Partial Differential Equations

Monday, 6 November 2023 12:56

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}$$

1-D Wave Equation

Hyperbolic P.D.E.
CLASS

$$\boxed{\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}}$$

1-D Diffusion Equation

Parabolic P.D.E.

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y)}$$

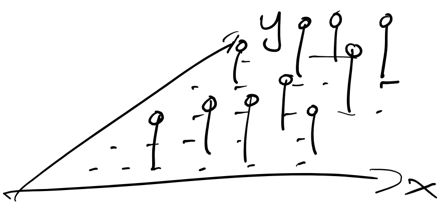
2-D Poisson Equation

Elliptic PDEs

$u(x,y) = ?$

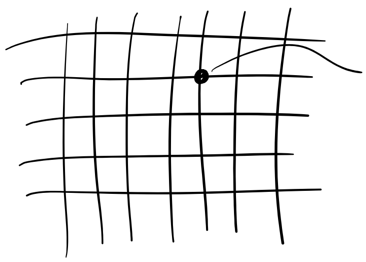
$\rho(x,y) = 0$ $\quad \nabla^2 u = 0$ Laplace Equation

Electrostatic Potential in a Vacuum.



Discrete Points in space
"discretize the problem"

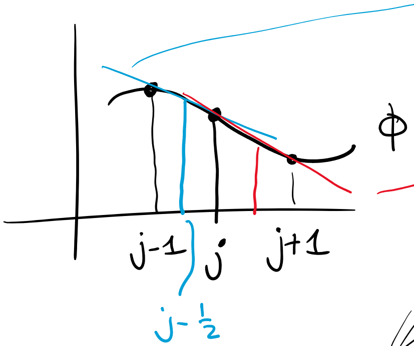
2-D Array of values u
at (x_j, y_l) $\begin{cases} j = 0 \dots J \\ l = 0 \dots L \end{cases}$



Grid spacing $\Delta x, \Delta y$
Simplify $\Delta \equiv \Delta x = \Delta y$

$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = ?$$



$$\left.\frac{\partial \Phi}{\partial x}\right|_{j-\frac{1}{2}} \approx \frac{\phi_j - \phi_{j-1}}{x_j - x_{j-1}} = \frac{\phi_j - \phi_{j-1}}{\Delta}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{j+\frac{1}{2}} \approx \frac{\phi_{j+1} - \phi_j}{\Delta}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right] \quad \left.\frac{\partial \phi}{\partial x}\right| - \left.\frac{\partial \phi}{\partial x}\right|$$

$j-\frac{1}{2}$

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right] \approx \frac{\left.\frac{\partial \phi}{\partial x}\right|_{j+\frac{1}{2}} - \left.\frac{\partial \phi}{\partial x}\right|_{j-\frac{1}{2}}}{\Delta}$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j \approx \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta^2}$$

$$\approx \frac{1}{\Delta^2}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_\ell \approx \frac{1}{\Delta^2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]_{j,\ell} \cong \frac{1}{\Delta^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$$ "Stencil"

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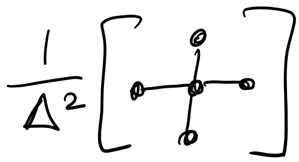
the equation reads $\nabla^2 \phi = 0$ Continuous

 $\phi = 0$ Discrete

$$\forall j,\ell : \quad \phi_{j+1,\ell} + \phi_{j-1,\ell} + \phi_{j,\ell+1} + \phi_{j,\ell-1} - 4\phi_{j\ell} = 0$$

$$\forall j,\ell : \phi_{j\ell}^{(new)} = \frac{1}{4}\left[\phi_{j+1,\ell}^{(old)} + \phi_{j-1,\ell}^{(old)} + \phi_{j,\ell+1}^{(old)} + \phi_{j,\ell-1}^{(old)}\right]$$

Iterate to Convergence

Jacobi Method

Jacobi Method very slowly converging.

$N_{iter} \sim \frac{1}{2} p N^2$ on an $N \times N$ grid ($J = L = N$)

to reduce the error by a factor of 10^{-p}

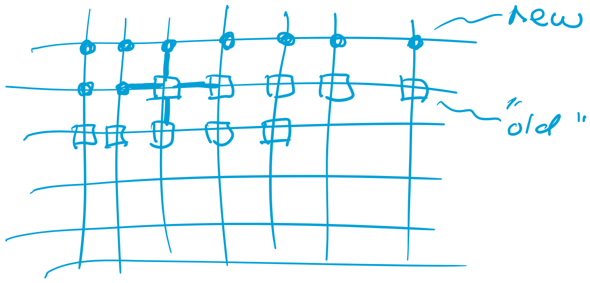
Very slow : $N_{ops} = N_{iter} \cdot N^2 \cdot \underbrace{N_{ops}/grid}_{4}$

$$\boxed{N_{ops} = 2 p N^4}$$

$\mathcal{O}(N^4)$

$\boxed{N_{ops} = \alpha p N} \quad \mathcal{O}(N^4)$

Gauss - Seidel
"sweeps"



$N_{iter} \sim \mathcal{O}(N)$ algorithm $\Rightarrow \mathcal{O}(N^3)$

$N_{ops} \sim \mathcal{O}(N^2 \log N) \Rightarrow$ FFT

$N_{iter} \sim \underset{\text{constant}}{\mathcal{O}(k)} \Rightarrow \mathcal{O}(N^2)$ algorithm

Multigrid best

SOR - Successive Over-Relaxation

$$\Phi_{je}^{(n+1)} = \Phi_{je}^{(n)} + \underbrace{\frac{1}{4}\left(\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}\right)^{(n)}}_{\text{correction}}$$

"< 1 volt"

instead

$$\Phi_{je}^{(n+1)} = \Phi_{je}^{(n)} + \frac{\omega}{4}\left(\begin{array}{ccc} & \circ & \\ \circ & \circ & \circ \\ & \circ & \end{array}\right)^{(n)}$$

$\omega = 1 \Rightarrow$ Jacobi

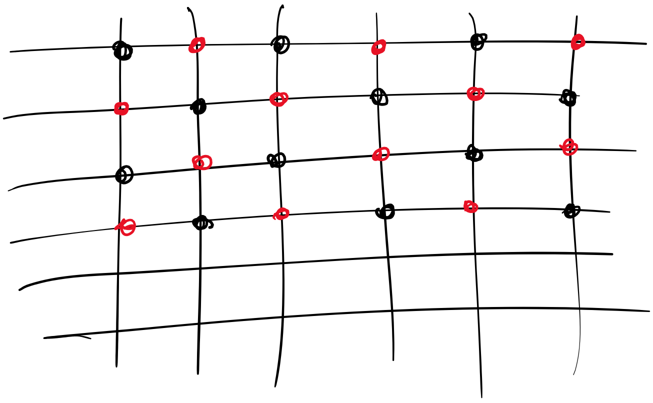
$\omega > 1 \Rightarrow$ SOR $\quad (\omega < 2)$

If ω is optimal $\boxed{N_{iter} \sim \frac{1}{3} p N}$!

$$\omega \cong \frac{2}{1 + \pi/N} \qquad \omega \to 1.7 - 1.9$$

"In place" (one grid of Φ)

Chess board pattern of Read and Write

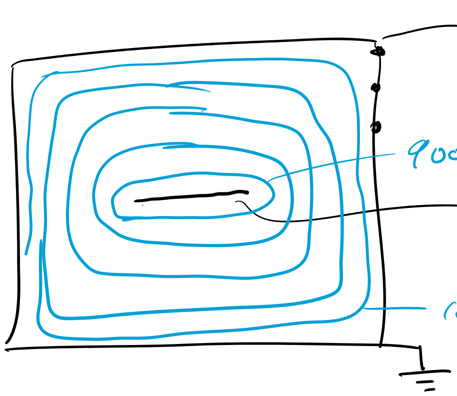


Every black point is independent of every other black point in the calculation.

Same is true of the red ones.

PDE on its own doesn't give a solution.

Need Boundary Conditions



$$\Phi_{je}^{(n+1)} = \Phi_{je}^{(n)} + \frac{\omega}{4}\left(\;\right)^{(n)}$$

for B.C. set this to $\emptyset$!

$$\Phi_{je}^{(n+1)} = \Phi_{je}^{(n)} + R_{je}\left(\;\right)^{(n)}$$

initially set every j,e to $\frac{\omega}{4}$

$$\mathrm{Max}_{je}\left[R_{je}\left(\;\right)\right] < 1 \text{ volt?}$$