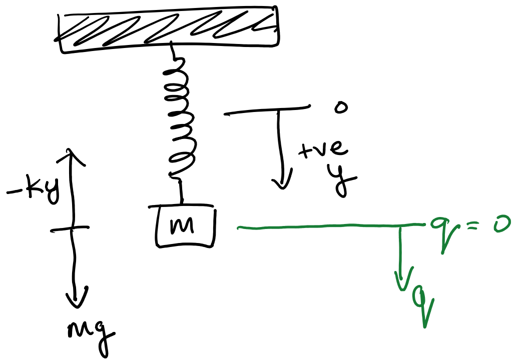
Symplectic Integrators

Monday, 24 October 2022 12:51

ODEs $\longrightarrow$ Runge-Kutta Methods "Black-box"

Harmonic Oscillator:



$$F = mg - ky$$

to make it easier

$m = 1 \quad k = 1$

$$F = g - y$$

Define $q = 0$ where $F = 0$

Now it is true

$$q = y - g = -F$$

Note: $\frac{dy}{dt} = \frac{dq}{dt} = \dot{y}$ ← velocity

Newton's Law $F = ma = m\frac{d^2y}{dt^2} \equiv m\ddot{y}$

$$\ddot{q} = \ddot{y} = F$$

$$\ddot{q} = F = -q \quad \Leftarrow \text{2nd order ODE}$$

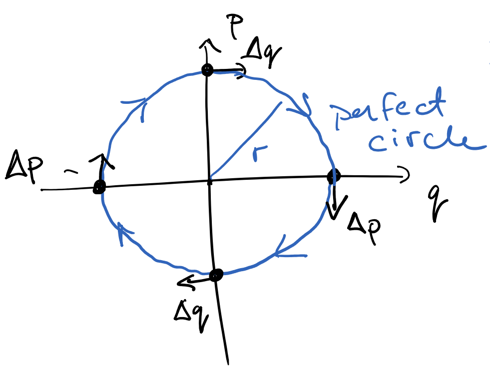
to get to 2 first order ODEs

introduce $p = mv = m\dot{y} = \dot{q}$

$$\dot{p} = \ddot{q}$$

$$\boxed{\begin{aligned}\dot{p} &= -q \\ \dot{q} &= p\end{aligned}}$$

Harmonic Oscillator



Phase diagram

$$r^2 = q^2 + p^2$$

equation of a circle

"Energy" $= q^2 + p^2$

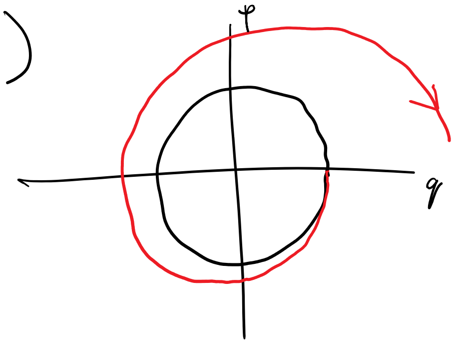
Should be conserved for all time (∞)

$$\left.\begin{aligned} q_{n+1} &= q_n + h p_n \\ p_{n+1} &= p_n - h q_n \end{aligned}\right\} \text{Equations for steps in Forward Euler Method}$$

$p_{n+1} = p_n - h q_n$ Forward Euler

$$q_{n+1}^2 + p_{n+1}^2 = (1+h^2)(q_n^2 + p_n^2)$$

$$r_{n+1}^2 = (1+h^2)\, r_n^2$$



Exponentially growing, or diverging away

Energy is increasing exponentially!

$$E = \underbrace{T}_{\text{Kinetic Energy}} + \underbrace{U}_{\text{Potential Energy}} = \frac{1}{2} m v^2 + \frac{k}{2} y^2 = \frac{1}{2}(p^2 + q^2)$$

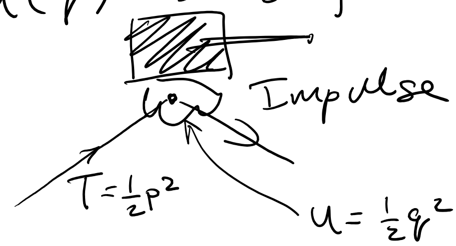
$$H = \frac{1}{2}(p^2+q^2) = T(p) + U(q)$$

seperability

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

$T(p) \longrightarrow$ "free motion" Drift

$U(q) \longrightarrow$ only changes the velocity/momentum



Kick

Leapfrog Method, Strömer-Verlet

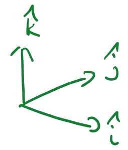
For Harmonic Oscillator

$$h = \Delta t \left\{ \begin{aligned} q_{n+\frac{1}{2}} &= q_n + \tfrac{1}{2} h\, p_n && \text{"half drift"} \\ p_{n+1} &= p_n - h\, q_{n+\frac{1}{2}} && \text{"full kick"} \\ q_{n+1} &= q_{n+\frac{1}{2}} + \tfrac{1}{2} h\, p_{n+1} && \text{"another half drift"} \end{aligned} \right.$$

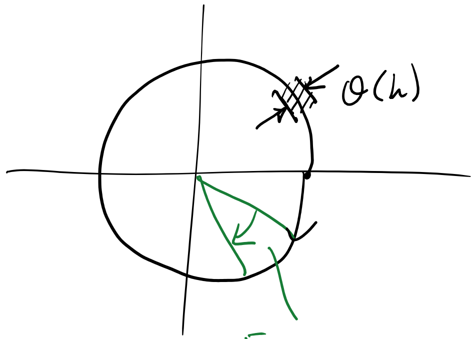
$\hookrightarrow \quad q_{n+1} = q_{n+\frac{1}{2}} + \frac{1}{2} h p_{n+1}$ "another half drift"

In terms of $\underline{x}$ and $\underline{v}$

$$\underline{x}_{\frac{1}{2}} = \underline{x}_0 + \frac{1}{2} h \underline{v}_0$$
$$\underline{v}_1 = \underline{v}_0 + h\left(-\nabla U(\underline{x}_{\frac{1}{2}})\right)$$
$$\underline{x}_1 = \underline{x}_0 + \frac{1}{2} h \underline{v}_1$$



$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$



but... there is no free lunch afterall...

And there is an $O(h^3)$ local error per step!

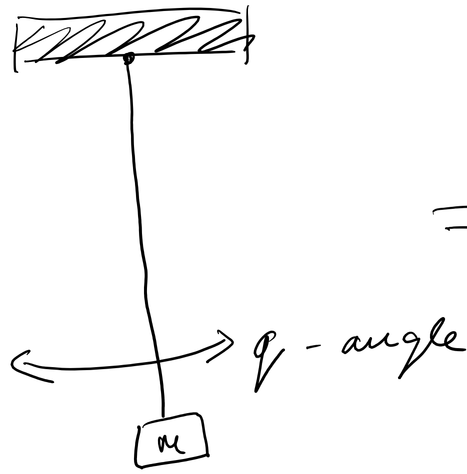
Error accumulates in the angle.

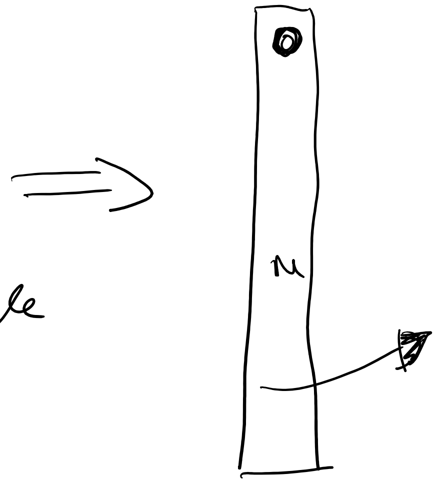
numerical → $\widetilde{\text{Period}} \neq \text{Period}$

$$\tilde{H} \neq H$$

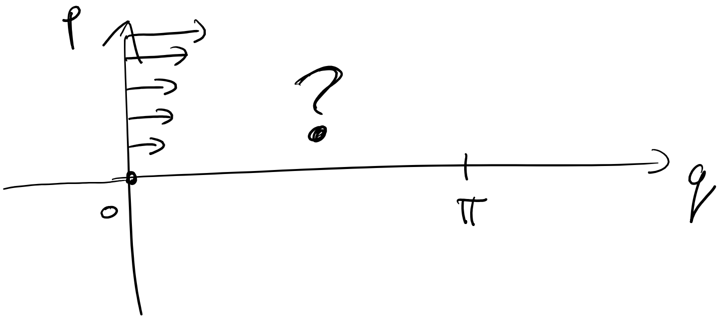
What shall we do?

$$H = \frac{1}{2}p^2 - \varepsilon \cos q$$

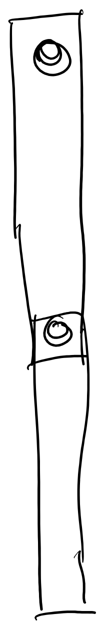




Make phase plots



Test leap-Frog against the mid point method ~~or~~/and RK4



Double pendulum

⇒ Chaotic