2. Conservation of Energy

$$\frac{de}{dt} = -\left(\frac{P}{\rho}\right)\nabla\cdot\underline{u}$$

Total Energy is given by

$$E = \rho\left(\tfrac{1}{2}\underline{u}\cdot\underline{u} + e\right)$$

e : Specific internal energy of the fluid parcel (per unit mass!)

Now we have $\rho\underline{u}$, ρ, e, P 6 variables and a system of 5 equations.

However we need to specify the equation of State (EOS) for our fluid.

e.g., Ideal gas :
$$e = \frac{P}{\rho(\gamma-1)}$$

$$\gamma = \frac{f+2}{f}$$

f is the number of degrees of freedom

Monatomic in 3D : $f=3$

in 2-D : $f=2$

$$\gamma = \underset{3D}{5/3}\,,\ \underset{2D}{2}$$

$$e = \frac{k_B T}{\mu\, m_H}$$

mean molar mass

$\mu = 0.5$ for a plasma

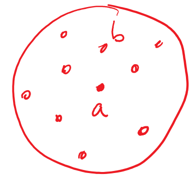
e^-, P

mass of hydrogen

$\underline{u}$ is the velocity of the fluid parcel or in the SPH case the velocity of the particle. Usually we use $\underline{v}$ for a particle velocity, let's use $\underline{v}$ from now on.

$$\nabla\cdot\underline{v} \overset{\sim}{=} \sum_b m_b \underline{V}_b \cdot \nabla W(|\underline{r}-\underline{r}_b|, h)$$

could use this, but better is:



$$\nabla\cdot\underline{v} = \frac{1}{\rho}\left[\nabla\cdot(\rho\underline{v}) - \underline{v}\cdot\nabla\rho\right]$$

$$\nabla\cdot\underline{v}_a \overset{\sim}{=} \frac{1}{\rho_a}\left[\sum_b m_b \frac{\cancel{\rho_b} v_b}{\cancel{\rho_b}}\cdot\nabla_a W_{ab} - \frac{v_a}{\rho_a}\cdot\sum \right]$$

$$\rho_a(\nabla\cdot\underline{v})_a = \sum_b m_b(\underline{v}_b - \underline{v}_a)\cdot\nabla_a W_{ab}$$

$$W(|\underline{r}_a-\underline{r}_b|, h)$$

$$\frac{de}{dt} = -\frac{P}{\rho}\nabla\cdot\underline{v}$$

$$\boxed{\frac{de_a}{dt} = \left(\frac{P_a}{\rho_a^2}\right)\sum_b m_b(\underline{v}_a - \underline{v}_b)\cdot\nabla_a W_{ab}}$$

Benz Formulation

Continuity Equation : consv. of mass

$$\rho_a = \sum_b m_b W_{ab}$$

Momentum Equation $\quad \frac{d\underline{v}_a}{dt} = -\frac{\rho_a \nabla P_a}{\rho_a^2}$

$$= -\frac{1}{\rho_a^2}\sum_b m_b (P_b - P_a)\nabla_a W_{ab}$$

Force → 0 for const. Pressure, but linear momentum and angular momentum are not conserved if you use this.

Instead:

$$\frac{\nabla P}{\rho} = \nabla\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2}\nabla\rho$$

$$\Rightarrow \boxed{\frac{d\underline{v}_a}{dt} = -\sum_b m_b\left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right)\nabla_a W_{ab}}$$

Symmetric between $a \Leftrightarrow b$
obey Newton's 3rd Law and
conserve momentum.

Add Artificial Viscosity to the SPH equations
to cause shocks (pile-ups of material)

$$\frac{d\underline{v}_a}{dt} = -\sum_b m_b\left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \Pi_{ab}\right)\nabla_a W_{ab}$$

$$\Pi_{ab} = \begin{cases} -\alpha\,\overline{c_{ab}}\,\mu_{ab} + \beta\,\mu_{ab}^2, & \underline{v}_{ab}\cdot\underline{r}_{ab} < 0 \\ 0, & \underline{v}_{ab}\cdot\underline{r}_{ab} > 0 \end{cases}$$

$$\mu_{ab} = \frac{h_{ab}\,\underline{v}_{ab}\cdot\underline{r}_{ab}}{r_{ab}^2 + \eta^2}$$

to avoid singularities

$$c_a = \sqrt{\frac{P_a\gamma}{\rho}} \qquad e_a = \frac{P_a}{\rho_a(\gamma-1)} \qquad c_a = \sqrt{\gamma(\gamma-1)e_a}$$

$$\downarrow$$

$$\frac{P}{\rho^2} = \frac{c^2}{\gamma\rho}$$

Variables for Particles: $\underline{r}, \underline{v}, e, c, \rho$

need to calculate $\underline{a}$ acceleration $\dot{e}$

$\underline{v}_{pred}$, e_{pred}

DRIFT1( ), DRIFT2( ), KICK( ), CALCFORCE( )

SPH:

```
DRIFT1(Δt=0)
CALCFORCE()
for (step = 0; step < NSTEP; ++step) {
    DRIFT1(Δt/2)
    CALCFORCE()
    KICK(Δt)
    DRIFT2(Δt/2)
}
```

---

where CALCFORCE() {

TREEBUILD()

NN-Density ← All particles calculate ρ

$\sqrt{\gamma(\gamma-1)\, e_{PRED}}$ → CALCSOUND ← All particles calculate c

NN-SPHFORCE ← All calc $\underline{a}$, $\dot{e}$

}

DRIFT1(Δt)

$$\underline{r} \mathrel{+}= \underline{v}\,\Delta t$$

$$\underline{V}_{pred} = \underline{v} + \underline{a}\,\Delta t$$

$$e_{pred} = e + \dot{e}\,\Delta t$$

DRIFT2(Δt)

$$\underline{r} \mathrel{+}= \underline{v}\,\Delta t$$

KICK(Δt)

$$\underline{V} \mathrel{+}= \underline{a}\,\Delta t$$

$$e \mathrel{+}= \dot{e}\,\Delta t$$

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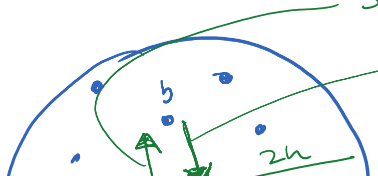
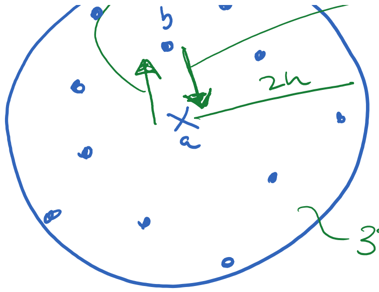
scatter operation

gather operation





32 nearest neighbors : $h_a \neq h_b$

$W_{ab} \neq W_{ba}$

$$W_{ab} = \frac{1}{2}\left(W(r_{ab}, h_a) + W(r_{ab}, h_b) \right)$$

Symmetrize the Kernel

or $h_{ab} = \frac{1}{2}(h_a + h_b)$

harder

$$\frac{d\underline{v}_a}{dt} = -\frac{1}{2}\sum_b m_b \boxed{F_{ab} \nabla_a W(r_{ab}, h_a)}$$

gather part

$$-\frac{1}{2}\sum_b m_b \boxed{F_{ab} \nabla_a W(r_{ab}, h_b)}$$

rewrite a's b's

$$\frac{d\underline{v}_b}{dt} \mathrel{+}= -\frac{1}{2}\sum_a m_a F_{ab} \nabla_b W(r_{ab}, h_a)$$

but $\nabla_b = -\nabla_a$

$$\mathrel{+}= \frac{1}{2}\sum_a m_a \boxed{F_{ab} \nabla_a W(r_{ab}, h_a)}$$

Scatter Term

So we only need to calculate the contribution $F_{ab} \nabla_a W(r_{ab}, h_a)$ once!

Typically we use 32 - 64 neighbors

Trade off between noise and resolution.

The Kernel:

$$W(r;h) = \frac{\sigma}{h^d} \begin{cases} 6\left(\frac{r}{h}\right)^3 - 6\left(\frac{r}{h}\right)^2 + 1, & 0 \leq \frac{r}{h} < \frac{1}{2} \\ 2\left(1 - \left(\frac{r}{h}\right)\right)^3, & \frac{1}{2} \leq \frac{r}{h} \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\quad 0 \quad , \text{otherwise}$$

$r = |\underline{r} - \underline{r}'|$ d- dimension

$$\sigma = \begin{cases} 4/3 & \text{in } 1\text{-D} \\ 40/7\pi & \text{in } 2\text{-D} \\ 8/\pi & \text{in } 3\text{-D} \end{cases}$$

$$\frac{\partial W(r;h)}{\partial r} = \frac{6\sigma}{h^{d+1}} \begin{cases} 3\left(\frac{r}{h}\right)^2 - 2\left(\frac{r}{h}\right), & \frac{r}{h} < \frac{1}{2} \\ -\left(1-\left(\frac{r}{h}\right)\right)^2, & \frac{1}{2} \le \frac{r}{h} \le 1 \end{cases}$$

Monaghan Kernel

↳ Clumping instability

☆ Wendtland Kernels are better