Physical Cosmology

Spring Semester 2025
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Problem Set 9

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Exercise 1 [5 points]

During the epoch of inflation, the Einstein–Hilbert action can be written as

$$\mathcal{S}\left[g^{\mu\nu},\phi\right] = \int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{R}{16\pi G} + \mathcal{L}_\phi\right], \tag{1}$$

where

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \tag{2}$$

is the Lagrangian density of the inflation field ϕ with potential $V(\phi)$.

Vary the action $\mathcal{S}$ with respect to ϕ and show that the background evolution of the inflation field is governed by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} = 0\,. \tag{3}$$

Hint: Neglect spatial derivatives of ϕ and assume a flat FLRW metric – this is a reasonable assumption because inflation drives the spatial curvature to zero. You may also need to remember that

$$\partial_\mu\left(\sqrt{-g}\,A^\mu\right) = \sqrt{-g}\,\nabla_\mu A^\mu = \sqrt{-g}\,\left(\partial_\mu A^\mu + \Gamma^\mu_{\mu\alpha}A^\alpha\right)\,. \tag{4}$$

Exercise 2 [5 points]

Assume that the potential energy of the inflation field reads

$$V(\phi) = \lambda\,\phi^4\,, \tag{5}$$

and recall the slow roll condition

$$\epsilon(\phi) = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{1}{V}\frac{\partial V}{\partial\phi}\right)^2 \ll 1\,, \qquad M_{\mathrm{Pl}} = \frac{1}{\sqrt{8\pi G}}\,, \tag{6}$$

where the field is rolling towards $\phi = 0$ from the positive side.

a) At what value of ϕ is this condition broken? Assuming that inflation ends at $\epsilon = 1$, show that the number of e-foldings,

$$\ell(t_i) = \ln\left(\frac{a[t(\epsilon=1)]}{a(t_i)}\right)\,, \tag{7}$$

occuring during inflation is given by

$$\ell(t_i) = \pi G\left(\phi_i^2 - 8M_{\mathrm{Pl}}^2\right)\,. \tag{8}$$

for some initial value $\phi_i = \phi(t_i)$.

Hint: What is $\dot{\ell}$? Recall the evolution equation (3) for the scalar field, and that we can neglect $\ddot{\phi}$ in the slow-roll approximation. Furthermore, recall that H is related to the energy density of the universe via the first Friedmann equation. What dominates the energy density during inflation?

b) Starting from equation (8) and the first Friedmann equation, show that

$$\phi = \phi_i \, \exp\left[-\sqrt{\frac{16\lambda M_{\mathrm{Pl}}^2}{3}}\,(t-t_i)\right] . \tag{9}$$

Then, again starting from the first Friedmann equation and using the solution above for ϕ, show that

$$a = a_i \, \exp\left\{\frac{\phi_i^2}{8M_{\mathrm{Pl}}^2}\left[1-\exp\left(-\sqrt{\frac{64\lambda M_{\mathrm{Pl}}^2}{3}}\,(t-t_i)\right)\right]\right\} . \tag{10}$$

c) Expand the solution above for small time differences $t - t_i$ and show that inflation is approximately exponential in the initial stage, i.e., $a \propto e^{\omega t}$. Show that the constant ω is equal to the Hubble parameter during inflation.