SPH Method

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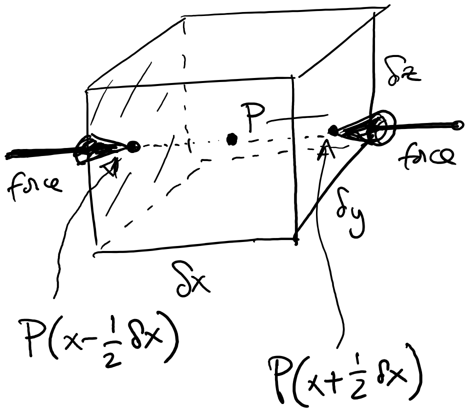
$$\frac{d\underline{u}}{dt} = \underline{g} - \frac{\nabla P}{\rho}$$



Pressure is a force per unit area

Force in +ve x-direction:

$$\underbrace{\left\{\cancel{P} - \frac{1}{2}\delta x \frac{\partial P}{\partial x}\right\}}_{\text{force/unit area}} \cdot \delta y \delta z$$

in -ve x-direction

$$\left\{\cancel{P} + \frac{1}{2}\delta x \frac{\partial P}{\partial x}\right\} \cdot \delta y \delta z$$ subtract this

$$= -\frac{\partial P}{\partial x} \cdot \delta x \delta y \delta z$$

this is the net force in the x-direction on the small cube of fluid due to the pressure gradient.

$$\rho \cancel{\delta x \delta y \delta z} \frac{d\underline{u}}{dt} = -\frac{\partial P}{\partial x} \cancel{\delta x \delta y \delta z}$$

$$\boxed{\frac{d\underline{u}}{dt} = -\frac{\nabla P}{\rho}}$$

Euler equation

Conservation of Momentum.

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}$$

What do we have? $\underbrace{\underline{u}, P \text{ and } \rho}_{5 \text{ variables}}$

So we need at least 2 more equations to close the system of equations.

1. Continuity Equation : cons of mass
2. Energy Equation : cons of energy

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1. Consider the infinitesimal cube

$$\left\{ \rho u - \frac{1}{2} \frac{\partial(\rho u)}{\partial x} \delta x \right\} \delta y \, \delta z$$ +ve - x-direction

net inflow $\Rightarrow$ $-\frac{\partial(\rho u)}{\partial x} \delta x \delta y \delta z$

Total net inflow $= -\nabla\cdot(\rho \underline{u}) \, \delta V$

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla\cdot(\rho \underline{u})}_{\underline{u}\cdot\nabla\rho \, + \, \rho\nabla\cdot\underline{u}} = 0$$ Continuity Equation

$$\frac{d\rho}{dt} + \rho \nabla\cdot\underline{u} = 0$$ ✓ Lagrangian Continuity Equation

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2. Conservation of Energy

$$\frac{de}{dt} = -\left(\frac{P}{\rho}\right) \nabla\cdot\underline{u}$$

e: Specific Internal Energy

per unit mass

Total Energy is given by

$$E = \rho\left(\frac{1}{2}\underline{u}\cdot\underline{u} + e\right)$$

What variables $\underline{u}, \rho, e, P$ 6 variables

and a system of 5 equations.

However we need to specify the equation of state (EOS) of our fluid.

e.g., Ideal gas:

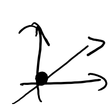
$$\boxed{e = \frac{P}{\rho(\gamma - 1)} \qquad \gamma = \frac{f+2}{f}}$$

f is the number of degrees of freedom.

$$e = \frac{k_B T}{\mathcal{N} \, m_H}$$

mass of ...

Monatomic



f=3 $\quad \gamma = \frac{5}{3}$, 2

ionized Hydrogen

molar mass hydrogen e.g. [illegible]

e^- • • p^+ } mean molar mass $\mu = 0.5$

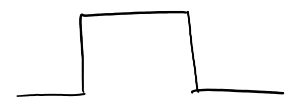
SPH - use particles to follow the flow and consider various integrals in the conservation laws and use interpolants for these.

$$A_I(\underline{r}) = \int_{\text{all space}} A(\underline{r}')\, W(\underline{r}-\underline{r}'; h)\, d\underline{r}'$$

"Kernel"

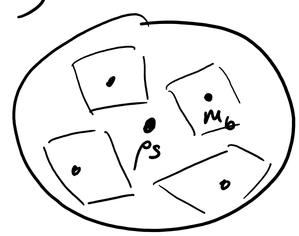
1. $\int W \, d\underline{r} = 1$
2. $\lim_{h\to 0} W(\underline{r}-\underline{r}'; h) = \delta(\underline{r}-\underline{r}')$

approximate this via summation



$$A_s(\underline{r}) = \sum_b m_b \frac{A_b}{\rho_b} W(\underline{r}-\underline{r}_b, h)$$

$$\boxed{\rho_s(\underline{r}) = \sum_b m_b W(\underline{r}-\underline{r}_b, h)}$$



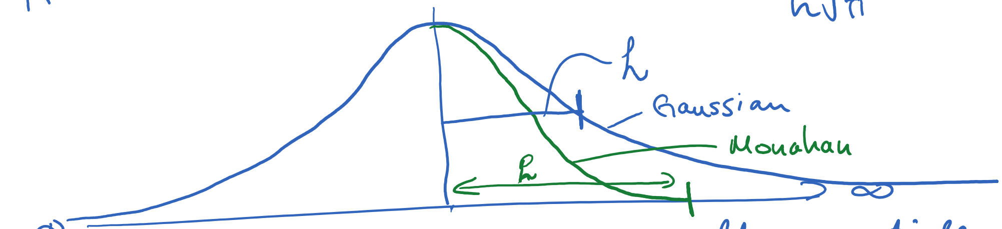
Also need to approximate gradients of this:

$$\nabla A_s(\underline{r}) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\underline{r}-\underline{r}_b, h)$$

although higher accuracy is obtained by

$$\rho \nabla A = \nabla(\rho A) - A \nabla \rho$$

Kernels: Gaussian $W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x^2}{h^2}\right)}$

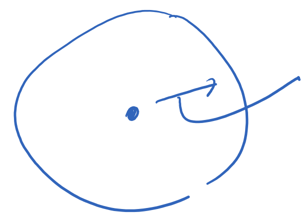


b must run over all particles

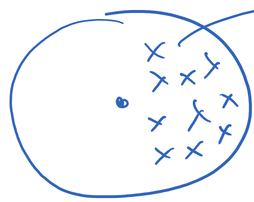
$-\infty$

b must run over all particles no matter how far away they are.

Compact Kernels, finite extent!



Kernel over the 32 nearest neighbors



BAD.

Compact, smooth, easy to evaluate

Monahan Kernel: "cubic-spline kernel"

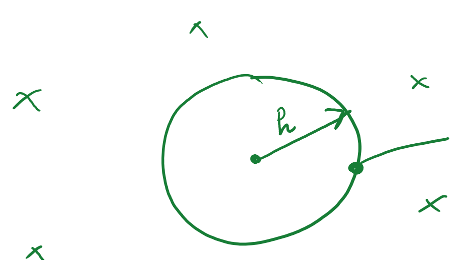
$$W(r;h) = \frac{\sigma}{h^d}\begin{cases} 6\left(\frac{r}{h}\right)^3 - 6\left(\frac{r}{h}\right)^2 + 1, & 0 \le \frac{r}{h} < \frac{1}{2} \\ 2\left(1-\left(\frac{r}{h}\right)\right)^3, & \frac{1}{2} \le \frac{r}{h} \le 1 \\ 0, & \frac{r}{h} > 1 \end{cases}$$

d- dimension

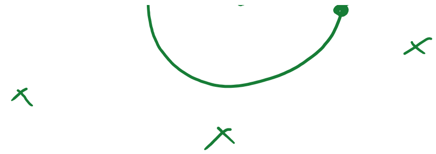
Normalizing constant $\sigma = \begin{cases} 4/3 & \text{in 1-D} \quad d=1 \\ 40/7\pi & \text{in 2-D} \\ 8/\pi & \text{in 3-D} \end{cases}$

$$\frac{\partial W(r;h)}{\partial r} = \frac{6\sigma}{h^{d+1}}\begin{cases} 3\left(\frac{r}{h}\right)^2 - 2\left(\frac{r}{h}\right), & \frac{r}{h} < \frac{1}{2} \\ -\left(1-\left(\frac{r}{h}\right)\right)^2, & \frac{1}{2} \le \frac{r}{h} \le 1 \end{cases}$$

- Calculate ρ, but use Monahan Kernel
  compare visually to the "top-hat" kernel result.



h = r of 32nd nearest neighbor



neighbor

$\left(\frac{r}{h}\right) = 1$ and $W = 0$!