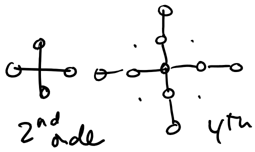
Multigrid Method

Monday, 3 April 2023 10:13

Recall $\nabla^2\phi = \rho$



2nd order 4th order

Now let, $\mathcal{L}_h \phi_h = \rho_h$, $\mathcal{L}$ be a linear operator on ϕ (h is the mesh size)

Let $\tilde{\phi}_h$ denote some approximate solution of the above and let ϕ_h be the exact solution

$$\phi_h = \mathcal{L}_h^{-1}\rho_h \quad \text{(solved via linear algebra)}$$

Then consider the <u>correction</u> to the solution given by,

$$v_h = \phi_h - \tilde{\phi}_h,$$

and the residual (or defect) be,

$$d_h = \mathcal{L}_h\tilde{\phi}_h - \rho_h$$

Then $d_h = \mathcal{L}_h\tilde{\phi}_h - \mathcal{L}_h\phi_h$

$d_h = \mathcal{L}_h(\tilde{\phi}_h - \phi_h)$ since $\mathcal{L}$ is linear

$$\boxed{\mathcal{L}_h v_h = -d_h}$$

- For the Laplace equation $\nabla^2\phi = 0$ $\rho_h = 0$ so $d_h = \mathcal{L}_h\tilde{\phi}_h$ in this case.

Iterative methods do this by using Jacobi or Gauss-Seidel to iterate

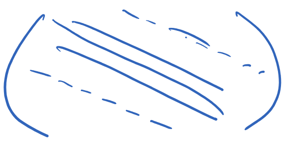
$$\hat{\mathcal{L}}_h \hat{v}_h = -d_h$$

e.g.
$$\hat{v}_i = -\frac{1}{L_{ii}}\left(\sum_{\substack{i=0\\ j\neq i}}^{N-1} L_{ij}\hat{v}_j + d_i\right)$$

for SOR $w > 1$

for N mesh points

i-th mesh point of $\hat{v}_h$

$\mathcal{L}_h =$ 

Then the next approximation is

$$\Phi_h = \bar{\Phi}_h + v_h$$

<u>Coarsening the grid</u>: approximate the operator by coarsening

$$\boxed{\mathcal{L}_H v_H = -d_H}$$

where H is a coarse mesh representation.

Typically we use $H = 2h$.

Since $\mathcal{L}_H$ is of smaller dimension it will be easier to solve (Recall SOR scales as $N^{1.5}$ if $M \times M$ mesh $\to M^3$)

↳ so 8× faster in 3-D

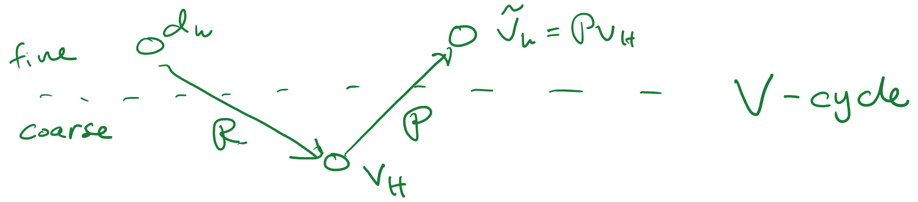
Now, how do we get the coarse version of the defect? This is called <u>restriction</u>, R.

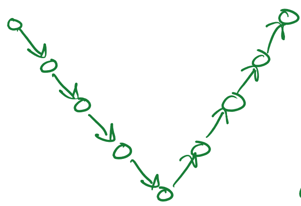
$$d_H = R\, d_h$$

After we solve for the correction, v_H, we need to map it back to the fine grid. This is called <u>prolongation</u>, P.

$$\tilde{v}_h = P\, v_H$$

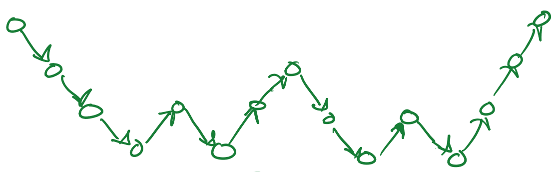
Finally $$\tilde{\Phi}^{new} = \tilde{\Phi}_h + \tilde{v}_h$$





you can nest the V-cyles

or nesting 2 V-cyles → W-cycle



<u>Steps</u>: ① Pre-smoothing: compute $\bar{\Phi}_h$ by applying a few steps of regular relaxation (G-S, SOR ...) to Φ_h

② Coarse grid correction

a) Compute defect on fine grid d_h

b) Restrict $d_H = R\, d_h$

c) "Solve for correction" (recursive)

$$\mathcal{L} v_H = -d_H$$

d) Prolong $\tilde{v}_h = P v_H$

e) Compute next approx: $\tilde{\phi}_h = \tilde{\phi}_h + \tilde{v}_h$

③ Post smooth $\tilde{\phi}_h^{new}$.

R and P: $\quad d_h = P R d_h$

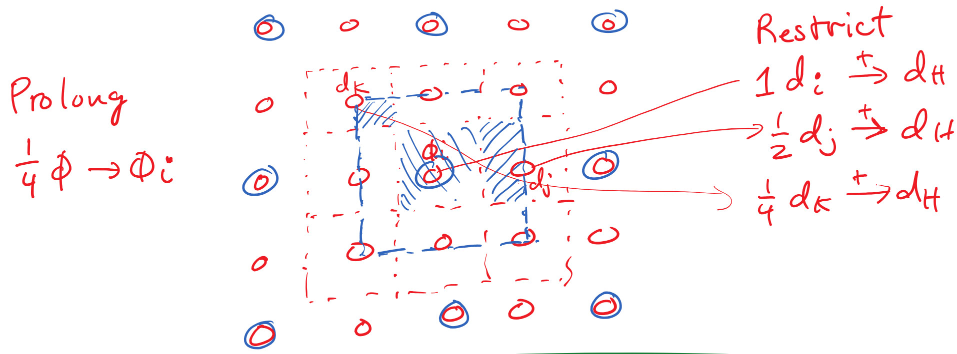
from which $R = P^+$, the adjoint operator to prolongation

For: $\mathcal{L} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad R = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$$P = \frac{1}{4} R^T = \frac{1}{4} \left[\text{" same "} \right]$$

P here is 2nd order operator, so is R and also $\mathcal{L}$

Make sure $\quad \underset{2}{O(P)} \underset{+}{+} \underset{2}{O(R)} \underset{✓}{>} \underset{2}{O(\mathcal{L})}$



☆ What about boundary conditions?

Let's assume that the BC lies on both the coarse and fine grid points

B.C. on left points

$$\mathcal{L} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \quad P = \begin{bmatrix} 0 & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$