SINS 12: Finite Volume Method, 1-D and 2-D

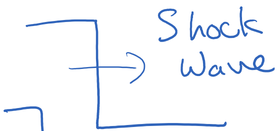
02 December 2019 09:44

Please complete the course evaluation (<5mins), Thanks!

[Evaluation of the Course: https://qmsl.uzh.ch/de/M44J9](https://qmsl.uzh.ch/de/M44J9)

Finite Difference $\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} = 0$

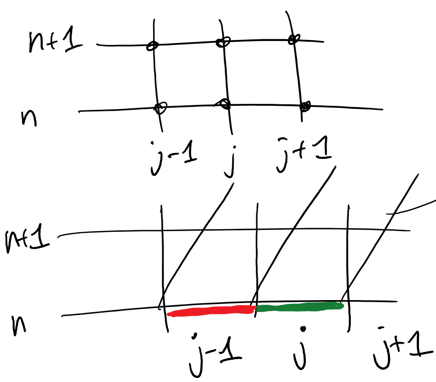
Integral equations:



$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[\underbrace{f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}}_{\text{Approximated numerically}}\right]$$

Integration over the fluxes should be exact.

For linear advection $f(\rho) = a \cdot \rho$



Finite difference

a: Characteristic of the equation

Finite Volume

$$\rho_j^{n+1} = \underbrace{\frac{a \cdot \Delta t}{\Delta x}}_{c} \rho_{j-1}^n + \underbrace{\left(1 - \frac{a \cdot \Delta t}{\Delta x}\right)}_{1-c} \rho_j^n$$

Godunov

$$\rho_j^{n+1} = c\, \rho_{j-1}^n + (1-c)\, \rho_j^n$$

$$\rho_j^{n+1} - \rho_j^n + c\left(\rho_j^n - \rho_{j-1}^n\right)$$

1ST order upwind scheme

"CIR Method"

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} = 0 \qquad \frac{\partial^2 \rho}{\partial x^2}$$

Numerical diffusion added

∂x^2 diffusion added

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + a\,\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} = 0$$

Taylor expand ρ in time to 2^{nd} order

$$\rho_j^{n+1} = \rho_j^n + \Delta t\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)$$

Taylor expand ρ in space to 2^{nd} order

$$\rho_{j+1}^n = \rho_j + \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

$$\rho_{j-1}^n = \rho_j - \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^{\cancel{2}}}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}{\cancel{\Delta t}} + \frac{a\left(\cancel{2\Delta x}\left(\frac{\partial \rho}{\partial x}\right)\right)}{\cancel{2\Delta x}} = 0$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -\frac{\Delta t}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right) + O(\Delta t^2, \Delta x^2)$$

$$\frac{\partial \rho}{\partial t} + a\left(\frac{\partial \rho}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \rho}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial t}\right)$$

$$-a\frac{\partial \rho}{\partial x}$$

$$\frac{\partial^2 \rho}{\partial t^2} = +a^2\frac{\partial^2 \rho}{\partial x^2}$$

$$\overbrace{\boxed{-a^2\frac{\Delta t}{2}}}^{D}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

Advection-diffusion

Unstable because it has a negative diffusion coefficient.

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -a^2\frac{\Delta t}{2}\frac{\partial^2 \rho}{\partial x^2}$$

Modified Equation

2-D Advection $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0$

2-D Advection

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

where $\underline{u} = \langle a, b \rangle$ $\quad a, b > 0$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = 0$$

A first order finite difference is:

$$\frac{\rho_{j\ell}^{n+1} - \rho_{j\ell}^{n}}{\Delta t} + a\,\frac{\rho_{j\ell}^{n} - \rho_{j-1,\ell}^{n}}{\Delta x} + b\,\frac{\rho_{j\ell}^{n} - \rho_{j,\ell-1}^{n}}{\Delta y} = 0$$

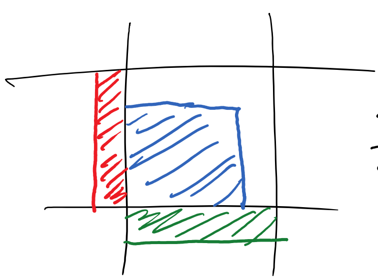
Stability Analysis shows that:

$C_a > 0$ $\quad \boxed{C_a = \frac{a\Delta t}{\Delta x}}$ $\quad \boxed{\text{AND}}$ $\quad \boxed{\frac{a\Delta t}{\Delta x} + \frac{b\Delta t}{\Delta y} \leq 1}$

$C_b > 0$

$$C_a + C_b \leq 1$$

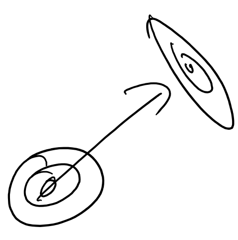
Sum of the Courant numbers in x and y must be less than 1.



Modified equation is interesting:

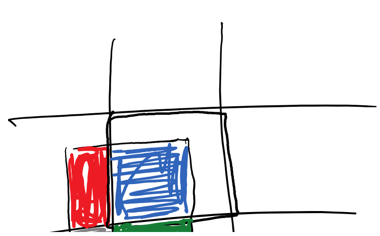
$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = \frac{a\Delta x}{2}(1 - C_a)\frac{\partial^2 \rho}{\partial x^2} + \frac{b\Delta y}{2}(1 - C_b)\frac{\partial^2 \rho}{\partial y^2} \boxed{- ab\,\Delta t\,\frac{\partial^2 \rho}{\partial x \partial y}}$$

won't preserve shape

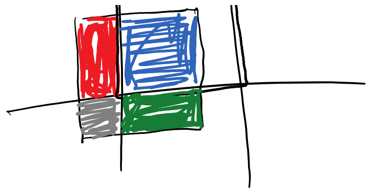


Diffusion in tangential and anti-diffusion in direction of motion.

Corner Transport Upwind



$$\rho_{j\ell}^{n+1} = \underline{(1 - C_a)(1 - C_b)\rho_{j\ell}^{n}}$$



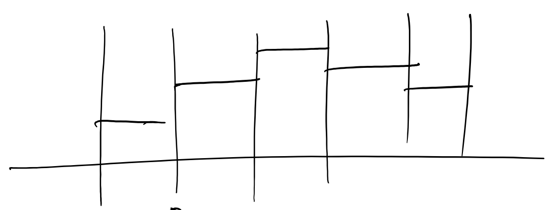
$$\rho_{jl}^{n+1} = (1-C_a)(1-C_b)\rho_{jl}^{n} + C_a(1-C_b)\rho_{j-1\,l}^{n} + (1-C_a)C_b\,\rho_{j\,l-1}^{n} + C_aC_b\,\rho_{j-1\,l-1}^{n} \quad !$$

Stability: $0 \leq C_a \leq 1 \quad 0 \leq C_b \leq 1$

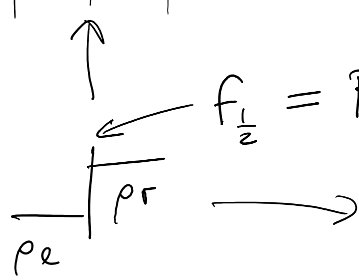
$$C_a = \frac{a\,\Delta t}{\Delta x} \qquad C_b = \frac{b\,\Delta t}{\Delta y}$$

Can also write this as 2 seperate steps:

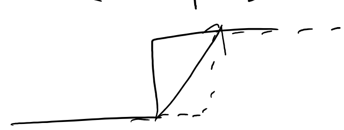
$$\boxed{\begin{aligned} \rho_{jl}^{*} &= (1-C_a)\rho_{jl}^{n} + C_a\,\rho_{j-1\,l}^{n} \\ \rho_{jl}^{n+1} &= (1-C_b)\rho_{jl}^{*} + C_b\,\rho_{j\,l-1}^{*} \end{aligned}}$$



piecewise constant



$f_{\frac{1}{2}} = RP(\rho_l, \rho_r)$



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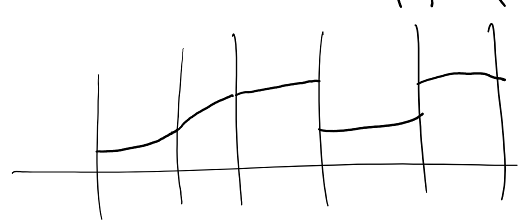
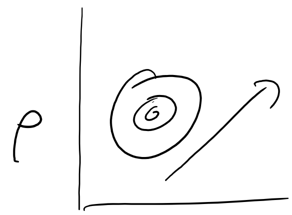
piecewise parabolic method PPM higher order



TEST 2 Methods in 2-D Finite Difference to CTU Method



$$f(x,y) = A \exp\left[-\left(\frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2}\right)\right]$$