^{(i)} = 1$.

Use the following cosmological parameters

$$\Omega_r = 9.22\times 10^{-5}\,, \qquad \Omega_m = 0.314\,, \qquad \Omega_\Lambda = 1 - \Omega_r - \Omega_m 4\,, \qquad h = 0.674\,. \tag{8}$$

- Rewrite the Einstein-Boltzmann system using the e-folding time $x = \log a$ instead of the conformal time η.
- Solve numerically (e.g., using odeint in Python, or GSL in C, or implementing a Runge-Kutta solver from scratch) the simplified Einstein-Boltzmann system for different values of $k \in (10^{-4}, 10)$ h Mpc^{-1}, setting the initial conditions at $a^{(i)} = 10^{-8}$.

[1] The points obtained by solving this exercise are bonus, it means that with a full score this problem set is worth 15/10. This exercise is heavily inspired by Ex. 2 of Sec. 8 (Sec. 7 in the first edition) of Dodelson's book "Modern Cosmology".

Hint: pay attention to Θ_0 and Θ_1 after matter-radiation equality: you could find numerical instabilities. After this time, you can simply set the multipoles to zero since their impact over the evolution of the gravitational potential is irrelevant.

- Plot the temporal evolution of these quantities for some wavenumbers; try to see if you can reproduce Fig. 8.6 (Fig. 7.6 in the first edition) in Dodelson's book. On very small scales ($k \sim 1 - 10$h Mpc^{-1}), during radiation domination your solution should coincide with the analytic expression:

$$\Phi(\eta, k) = \frac{3j_1(k\eta/\sqrt{3})}{k\eta/\sqrt{3}} \Phi(\eta^{(i)}, k) \,. \tag{9}$$

  Compare it with the numerical result, recalling that during radiation domination $\eta = 1/\mathcal{H}$.

- Compute the matter power spectrum at $z = 0$. Start by considering that at late times the Poisson equation becomes

$$k^2 \Phi = 4\pi G a^2 \rho_m \delta \,. \tag{10}$$

  Use this equation to obtain a relation between $P_\Phi(z, k)$ and $P_\delta(z, k)$. The primordial power spectrum reads

$$P_\Phi^{(i)}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_{\text{pivot}}}\right)^{n_s - 1} \,, \tag{11}$$

  with $A_s = 2.1 \times 10^{-9}$, $n_s = 0.966$, $k_{\text{pivot}} = 0.05$ h Mpc^{-1}. Compute and plot the power spectrum both for Φ and δ.

- Finally, you can compare your results with the solution of the full Einstein-Boltzmann system computed by the code CLASS (http://class-code.net/). Its Python wrapper can be easily installed in a Linux system by typing *pip install classy* in the terminal. Does the overall shape agree? What is the main difference with your result? You can plot the CLASS linear matter power spectrum with the following snippet:

```
import numpy as np
import matplotlib as plt
import classy
from classy import Class
LambdaCDM = Class()
#setting the relevant cosmological parameters
LambdaCDM.set({'Omega_b':0.049,'Omega_cdm':0.265,'h':0.6732,
        'ln10^{10}A_s':3.0448,'n_s':0.96605})
#telling class that I want the matter power spectrum,
#and k_max = 10 h/Mpc
LambdaCDM.set({'output':'mPk','P_k_max_1/Mpc':10})
# run class
LambdaCDM.compute()
#wrapping out the result of the computation:
#this gives the matter power spectrum at z = 0
def P_m_class_scalar(k):
h = LambdaCDM.h()
#needed to recover a result with the units of Mpc/h:
return LambdaCDM.pk(k*h, 0)*(h**3)
#we want a function able to take vectors as input
P_m_class = np.vectorize(P_m_class_scalar)
k = 10**np.arange(-4, 1, 0.1) #Mpc/h
plt.loglog(k, P_m_class(k))
plt.show()
```