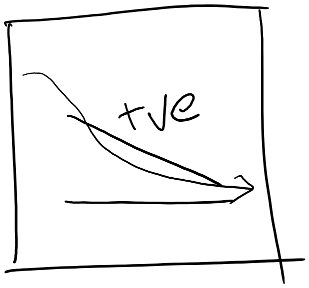
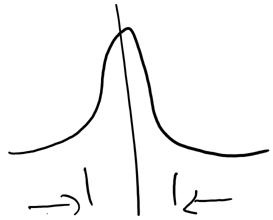
Hydrodynamics in 1-D

13 December 2021 12:28

Noé Zimmermann winner!





☆ Oral Exam Schedule is set via doodle

Recall: LAX Method:

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{1}{2}C\left(\rho_{i+1}^{(n)} - \rho_{i-1}^{(n)}\right)$$

$$C = \frac{\Delta t\, u}{\Delta x}$$

For linear advection equation, the flux is given by $f(\rho) = \rho u$

Rewrite the above equation using fluxes:

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{\Delta t}{2\Delta x}\left(f_{i+1}^{(n)} - f_{i-1}^{(n)}\right)$$

Rewrite the 1st order Upwind Method:

$$\rho_i^{(n+1)} = \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_i^{(n)} - f_{i-1}^{(n)}\right) \quad \text{for } u \geq 0$$

$$\rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_{i+1}^{(n)} - f_i^{(n)}\right) \quad \text{for } u < 0$$

Now for 1-D hydrodynamics, we have 3 conservation equations:

1. Conservation of mass: $F = \rho u$

2. Conservation of momentum: $F = \rho u^2 + P$
3. Conservation of energy: $F = u(E + P)$

P is the pressure.

Now let's use a "State Vector"

$$\underline{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$$

$$\underline{F}(\underline{U}) = [\rho u, \ \rho u^2 + P, \ u(E+P)]$$

little u is the velocity in the x-direction!

$$\boxed{\frac{\partial \underline{U}}{\partial t} + \nabla \cdot F(\underline{U}) = 0}$$

Very general equation of "stuff".

We have 3 equations, but we have 4 unknowns: ρ, u, P, E!

E = Kinetic Energy + Thermal Energy Density

$= \frac{1}{2}\rho u^2 + \rho e$ (specific)

thermal energy density

Specific thermal energy

Still have 4 unknowns...

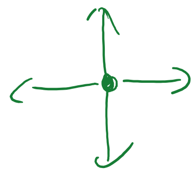
ρ, u, P, e

E.O.S. Equation of State <u>Ideal Gas</u>

$$e = \frac{P}{(\gamma - 1)\rho} \qquad \gamma = \frac{f+2}{f}$$

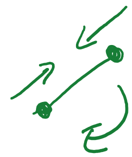
where f is the number of degrees of freedom

Mon atomic Gas



in 2-D Monoatomic Gas

$f = 2 \quad \boxed{\gamma = 2}$



in 1-D $\gamma = 3$ ← we will consider this

$$E = \frac{1}{2}\rho u^2 + \frac{1}{2}P$$

$$\underline{F}(\underline{u}) = \left[\rho u,\ \rho u^2 + P,\ u\Big(\underbrace{\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}P}_{E+P}\Big)\right]$$

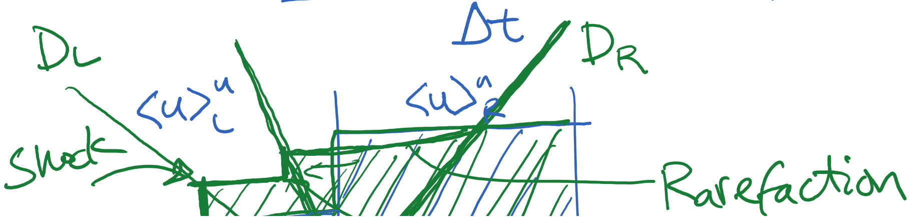
In finite volume let the values of $\underline{u}$ in each cell be the <u>average</u> of the "material" in cell:

$$\langle u \rangle_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)\, dx$$

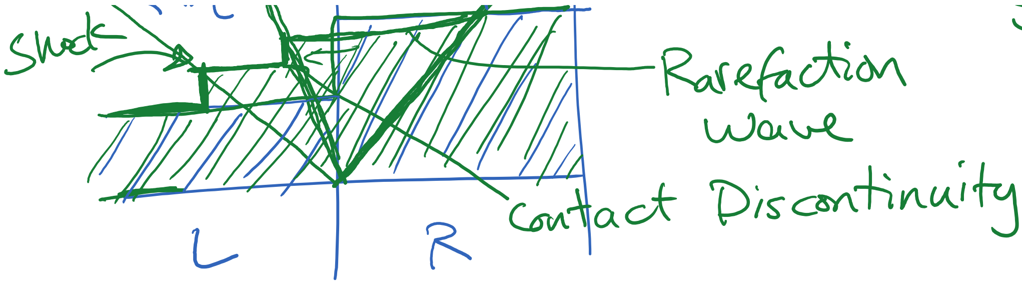
More complicated than for linear advection because information flows in more than one direction

$$\frac{\partial \underline{u}}{\partial t} + \nabla . F(\underline{u}) = 0$$

$$\frac{\langle u \rangle_i^{n+1} - \langle u \rangle_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^{n+1/2} - F_{i-\frac{1}{2}}^{n+1/2}}{\Delta x} = 0$$



3 Conserved Quantities



3 Conserved Quantities $\Rightarrow$ 3 waves
Characteristics the 3 lines of information flow.

$$F(\underline{U}_L, \underline{U}_R)$$ ← Riemann Solver

D_L, D_R ← discontinuity speed of the left and right wave.

Approximate Riemann Solver:

$D_L = -D_R$ simplification by assuming set to the maximum discontinuity speed, D_{max}

$$D = |u| + C_s$$ ← sound speed

$$C_s = \sqrt{\gamma \frac{P}{\rho}}$$ speed of sound

$$D_{max} = \max(D_L, D_R)$$

Lax-Friedrichs Riemann Solver

$$\boxed{F^n_{i-\frac{1}{2}} = \frac{1}{2}\left[\underline{F}(\underline{U}^n_i) + \underline{F}(\underline{U}^n_{i-1})\right] - \frac{1}{2} D_{max}\left[\underline{U}^n_i - \underline{U}^n_{i-1}\right]}$$

Back to our time-space centered approximation of the averages in cells:

Back to our approximation of the averages in cells:

$$\underline{U}_i^{n+1} = \underline{U}_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right]$$

* for $n+\frac{1}{2}$ we could just use n for this $\Longrightarrow$ Method $\boxed{B}$

How could I get the flux at $n+\frac{1}{2}$:

We make a predictor step to $n+\frac{1}{2}$ in order to calculate these fluxes:

$${}^{*}\underline{U}_i^{n+\frac{1}{2}} = \underline{U}_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right] \quad \text{predictor step}$$

we follow this with an update step:

$$\underline{U}_i^{n+1} = \underline{U}_i^n - \frac{\Delta t}{\Delta x}\left[F\left({}^{*}\underline{U}_i^{n+\frac{1}{2}}, {}^{*}\underline{U}_{i+1}^{n+\frac{1}{2}}\right) - F\left({}^{*}\underline{U}_{i-1}^{n+\frac{1}{2}}, {}^{*}U_i^{n+\frac{1}{2}}\right)\right]$$

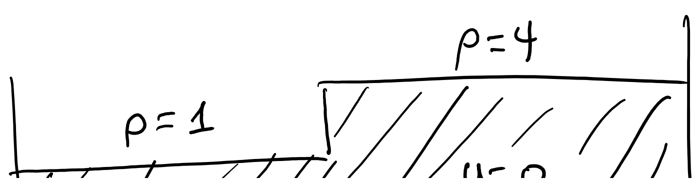
Method $\boxed{C}$

Method $\boxed{A}$: the original LAX Method

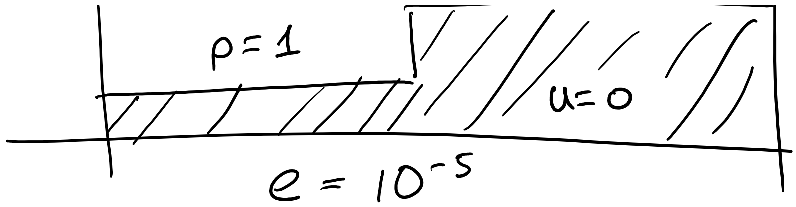
$$U_i^{n+1} = \frac{1}{2}\left(U_{i+1}^n + U_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right)$$

* No Riemann Solver involved!

Try is a "shock tube" problem



Methods $\boxed{A}$, $\boxed{B}$ and



$\boxed{A}$, $\boxed{B}$ and $\boxed{C}$

$\hookrightarrow$ solve for P given ρ

Sedov-Taylor Blast wave solution

one cell has $e = 1$ (like bomb)

$\rho = 1$, $u = 0$
$e = 10^{-5}$

$\Delta t = ?$ Can use Dmax like the Courant condition

$$\boxed{D_{max} < \frac{\Delta x}{\Delta t}}$$