Recall: LAX Method:

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{1}{2} C \left(\rho_{i+1}^{(n)} - \rho_{i-1}^{(n)}\right)$$

$$C = \frac{\Delta t \cdot u}{\Delta x}$$

For linear advection equation, the flux is given by $f(\rho) = \rho u$

Rewrite the above using fluxes:

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{\Delta t}{2 \Delta x}\left(f_{i+1}^{(n)} - f_{i-1}^{(n)}\right)$$

(Method A)

Rewrite Upwind scheme (1st order):

$$\rho_i^{(n+1)} = \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_i^{(n)} - f_{i-1}^{(n)}\right) \quad \text{for } u \geq 0$$

$$\rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_{i+1}^{(n)} - f_i^{(n)}\right) \quad \text{for } u < 0$$

Now for 1-D hydrodynamics, 3 conservation equations:

1. Conservation of mass: $F = \rho u$
2. Conservation of momentum: $F = \rho u^2 + P$
3. Conservation of energy: $F = u(E + P)$

P is pressure, now let's use a "State Vector"

$$\underline{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \qquad \underline{F}(\underline{U}) = [\rho u, \rho u^2 + P, u(E+P)]$$

velocity

$$\boxed{\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0}$$

Very general

$$\boxed{\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0}$$

Very general equation of conservation of "Stuff"

3 equations, but 4 unknowns

ρ, u, P, E

E = Kinetic Energy + Thermal Energy Density

$= \frac{1}{2}\rho u^2 + \rho e$

e ← specific internal energy

Still have 4 unknowns!

ρ, u, P, e

Ideal Gas: Equation of State

$$e = \frac{P}{(\gamma-1)\rho} \qquad \gamma = \frac{f+2}{f}$$

where f is the number of degrees of freedom

2-D Monoatomic Gas

$f = 2 \quad \boxed{\gamma = 2} \qquad$ 3-D: $\gamma = 5/3$

in 1-D $\gamma = 3$

$$\Rightarrow E = \frac{1}{2}\rho u^2 + \frac{1}{2}P$$

for 1-D "gas"

$$\boxed{\underline{F}(\underline{U}) = \left[\rho u, \rho u^2 + P, u\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\right]}$$

for 1-D Monoatomic gas

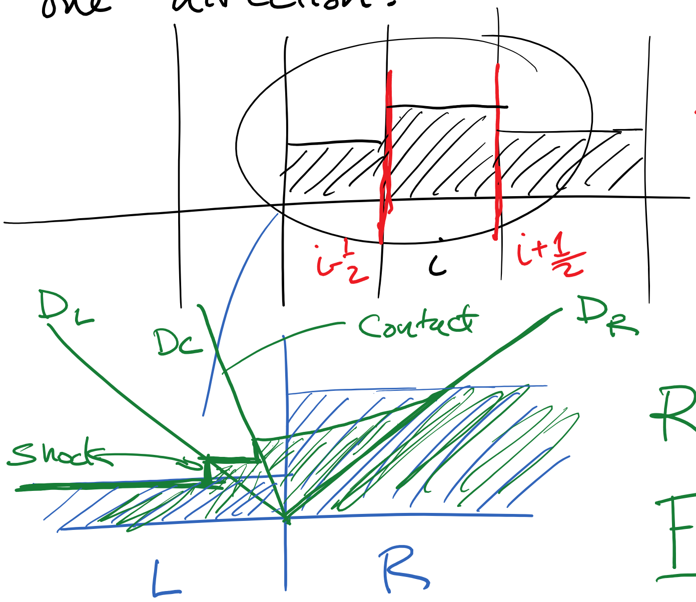
Method $\boxed{A}$: the original LAX Method

$$U_i^{(n+1)} = \frac{1}{2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^{(n)} - F_{i-1}^{(n)}\right)$$

In finite Volume let the values of U in each cell be the average of the "material" in the cell

$$\langle u \rangle_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)\, dx$$

More complicated than for linear advection because information flows in more than one direction.



Godonov

$$\frac{\langle u \rangle_i^{n+1} - \langle u \rangle_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^{n+1/2} - F_{i-\frac{1}{2}}^{n+1/2}}{\Delta x} = 0$$

Riemann Solver

$$\underline{F}(\underline{u}_L, \underline{u}_R)$$

Approximate the Riemann Solver

Only 2 discontinuities, D_L, D_R

Also that $D_L = -D_R$!

and the maximum value of the discontinuity.

$$D = |u| + c_s \quad \text{Sound speed}$$

$$c_s = \sqrt{\gamma \frac{P}{\rho}}$$

$$D_{max} = \max(D_L, D_R)$$

Lax-Friedrich Riemann Solver

$$\boxed{\underline{F}^{n}_{i-\frac{1}{2}} = \frac{1}{2}\left[\underline{F}(\underline{U}^n_i) + \underline{F}(\underline{U}^n_{i-1})\right] - \frac{1}{2} D_{max}\left[\underline{U}^n_i - \underline{U}^n_{i-1}\right]}$$

Recall:

$$\underline{U}_i^{n+1} = \underline{U}_i^n - \frac{\Delta t}{\Delta x}\left[F^{n}_{i+\frac{1}{2}} - F^{n}_{i-\frac{1}{2}}\right]$$

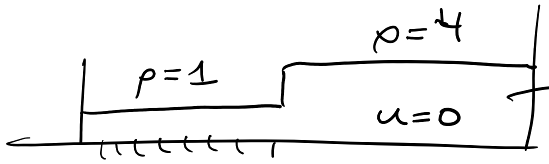
using n where we had $n+\frac{1}{2}$

$\Rightarrow$ Method $\boxed{B}$

Method $\boxed{A}$ does not involve a Riemann solver, but Method $\boxed{B}$ does.

"Shock Tube" Problems

Methods $\boxed{A}$ and $\boxed{B}$



$e = 10^{-5}$

solve for P given ρ.

Sedov-Taylor Blast wave solution

others $e = 10^{-5}$

$\rho = 1$, $u = 0$ ← all have

$\Delta t = ?$ Can use D_{max}

Courant Condition $\boxed{D_{max} < \frac{\Delta x}{\Delta t}}$