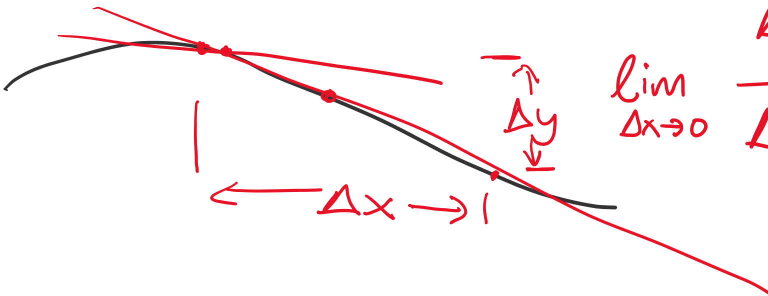
SINS 1 - Ordinary Differential Equations

30 September 2019 13:00



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$ ← not quite an O.D.E.

rewrite this

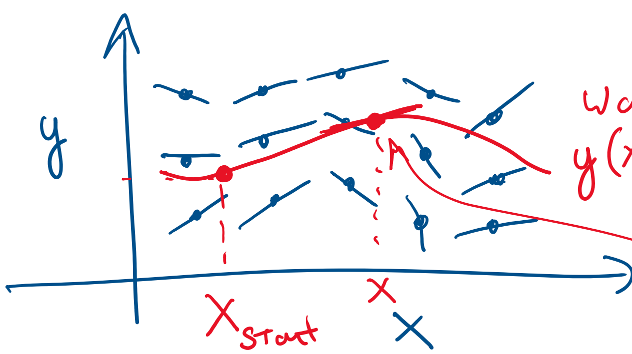
$$\frac{dy}{dx} = z(x)$$

$$\frac{dz}{dx} = r(x) - q(x) \cdot z(x)$$

In general: N-Functions and N-equations

$$\frac{d y_i(x)}{dx} = f_i(x, y_0, y_1, \dots, y_{N-1})$$

$$i = 0, \dots, N-1$$



Boundary Conditions

Initial Conditions

$$\frac{dy}{dx}(x_{start}) = \text{Slope}$$

von Neumann B.C.

$$y(x_{start}) = y_s$$

Dirchlet B.C.

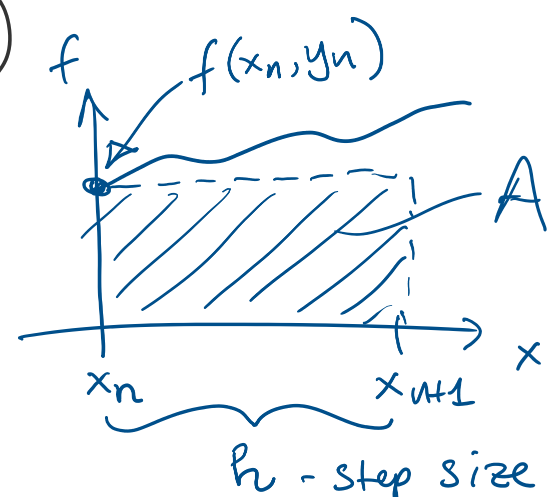
$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\, dx$$

We want $y(x)$ with a certain accuracy:

$$\frac{dy}{dx} = \underline{f}(x, y)$$

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

?



$$y_{n+1} - y_n = h \cdot f(x_n, y_n)$$

$$\boxed{y_{n+1} = y_n + h \cdot \underline{f}(x_n, y_n)}$$

$$y_0 \to y_1 \to y_2 \to y_3 \cdots \to$$

$$y(x) \approx y_0, y_1, y_2$$

Truncation Error: Error associated with the algorithm or method, and not the precision of the floating point calculation.

"h"

Local Error: Error of one Step

Global Error: Error over a fixed interval.

Δx a global interval

$$\int_{x_n}^{x_{n+1}} f(x,y)\,dx = \int_{x_n}^{x_{n+1}} \left[f(x,y_n) + f'(x,y_n)\cdot(y_{n+1}-y_n) \right] dx$$

$$= h\cdot f(x_n,y_n) + h\cdot(y_{n+1}-y_n)\cdot f'(x_n,y_n)$$

Substitute $y_{n+1}-y_n = h\cdot f(x_n,y_n)$

$$= h\cdot f(x_n,y_n) + h^2 f(x_n,y_n) f'(x_n,y_n)$$

$\boxed{\text{Local Error of } \mathcal{O}(h^2)}$

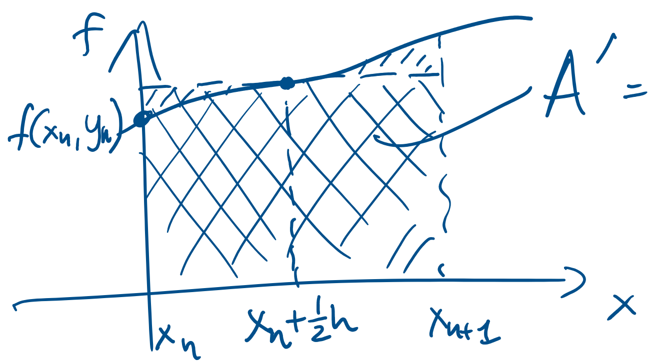
How many Steps $N_{steps} = \dfrac{\Delta X}{h}$

$\boxed{\text{Global Error} \Rightarrow \mathcal{O}(h)}$

Forward Euler Method

Never use except in assignments :)

Can we do better?



$$A' = h\cdot f\Big(x_n+\frac{h}{2}, \underbrace{y\big(x_n+\frac{h}{2}\big)}_{\text{unknown}}\Big)$$

$y_{n+\frac{1}{2}}$

$$y_{n+\frac{1}{2}} \cong y_n + \frac{h}{2} f(x_n, y_n)$$

Then,

$$\boxed{y_{n+1} - y_n = h\cdot f\Big(x_n+\frac{h}{2}, y_n + \frac{h}{2} f(x_n,y_n)\Big)}$$

Midpoint Runge-Kutta

Local error: $\mathcal{O}(h^3)$

Local error: $\mathcal{O}(h^3)$
Global error: $\mathcal{O}(h^2)$

4th - Order Runge-Kutta

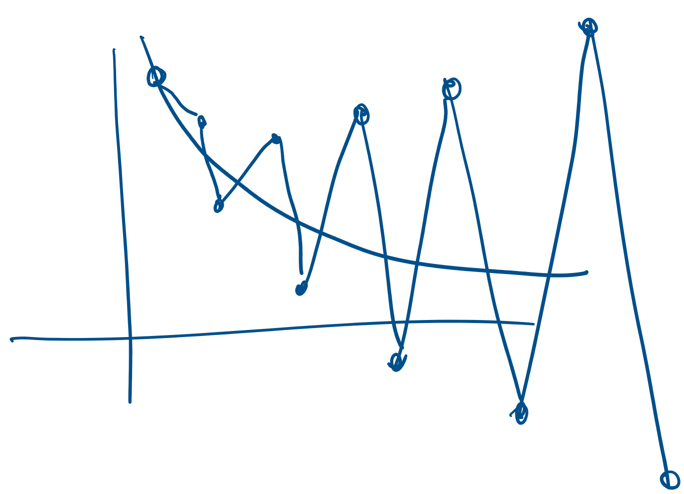
$$\underline{k}_1 = h \cdot \underline{f}(x_n, y_n)$$
$$\underline{k}_2 = h \cdot \underline{f}\left(x_n + \frac{h}{2}, y_n + \frac{\underline{k}_1}{2}\right)$$
$$\underline{k}_3 = h \cdot \underline{f}\left(x_n + \frac{h}{2}, y_n + \frac{\underline{k}_2}{2}\right)$$
$$\underline{k}_4 = h \cdot \underline{f}(x_n + h, y_n + \underline{k}_3)$$

$$y_{n+1} = y_n + \frac{\underline{k}_1}{6} + \frac{\underline{k}_2}{3} + \frac{\underline{k}_3}{3} + \frac{\underline{k}_4}{6} + \mathcal{O}(h^5)$$

Expicit

$$y_{n+1} = y_n + h \cdot f\left(\frac{1}{2}(x_n + x_{n+1}), \frac{1}{2}(y_n + y_{n+1})\right)$$

Implicit Method



Unstable

Predator - Prey Verhalten

Lotka-Voltera Model (1920)

Lotka-Voltera Model (1926)

Foxes and Mice

f $\qquad$ m

Without foxes the mice population grows without limitation.

$$\frac{\Delta m}{m} = k_m \cdot \Delta t$$

↖ Birth rate (constant)

But if foxes are around then the population reduces proportional to the number of foxes.

$$\frac{\Delta m}{m} = k_m \Delta t - k_{mf} \cdot f \cdot \Delta t$$

$$\Delta m = (k_m \cdot m - k_{mf} \cdot \underbrace{mf}_{\text{Number of encounters}}) \Delta t$$

$$\frac{\Delta f}{f} = -k_f \Delta t$$

↖ Death rate for foxes

$$\frac{\Delta f}{f} = -k_f \Delta t + k_{fm}\, m\, \Delta t$$

$$\Delta f = (-k_f \cdot f + k_{fm} f\, m) \Delta t$$

$$dm = k_m \cdot m - k_{mf} \cdot m \cdot f$$

$$\frac{dm}{dt} = k_m \cdot m - k_{mf} \cdot m \cdot f$$

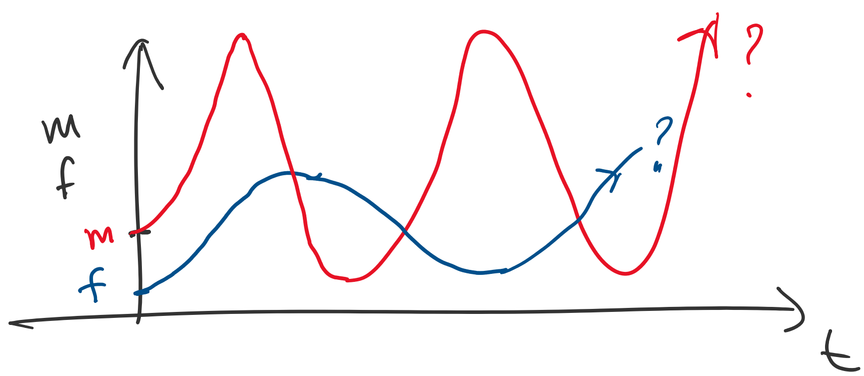
$$\frac{df}{dt} = -k_f \cdot f + k_{fm} \cdot f \cdot m$$

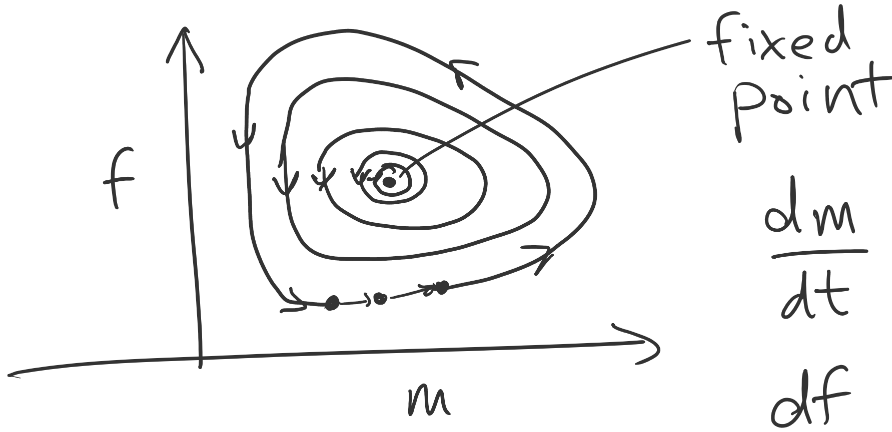
$k_m = 2$
$k_{mf} = 0.02$
$k_{fm} = 0.01$
$k_f = 1.06$

$m(0) = 100$

$f(0) = 15$

2 Plots Please:





$$\frac{dm}{dt} \overset{!}{=} 0$$

?

$$\frac{df}{dt} \overset{!}{=} 0$$