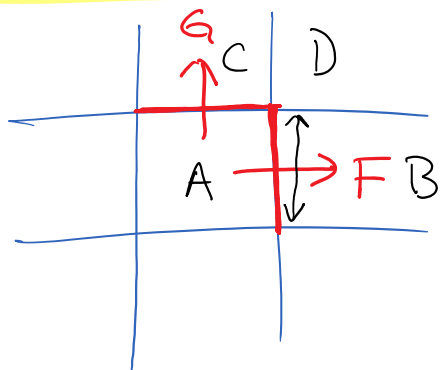


2D again:
$$\frac{\partial \underline{u}}{\partial t} + \nabla \cdot \underline{F}(\underline{u}) = 0 \quad \underline{F} = \begin{bmatrix} F \\ G \end{bmatrix}$$

Piecewise constant states averaged over cell $\langle u \rangle_{i,j}^n$

$$\langle u \rangle_{i,j}^n = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x, y, t^n) dx dy$$

$$\langle u \rangle_{i,j}^{n+1} - \langle u \rangle_{i,j}^n + \Delta x \left[F_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - F_{i-\frac{1}{2},j}^{n+\frac{1}{2}} \right] + \frac{\Delta t}{\Delta y} \left[G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - G_{i,j-\frac{1}{2}}^{n+\frac{1}{2}} \right] = 0$$



$$F_{i+\frac{1}{2},j}^{n+\frac{1}{2}} = \frac{1}{\Delta t} \frac{1}{\Delta y} \int_{t^n}^{t^{n+1}} dt \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} dy F(x_{i+\frac{1}{2}}, y, t)$$

$$G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} dt \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} dx G(x, y_{j+\frac{1}{2}}, t)$$

2-D LAX Scheme:

$$u_{i,j}^{n+1} = \frac{1}{4} \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) - \frac{\Delta t}{2\Delta x} (F_{i+1,j}^n - F_{i-1,j}^n) - \frac{\Delta t}{2\Delta y} (G_{i,j+1}^n - G_{i,j-1}^n)$$

$$\pi = \left(\frac{\rho}{\rho u} \right) \quad F(\underline{u}) = \left[\rho u \quad \rho u^2 + P \quad \rho u v \quad u(E+P) \right]$$

$$\underline{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \quad \underline{F}(\underline{u}) = \begin{bmatrix} \rho u & \rho u^2 + P & \rho uv & u(E+P) \\ \rho v & \rho uv & \rho v^2 + P & v(E+P) \end{bmatrix}$$

$$\underline{E}(\underline{u}) = \begin{bmatrix} F \\ G \end{bmatrix} \quad \begin{array}{l} \text{Flux in x-direction} \\ \text{Flux in y-direction} \end{array}$$

$$E = \frac{1}{2} \rho (u^2 + v^2) + \rho e \quad \begin{array}{l} \text{thermal energy per} \\ \text{unit mass} \end{array}$$

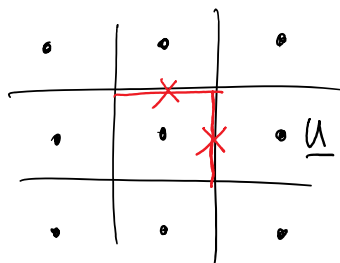
Introduce an equation of state (EOS)

$$e = \frac{P}{(\gamma-1)\rho} \quad \text{Ideal Gas} \quad \gamma = \frac{f+2}{f} = \frac{4}{2}$$

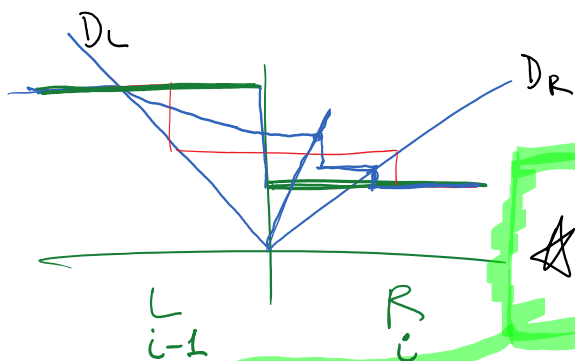
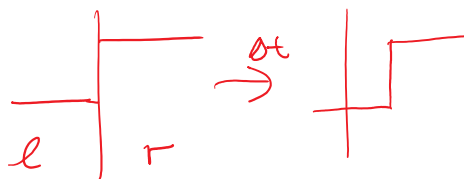
f degrees of freedom
↕

$$\frac{\partial \underline{u}}{\partial t} + \nabla \cdot \underline{F}(\underline{u}) = 0$$

How do we solve the fluxes (Approximate):



Riemann Problem:



★ Lax-Friedrichs Riemann Solver

$$D_L = -D_R$$

$$F_{i-\frac{1}{2}}^n = \frac{1}{2} (F_{i-1}^n + F_i^n) - \frac{1}{2} D_{\max} (u_i^n - u_{i-1}^n)$$

→ $D = |u| \pm c_0$ ← sound speed

Discontinuity Speed $\rightarrow D = |u| + C_s \leftarrow$ sound speed

$$C_s = \sqrt{\gamma \frac{P}{\rho}}$$

$$D_{\max} = \max(D_L, D_R)$$

Corner Transport Upwind:

① Flux in x-directions

$$① F_{i-\frac{1}{2},j}^{n+\frac{1}{2}} = \frac{1}{2} (*F_{i-1,j}^{n+\frac{1}{2}} + *F_{i,j}^{n+\frac{1}{2}}) - \frac{1}{2} D_{\max}^u (*U_{i,j}^{n+\frac{1}{2}} - *U_{i-1,j}^{n+\frac{1}{2}})$$

also need $F_{i+\frac{1}{2},j}^{n+\frac{1}{2}}!$

$$*U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2\Delta y} (G_{i,j+\frac{1}{2}}^n - G_{i,j-\frac{1}{2}}^n) \leftarrow \text{y-direction prediction step}$$

$$G_{i,j+\frac{1}{2}}^n = \frac{1}{2} (G_{i,j+1}^n + G_{i,j}^n) - \frac{1}{2} D_{\max}^v (U_{i,j+1}^n - U_{i,j}^n)$$

$$G_{i,j-\frac{1}{2}}^n = \dots$$

② Flux in y-direction

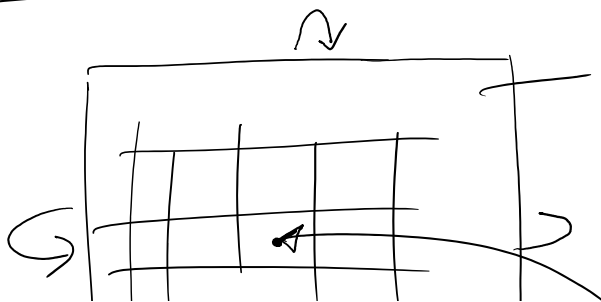
$$② G_{i,j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} (*G_{i,j-1}^{n+\frac{1}{2}} + *G_{i,j}^{n+\frac{1}{2}}) - \frac{1}{2} D_{\max}^v (*U_{i,j}^{n+\frac{1}{2}} - *U_{i,j-1}^{n+\frac{1}{2}})$$

also need $G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}!$

$$*U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2\Delta x} (F_{i+\frac{1}{2},j}^n - F_{i-\frac{1}{2},j}^n)$$

X-direction prediction step!

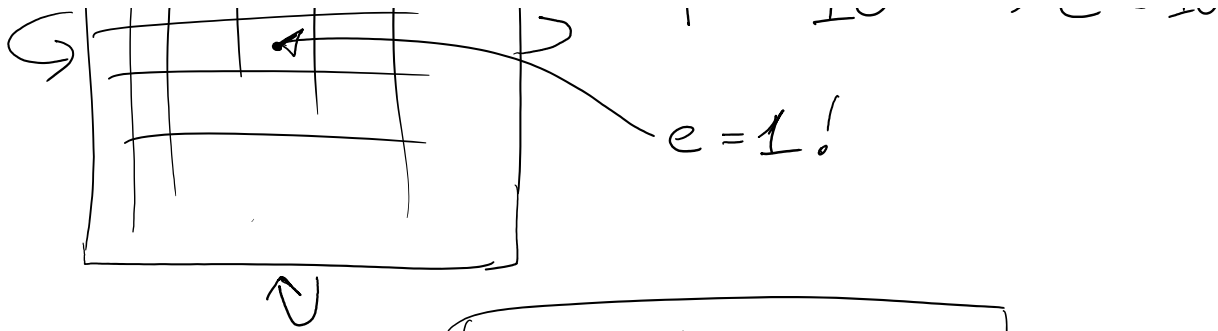
$$F_{i+\frac{1}{2},j}^n = \frac{1}{2} (F_{i+1,j}^n + F_{i,j}^n) - \frac{1}{2} D_{\max}^u (U_{i+1,j}^n - U_{i,j}^n)$$



$$u=0 \quad \rho=1$$

$$v=0$$

$$P = 10^{-5} \rightarrow e = 10^{-5}$$



Colella, 1990